MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

PRACTICE PROBLEMS FOR CHAPTER 14

1. Let $(x_n)_n$ be a sequence and a, b be reals with $a < b$. Suppose that for each $N \in \mathbb{N}$ there is $n \geq N$ such that $x_n \in [a, b]$. Prove that $(x_n)_n$ has a convergent subsequence whose limit is in $[a, b]$.

2. Suppose that for each $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{2^n}$, and prove that $(x_n)_n$ is Cauchy. Deduce that $(x_n)_n$ converges.

3. Let $(x_k)_k$ be a sequence.

(a) Define another sequence $(a_n)_n$ such that for each $k \in \mathbb{N}$, x_k is the k^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$.

(b) Once again, suppose that for each $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq \frac{1}{2^n}$, and, using part (a) and the comparison test **for series**, prove that $(x_n)_n$ converges.

Hint: By part (a), the sequence $(x_k)_k$ converges if and if $\sum_{n=1}^{\infty} a_n$ converges.

4. Fix a real $\lambda > 1$.

(a) Prove that for any fixed integer $k \geq 0$, $\lim_{n\to\infty} \dfrac{n^k}{\lambda^n} = 0$.

(b) Conclude that for any polynomial $p(x)$, $\lim_{n\to\infty} \dfrac{p(n)}{\lambda^n} = 0$.

(c) Also prove that for any polynomial $p(x)$, $\sum_{n=1}^{\infty} \dfrac{p(n)}{\lambda^n}$ converges.

5. Let $(a_n)_n$ be a sequence of non-negative reals. Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then for any $k \geq 1$, $\sum_{n=1}^{\infty} a_n^k$ also converges.

6. (a) Suppose that $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ both converge, and prove that $\sum_{n=1}^{\infty} a_n b_n$ converges.

(b) For each real $\lambda > 2$, construct an example of $(a_n)_n, (b_n)_n$ such that $\sum_{n=1}^{\infty} a_n^{\lambda}$ and $\sum_{n=1}^{\infty} b_n^{\lambda}$ both converge, but $\sum_{n=1}^{\infty} a_n b_n$ diverges.

7. (Tricky) Let $(x_n)_n$ be sequence and $L \in \mathbb{R}$. Suppose that any subsequence $(x_{n_k})_k$ has a further subsequence $(x_{n_{k_l}})_l$ that converges to L. Prove that $(x_n)_n$ converges to L.

Hint: Prove the contrapositive. Assume that $(x_n)_n$ doesn't converge to L and build a subsequence "far" from L.