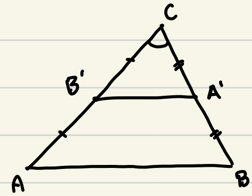
PRACTICE MIDTERM

Ex 2. a. Let ABC be a triangle and let A', B', C' be the midpoints of the sides BC, AC, AB respectively. Show that the segment A'B' is parallel to AB.

b. Let ABCD be a quadrilateral and let P, Q, R, S be the midpoints of AB, BC, CD, AD respectively. Using a, show that PQRS is a parallelogram.

Sol a. Consider the two triangles ABC and A'B'C. They share a common angle, namely $\angle ACB = \angle A'CB'$. Moreover, we have that

$$|AC|/|B'C| = |BC|/|A'C|$$

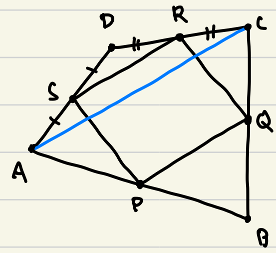


and by the SAS criterion, the triangles are similar. Consequently,

$$\angle A'B'C = \angle BAC \Rightarrow \angle BAC + \angle AB'A' = 2R$$

meaning AB is parallel to A'B'.

b. Consider the segment AC. By a, looking at the triangles DRS and ACD, we have that AC is parallel to RS. Same procedure shows that AC is parallel to PQ. Thus, PQ is parallel to RS.
An identical argument shows PS // QR. Consequently, PQRS is a parallelogram by definition.

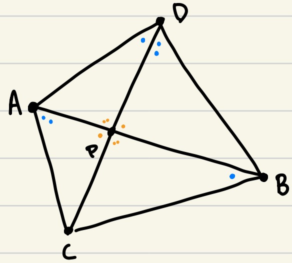


Ex3. Let A, B, C, D be four points such that AB and CD intersect at a point P and such that $|AP| \cdot |BP| = |CP| \cdot |DP|$.

a. Show that the triangles ADP and BCP are similar.

b. Using the characterization of cyclic quadrilaterals, conclude that A, B, C, D all lie on the same circle.

Sol. a. We know that $\angle ADP = \angle CPD$. Since $\frac{|AP|}{|CP|} = \frac{|DP|}{|BP|}$, we conclude by the SAS criterion that the triangles ADP and BCP are similar.



b. One can apply the same reasoning as in a to show that ACP and BDP are similar. Consequently,

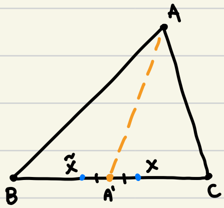
$$\begin{aligned}\angle ACB + \angle ADB &= \angle ACB + (\angle ADP + \angle BDP) \quad \text{By similarity} \\ &= \angle ACB + (\angle CBP + \angle CAP) \\ &= \angle ACB + \angle ABC + \angle BAC = 2\llcorner\end{aligned}$$

since angles in a triangle sum to $2\llcorner$. Thus, by the characterization of cyclic quadrilaterals, ABCD lie on a circle

Ex4. Given a triangle ABC and three cevians AX, BY, CZ let $\tilde{X}$ be the point on BC such that $X \neq \tilde{X}$ and $|A'X| = |A'\tilde{X}|$ where A' is the bisector of BC. Define $\tilde{Y}$ and $\tilde{Z}$ similarly. Apply Ceva's thm to show that AX, BY, CZ are concurrent iff $A\tilde{X}, B\tilde{Y}, C\tilde{Z}$ are.

Sol. Suppose without loss of generality that $X \neq A'$ since in that case, $\tilde{X} = X$. Moreover, assume $|BX| < |CX|$. We have by definition of $\tilde{X}$ that

$$|B\tilde{X}| = |A'B| + |A'\tilde{X}| = \underbrace{|A'B| + |A'X| = |A'C|}_{A' \text{ bisects } BC} + |A'X| = |CX|.$$



In much the same way, $|C\tilde{X}| = |BX|$. Now, applying the same reasoning to $\tilde{Y}$ and $\tilde{Z}$, we get $|A\tilde{Y}| = |CY|$, $|C\tilde{Y}| = |AY|$, $|A\tilde{Z}| = |BZ|$, $|B\tilde{Z}| = |AZ|$. Thus, if we let

$$\star = \frac{|BX|}{|CX|} \cdot \frac{|CY|}{|AY|} \cdot \frac{|AZ|}{|BZ|} \qquad \tilde{\star} = \frac{|B\tilde{X}|}{|C\tilde{X}|} \cdot \frac{|A\tilde{Y}|}{|C\tilde{Y}|} \cdot \frac{|A\tilde{Z}|}{|B\tilde{Z}|}$$

we conclude that $\star = \frac{1}{\tilde{\star}} \Rightarrow (\star = 1 \Leftrightarrow \tilde{\star} = 1)$. By Ceva's thm, AX, BY, CZ concurrent iff $\star = 1$ iff $\tilde{\star} = 1$ iff $A\tilde{X}, B\tilde{Y}, C\tilde{Z}$ concurrent.

Ex 5. Let ABC be an acute triangle and let D, E, F be its orthic triangle where D, E, F are the feet of the altitudes at A, B, C respectively. Denote the orthocenter of ABC by H.

a. Show that A, E, F, H all lie on a common circle.

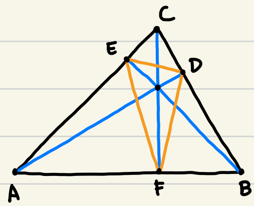
b. Show that $\angle EFH = \angle EAH$, $\angle CAD = \frac{\pi}{2} - \angle ACD$ and conclude that $\angle EFH = \frac{\pi}{2} - \angle ACD$.

c. Show that CF is the angle bisector of $\angle DFE$.

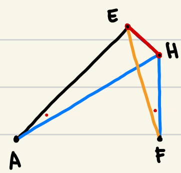
d. Show that the orthocenter of an acute triangle coincides with the incenter of its orthic triangle.

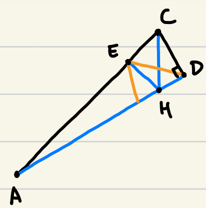
Sol. a. We have $\angle AFH = \frac{\pi}{2}$ and $\angle AEH = \frac{\pi}{2}$ by construction. Thus,

$$\angle AFH + \angle AEH = 2\cdot\frac{\pi}{2} \overset{\text{cyclic}}{\underset{\text{quad}}{\Rightarrow}} AEFH \text{ circumscribed.}$$



b. Since EH is the chord of a circle and F, A are both points on this circle on the same side of EH, it follows that $\angle EFH = \angle EAH$. We see that $\angle EAH = \angle CAD$ and ACD is a right angle triangle $\Rightarrow \angle ACD + \angle CAD = \frac{\pi}{2}$.
Thus, $\angle EFH = \angle EAH = \angle CAD = \frac{\pi}{2} - \angle ACD$.





c. Same reasoning as in a and b can be applied to vertices B, D, F, H. This shows $\angle DFH = \frac{\pi}{2} - \angle BCE$. But, $\angle BCE = \angle ACD = \angle ACB$ which means $\angle EFH = \frac{\pi}{2} - \angle ACB = \angle DFH$ and CF bisects $\angle DFE$.

d. Applying c to $\angle DEF$ and $\angle EDF$, we see that AD, BE, CF are all angle bisectors of DEF thus intersecting at the incenter of DEF. But, AD, BE, CF intersect at the orthocenter of ABC.