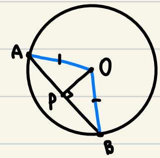
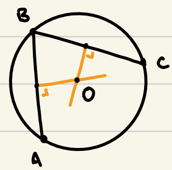
MIDTERM

Part 1.

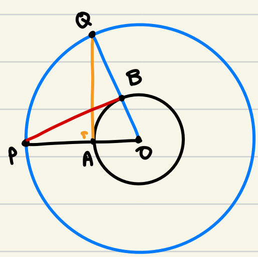
q3. a. The triangle ABO is isoceles since AO and BO are both radii. Thus, $\angle OAP = \angle OBP$ and $\angle APO = \angle BPO = \llcorner$. This implies $\angle AOP = \angle BOP$ and by SAS, AOP is congruent to $BOP \Rightarrow |AP| = |BP|$.



b. Pick three (distinct) points on the circle A, B, C. Draw segments AB and BC. We show in class that one can bisect a segment and erect a perp line using a straight-edge and compass. Thus, erect the perp bisectors of AB and BC. Since A, B, C are distinct, the perp bisectors are not parallel and they meet at one point. Call this point O.
By construction, ABO and BCO are isoceles (SAS again) which shows $|AO| = |BO| = |CO|$. By this same argument, if P is any point on the circle, $|OP| = |AO|$. This shows O is the center.



q4. a. We have $|AO| = |BO| =$ radius of the circle and $|OP| = |OQ|$ by hypothesis. Moreover, AOQ and BOP share $\angle AOB$. By SAS, AOQ is congruent to BOP. This implies $\angle OBP = \angle OAQ$.



b. Let P be a point outside a given circle with center O. Draw segment OP. Let A be the intersection of OP with the given circle. Erect perp line at A. Draw a circle with center O and radius OP. Let B be an intersection of the erected perp line with the circle passing through P. Draw segment QO and let B be the intersection of OQ with the given circle. By part a, since $|OQ| = |OP|$ we have $\angle OBP = \angle OAQ = \llcorner$ by construction. But, BP is then tangent to the given circle, as shown in class.

95. a. Let F be the foot of the altitude of C. Then, $\cos\alpha = |AF|/b$, $\cos\beta = |BF|/a$.

Since $|AF|+|BF| = c$, we get $c = b\cos\alpha + a\cos\beta$. Multiplying by c, we get $c^2 \overset{*}{=} ac\cos\beta + bc\cos\alpha$.

b. We have similarly $a^2 \overset{*}{=} ab\cos\gamma + ac\cos\beta$ and $b^2 \overset{*}{=} ab\cos\gamma + bc\cos\alpha$. Thus,

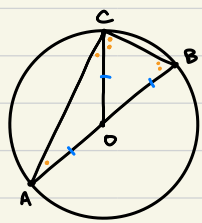
$$a^2+b^2-2ab\cos\gamma \overset{**}{=} (ab\cos\gamma + ac\cos\beta) + (ab\cos\gamma + bc\cos\alpha) - 2ab\cos\gamma$$
$$= ac\cos\beta + bc\cos\alpha \overset{*}{=} c^2$$

as required.

96. Let O be the center of the circle. Then, $|AO| = |BO| = |CO|$ since they are all radii. Thus, $\angle CAO \overset{*}{=} \angle ACO$ and $\angle CBO \overset{*}{=} \angle BCO$. But,

$$\angle ACB = \angle ACO + \angle BCO \overset{**}{=} \angle CAO + \angle CBO \underset{O \text{ lies on } AB}{=} \angle CAB + \angle ABC \underset{\text{sum angles in } \Delta = 2\llcorner}{=} 2\llcorner - \angle ACB$$

which implies $2\angle ACB = 2\llcorner \Leftrightarrow \angle ACB = \llcorner$.

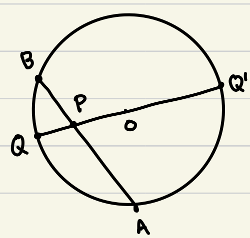


<u>PART 2.</u>

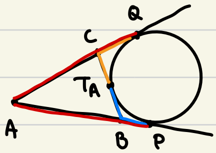
91. a. Since BD and AC are chords in a circle, $\angle BAD = \angle BCD$ and $\angle ABC = \angle ADC$. Moreover, $\angle APD = \angle BPC$ since AB is a secant of CD. We conclude by AAA.

b. Since ADP is similar to BCP, $\frac{|AP|}{|CP|} = \frac{|DP|}{|BP|} \Rightarrow |AP|\,|BP| = |CP|\cdot|DP|$.

c. Let Q, Q' be the intersection points of OP with the circle. By b, $|AB|\cdot|BP| = |PQ|\cdot|PQ'|$, but $|PQ| = R - |OP|$ and $|PQ'| = |OP| + R$ and we get $|PQ|\cdot|PQ'| = (R-|OP|)(R+|OP|) = R^2 - |OP|^2$.



q2. Recall that given a circle and a point outside the circle P, the two tangents to the circle passing through P have equal lengths. Applying this result to A, we get: $|AP| = |AQ|$. Applying now to B: $|BP| = |BT_A|$. And to C: $|CQ| = |CT_A|$. Now,



$$|AB| + |BT_A| = |AB| + |BP| = |AP| = |AQ| = |AC| + |CT_A|$$

$$2(|AB| + |BT_A|) = (|AB| + |BT_A|) + (|AC| + |CT_A|) = |AB| + |AC| + |BC| = \text{perimeter}.$$

and we get $|AB| + |BT_A| = \frac{1}{2}$ perimeter, as required.

q3. a. Since angles in a Δ sum to $2\llcorner$ and since $\angle BEC = \angle ADC = \llcorner$,

$$\angle CAD = 2\llcorner - (\angle ADC + \angle ACD) = 2\llcorner - (\angle BEC + \angle BCE) = \angle CBE.$$

b. Since CH' is a chord and A, B lie on the circle,

$\angle CBH' = \angle CAH'$.

c. From part a and b, $\angle DBH' = \angle DBH$.

d. We know $\angle BDH = \angle BDH' = \llcorner$. By ASA, BDH is congruent to BDH'

$\Rightarrow |DH| = |DH'|$.

q4. If XY is parallel to AB, then by AAA, ABC is similar to CXY. Thus $|CX|/|BC| \overset{*}{=} |CY|/|AC|$. Since the cevians are concurrent, by Ceva's thm,

$$1 = \frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = \frac{(|BC| - |CX|)}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|}$$

$$= \left(\frac{|BC|}{|CX|} - 1\right) \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} \overset{*}{=} \left(\frac{|AC|}{|CY|} - 1\right) \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|}$$

$$= \left(\frac{|AC| - |CY|}{|CY|}\right) \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = \frac{|AY|}{|CY|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = \frac{|AZ|}{|BZ|}$$

and we get $|AZ| = |BZ| \Leftrightarrow Z$ is the midpoint of AB.

If Z is the midpoint of AB, then $|AZ| = |BZ|$ and by

Ceva's thm,

$$1 = \frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} = \frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} = \frac{(|BC|-|CX|)\,|CY|}{|CX|\,(|AC|-|CY|)}$$

$$\Rightarrow \frac{|BC|-|CX|}{|CX|} = \frac{|AC|-|CY|}{|CY|} \Rightarrow \frac{|BC|}{|CX|} - 1 = \frac{|AC|}{|CY|} - 1 \Rightarrow \frac{|BC|}{|CX|} = \frac{|AC|}{|AY|}.$$

and by SAS (with proportions), CXY is similar to ABC. Thus, $\angle CYX = \angle BAC \Rightarrow XY$ is parallel to AB.