Midterm 1

Math 347, Fall 2015

Section G1

Last (Family) Name: ____________________

First (Given) Name: ____________________

NetID: ____________________

Instructions

All cell phones, calculators, and other devices must be turned off and out of reach. Also, all books, notebooks, and scratch papers must be out of reach. The last page of the test is scratch paper, use it if needed, but it will not be graded.

Problem	Score
1	/ 10
2	/ 14
3	/ 14
4	/ 12
Total	/ 50

1. (10 points) Prove that for all sets A, B, the following holds:

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B).$$

Solution. $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$: Fix arbitrary $x \in (A - B) \cup (B - A)$. Then $x \in A - B$ or $x \in B - A$. By symmetry, we may assume without loss of generality that $x \in A - B$. Thus $x \in A$ and $x \notin B$. Because $x \in A$, we have $x \in A \cup B$. Also, because $x \notin B$, it follows that $x \notin A \cap B$. Thus, by the definition of $-$, $x \in (A \cup B) - (A \cap B)$.

$(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$: Fix arbitrary $x \in (A \cup B) - (A \cap B)$. Then $x \in A \cup B$ and $x \notin A \cap B$. Because $x \in A \cup B$, we have that $x \in A$ or $x \in B$. By symmetry, we may assume without loss of generality that $x \in A$. Because $x \notin A \cap B$, we deduce that $x \notin B$. Hence $x \in A - B$. $\square$

2. (14 points) Let $f : X \to Y$ be a function from a set X to a set Y.

(a) (5 points) For a set $A \subseteq X$, fill in the space below to complete the definition of $f(A)$:

$$f(A) := \{y \in Y : \exists x \in A \ f(x) = y\}.$$

Incorrect solution 1: $f(A) := \{y \in Y : I_f(y) \in A\}$. This simply doesn't make sense because, by definition, $I_f(y)$ is a subset of X and not an element of X. Namely, $I_f(y) = \{x \in X : f(x) = y\}$. □

Incorrect solution 2: $f(A) := \{y \in Y : I_f(y) \subseteq A\}$. This makes sense but is still incorrect, and here is an example of a function that shows the incorrectness. Let $f : \mathbb{R} \to \mathbb{R}$ be the usual absolute value function $f(x) = |x|$. Taking $A := [-1, 0)$, we see that $f(A) = (0, 1]$. Now take any $y \in f(A) = (0, 1]$ and note that $I_f(y) = \{-y, y\} \nsubseteq A$. □

(b) (9 points) Prove that for all $A, B \subseteq X$, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Solution. Fix arbitrary $y \in f(A \cap B)$. By the above definition, this means that there is $x \in A \cap B$ such that $f(x) = y$. Because $x \in A$, we have $f(x) \in f(A)$. Similarly, because $x \in B$, we have $f(x) \in f(B)$. Thus, $y = f(x) \in f(A) \cap f(B)$. □

3. (14 points) Consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = 2n^3$. Determine whether the following are true or false, and prove your answer in either case.

(a) (7 points) $\forall n \in \mathbb{N}\ \exists B \in \mathbb{N}\ (x_n \leq B)$.

Solution. This is always true regardless of what $(x_n)_{n\in\mathbb{N}}$ is. Here is a proof: fix arbitrary $n \in \mathbb{N}$ and take $B = |x_n|$. Thus, in particular, $x_n \leq B$. □

(b) (7 points) $\exists B \in \mathbb{N}\ \forall n \in \mathbb{N}\ (x_n \leq B)$.

Solution. This is false, and to show it, we prove the negation, namely:

$$\forall B \in \mathbb{N}\ \exists n \in \mathbb{N}\ (x_n > B).$$

Fix arbitrary $B \in \mathbb{N}$. We need to find $n \in \mathbb{N}$ such that $2n^3 > B$. But the latter inequality is equivalent to $n > \sqrt[3]{\frac{B}{2}}$, so taking $n = \left\lceil \sqrt[3]{\frac{B}{2}} \right\rceil + 1$ works. Indeed, for this n, we have

$$x_n = 2n^3 = \left(\left\lceil \sqrt[3]{\frac{B}{2}} \right\rceil + 1\right)^3 > 2\left\lceil \sqrt[3]{\frac{B}{2}} \right\rceil^3 \geq 2\frac{B}{2} = B.$$

□

4. (12 points) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers satisfying

$$\begin{aligned}&a_1 = 1, a_2 = 8\\ &a_{n+2} = a_{n+1} + 2a_n, \text{ for every } n \in \mathbb{N}.\end{aligned}$$

Prove that for every $n \in \mathbb{N}$, $a_n = 3 \cdot 2^{n-1} + (-1)^n \cdot 2$.

NOTE: $(-1)^n = (-1)^{n+2}$.

Solution. We prove by strong induction on n.

Base cases: For $n = 1$,

$$3 \cdot 2^{n-1} + (-1)^n \cdot 2 = 3 \cdot 2^0 - 2 = 1 = a_1.$$

For $n = 2$,

$$3 \cdot 2^{n-1} + (-1)^n \cdot 2 = 3 \cdot 2^1 + 2 = 8 = a_2.$$

Step: Fix arbitrary $k \geq 3$ and suppose that for every natural number $i < k$,

$$a_i = 3 \cdot 2^{i-1} + (-1)^i \cdot 2.$$

We need to prove that

$$a_k = 3 \cdot 2^{k-1} + (-1)^k \cdot 2.$$

Because $k \geq 3$, the number $n := k - 2$ is at least 1, so $n \in \mathbb{N}$. Thus, $k = n + 2$, so we have

$$a_k = a_{n+2} = a_{n+1} + 2a_n.$$

By the induction hypothesis applied to $i = n + 1$ and $i = n$, we get

$$\begin{aligned}a_k = a_{n+2} = a_{n+1} + 2a_n &= 3 \cdot 2^n + (-1)^{n+1} \cdot 2 + 2\big(3 \cdot 2^{n-1} + (-1)^n \cdot 2\big)\\ &= 3 \cdot 2^n + (-1) \cdot (-1)^n \cdot 2 + 3 \cdot 2^n + 2 \cdot (-1)^n \cdot 2\\ &= 2 \cdot 3 \cdot 2^n + (-1 + 2) \cdot (-1)^n \cdot 2\\ &= 3 \cdot 2^{n+1} + (-1)^n \cdot 2\\ &= 3 \cdot 2^{n+1} + (-1)^{n+2} \cdot 2\\ &= 3 \cdot 2^{k-1} + (-1)^k \cdot 2.\end{aligned}$$

□

Scratch paper