Midterm 2

Math 347, Fall 2015

Section G1

Last (Family) Name: ________________________

First (Given) Name: ________________________

NetID: ________________________

Instructions

All cell phones, calculators, and other devices must be turned off and out of reach. Also, all books, notebooks, and scratch papers must be out of reach. The last page of the test is scratch paper, use it if needed, but it will not be graded.

Problem	Score
1	/ 10
2	/ 10
3	/ 10
4	/ 10
Total	/ 40

1. (10 points)

(a) Complete the following definitions:

(i) (2 points) A function $f : X \to Y$ is said to be *injective* if $\forall x_1, x_2 \in X \left[x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)\right]$.

(ii) (2 points) A function $f : X \to Y$ is said to be *surjective* if $\forall y \in Y \; \exists x \in X \; f(x) = y$.

(b) Let $f : \mathbb{N} \to \mathbb{N}$ be defined by

$$f(n) := \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases} .$$

(i) (3 points) Is f injective? Prove your answer.

No. For example, $f(8) = 4 = f(1)$.

(ii) (3 points) Is f surjective? Prove your answer.

Yes. Fix an arbitrary m in the target set $\mathbb{N}$. Then $f(2m) = m$.

2. (10 points)

(a) (3 points) Fill in the empty boxes and complete the following definition.

Definition. A function $g : \boxed{Y} \to \boxed{X}$ is called an *inverse* of a function $f : X \to Y$ if $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$, i.e. $\forall x \in X\ g(f(x)) = x$ and $\forall y \in Y\ f(g(y)) = y$.

(b) (7 points) Let $f : X \to Y$ and $g : Y \to Z$ be invertible functions (i.e. have inverses f^{-1} and g^{-1}). Write the inverse of $g \circ f$ in terms of f^{-1} and g^{-1} in the empty box below and prove that what you wrote is indeed the inverse of $g \circ f$.

$$(g \circ f)^{-1} = \boxed{f^{-1} \circ g^{-1}}$$

Put $h := f^{-1} \circ g^{-1}$.

To check that $\forall x \in X\ h \circ (g \circ f)(x) = x$, fix an arbitrary $x \in X$. Then $h \circ (g \circ f)(x) = f^{-1}(g^{-1}(g(f(x)))) = f^{-1}(f(x)) = x$.

To check that $\forall z \in Z\ (g \circ f) \circ h(z) = z$, fix an arbitrary $z \in Z$. Then $(g \circ f) \circ h(z) = g(f(f^{-1}(g^{-1}(z)))) = g(g^{-1}(z)) = z$.

3. (10 points)

(a) Complete the following definitions:

(i) (2 points) Sets X and Y are said to *have the same cardinality* if there is a bijection $f : X \xrightarrow{\sim} Y$.

(ii) (2 points) A set X is said to be *countable* if X has the same cardinality as $\mathbb{N}$, i.e. there is a bijection $f : X \xrightarrow{\sim} \mathbb{N}$.

(b) (6 points) Let X be the set of all negative integers that are cubes of other integers, i.e.

$$X := \left\{ x \in \mathbb{Z} : x < 0 \text{ and } \exists y \in \mathbb{Z} \; (x = y^3) \right\}.$$

Prove that X is countable. Do this directly without using any theorems proved in class.

Hint: It would be easier to define a function from $\mathbb{N}$ to X (rather than from X to $\mathbb{N}$).

Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) := (-n)^3$. By the very definition of X, it is clear that $f(\mathbb{N}) \subseteq X$. Thus, we can replace the target set with X, i.e. write $f : \mathbb{N} \to X$. It remains to show that f is injective and surjective.

Injectivity: Fix arbitrary $n_1, n_2 \in \mathbb{N}$ and suppose $f(n_1) = f(n_2)$. We need to show that $n_1 = n_2$. But $f(n_1) = f(n_2)$ means $(-n_1)^3 = (-n_2)^3$, so $-n_1^3 = -n_2^3$ and hence $n_1^3 = n_2^3$. Applying the cube-root function, we get $n_1 = n_2$.

Surjectivity: Fix an arbitrary $x \in X$. By the virtue of being in X, $x < 0$ and there is $y \in \mathbb{Z}$ with $x = y^3$. Since $x < 0$, y must also be negative, so $y = -n$ for some $n \in \mathbb{N}$. Thus, $x = (-n)^3 = f(n)$.

4. (10 points)

(a) Complete the following definitions:

(i) (2 points) A **nonempty** set X is said to be *finite* if there is a bijection $f : X \xrightarrow{\sim} [n]$ for some $n \in \mathbb{N}$.

(ii) (2 points) For a **nonempty finite** set X, we denote by $|X|$ the unique $n \in \mathbb{N}$ such that there is a bijection $f : X \xrightarrow{\sim} [n]$.

(b) (6 points) Prove that for **disjoint nonempty** sets X and Y, $|X \cup Y| = |X| + |Y|$.

Putting $n := |X|$ and $m := |Y|$, we need to define a bijection $f : X \cup Y \to [n+m]$. Because $n = |X|$ and $m = |Y|$, there are bijections $g : X \xrightarrow{\sim} [n]$ and $h : Y \xrightarrow{\sim} [m]$. We use these bijection to define a desired f: define $f : X \cup Y \to [n+m]$ as follows: for $z \in X \cup Y$,

$$f(z) := \begin{cases} g(z) & \text{if } z \in X \\ n + h(z) & \text{if } z \in Y \end{cases}.$$

This is a well-defined function since X and Y are disjoint. We now check that f is a bijection. (I didn't take off points for not checking this.) To do this, we show that f has an inverse. Indeed, define $j : [n+m] \to X \cup Y$ as follows: for $k \in [n+m]$, put

$$j(k) := \begin{cases} g^{-1}(k) & \text{if } k \leq n \\ h^{-1}(n-k) & \text{if } k > n \end{cases}.$$

It is now straightforward to verify that j is indeed the inverse of f.

Scratch paper