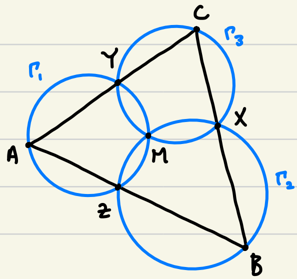
FINAL

Q3. Since Γ_2 and Γ_3 are not tangent, let M be their intersection, with $M \neq X$. We show that $AYMZ$ is a cyclic quadrilateral $\Leftrightarrow \angle AZM + \angle AYM = 2\llcorner$ by a thm seen in class. We have



$$\angle AZM = 2\llcorner - \angle BZM \quad \text{and} \quad \angle AYM = 2\llcorner - \angle CYM. \qquad *$$

But, since $CXMY$ and $BXMZ$ are cyclic, it follows that

$$\angle BZM + \angle BXM = 2\llcorner \quad \text{and} \quad \angle CYM + \angle CXM = 2\llcorner. \qquad **$$

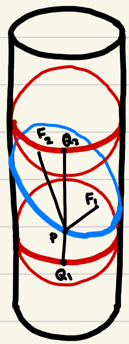
Putting $*$ and $**$ together, we have

$$\angle AZM \overset{*}{=} 2\llcorner - \angle BZM \overset{**}{=} \angle BXM$$
$$\angle AYM \overset{*}{=} 2\llcorner - \angle CYM \overset{**}{=} \angle CXM$$

but $\angle BXM + \angle CXM = 2\llcorner$

which shows that $\angle AZM + \angle AYM = 2\llcorner \Rightarrow M \in \Gamma_1$.

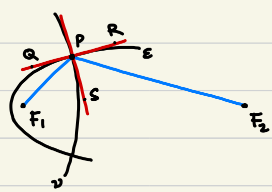
Q4. Let F_1, F_2 be the points of tangency of the spheres with the plane. Let P be any point on the intersection of the plane and the cylinder. Let Q_1, Q_2 be points on the spheres, both on the circle of tangency with the cylinder, such that the line containing Q_1, Q_2 is parallel to the axis of the cylinder (F_i, Q_i on the same sphere). By properties of tangents, $|F_1P| = |PQ_1|$ and $|F_2P| = |PQ_2|$. This implies



$$|F_1P| + |F_2P| = |Q_1Q_2| = \text{distance between the circles}$$

which is independent of P. By definition, the intersection of the plane and the cone is an ellipse.

Q5. Let P be an intersection point. We showed in class that tangents of ν bisect the angle $\angle F_1PF_2$. Moreover, we saw that tangents to ε were such that $\angle F_1PQ = \angle F_2PR$ (see figure). But,

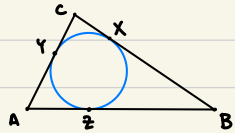


$$\underbrace{\angle QPF_1}_{} + \underbrace{\angle F_1PS}_{} + \underbrace{\angle F_2PS}_{} + \underbrace{\angle F_2PR}_{} = 2\llcorner.$$

$=$ $=$

which shows that $\angle QPF_1 + \angle F_1PS = \llcorner$, as desired.

Q6. By properties of tangents to circles, we have $|AY|=|AZ|$, $|BX|=|BZ|$, $|CX|=|CY|$ and thus, $\frac{|BX|}{|CX|}\frac{|CY|}{|AY|}\frac{|AZ|}{|BZ|} = \frac{|BX|}{|CX|}\frac{|CX|}{|AY|}\frac{|AY|}{|BX|} = 1$. By Ceva's thm, it follows that the cevians AX, BY, CZ are concurrent.

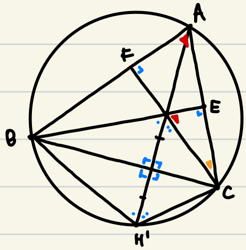


Q7. Let H' be the reflection of H through BC. We show that $ABH'C$ is cyclic. By properties of reflections, $\angle BH'H \overset{*}{=} \angle BHH'$ and $\angle CH'H \overset{**}{=} \angle CHH'$. Let E, F be the feet of the altitudes at B. Then,

$$\angle CHE + \angle ECH = \llcorner \quad \text{and} \quad \underbrace{\angle CAF}_{=\angle CAB} + \underbrace{\angle ACF}_{=\angle ECH} = \llcorner \quad \Rightarrow \angle CAB \overset{*}{=} \angle CHE.$$

but $\overset{*}{\angle CHE} + \overset{**}{\angle CHH'} + \overset{*}{\angle BHH'} = 2\llcorner \Rightarrow \angle CAB + \underbrace{\angle BH'H + \angle CH'H}_{=\angle BH'C} = 2\llcorner$

which shows that $ABH'C$ is cyclic.



Q8. Since $A*B*C$ and $A*D*E$, by B1, $\exists \ell_1, \ell_2$ with $A,B,C \in \ell_1$ and $A,D,E \in \ell_2$. Moreover, since A,B,D are not collinear, $\ell_1 \neq \ell_2$ and by I1, $\ell_1 \cap \ell_2 = \{A\}$. It follows that A, C, D are not collinear. By I1, let m be the line containing B, E. Again, $\ell_1 \neq m$ and $\ell_2 \neq m$ by I1 and since A, B, D are not collinear. It follows that $m \cap \ell_1 = \{B\}$ and $m \cap \ell_2 = \{E\}$. Thus, $A, D, C \notin m$. Also, since $B \in m$ and $A*B*C$, AC intersects m.

By B4, $\exists P \in m$ s.t. $A*P*D$ or $C*P*D$, but not both.

If $A*P*D$, then $P \in \ell_2 \Rightarrow P \in m \cap \ell_2 \Rightarrow P = E$. But, $A*D*E$ which contradicts B3 thus, $C*P*D$.

We show that $P \in BE \subset m$. By the plane sep. theorem, ℓ_2 divides the plane into two subsets, say S_1 and S_2. Since $B \notin \ell_2$, suppose $B \in S_1$. Then, since $A*B*C$, $C \in S_1$. Since $C*P*D$, $P \in S_1$. Thus, BP does not intersect ℓ_2. By B3, $B*P*E$ and BE and CD intersect at P.