# MATH 347: FUNDAMENTAL MATHEMATICS, FALL 2015

HOMEWORK 10

Due date: Dec 2 (Wed)

**Exercises from the textbook.** 14.1, 14.2, 14.3, 14.4

**Out-of-the-textbook exercises** (these are as mandatory as the ones from the textbook)**.**

**1.** Let $(x_n)_n$ be a sequence of positive reals. Prove that the sequence $\left(\frac{1}{x_n}\right)_n$ is bounded if and only if there exists $\lambda > 0$ such that $\forall n \in \mathbb{N}$ $\lambda < x_n$.

**2.** (a) Let $\lambda > 1$. Prove that for all $n \in \mathbb{N}$, $\lambda^n \geq 1 + n(\lambda - 1)$. Conclude that the sequence $(\lambda^n)_n$ is unbounded. Thus, what do you conclude about its convergence?

(b) Prove that for $0 \leq \lambda < 1$, the sequence $(\lambda^n)_n$ converges to 0.

**3.** Determine whether the following sequences converge and prove your answers.

(a) $\left(\frac{n}{3\sqrt{n}-2}\right)_n$

(b) $\left(\frac{n}{3n-2}\right)_n$

(c) $\left(\frac{n}{3n^2-2}\right)_n$

(d) $\left(\frac{n^2+1}{2n^2+n+3}\right)_n$

(e) $\left(\sqrt{n(n+1)} - n\right)_n$

(f) $\left(\sqrt{n(n+1)} - \sqrt{n}\right)_n$

**4.** Let $(x_n)$ be a sequence of positive reals such that

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L > 1.$$

Follow the steps (i)–(iii) below to prove that $(x_n)_n$ is unbounded, and hence divergent.

(i) Prove that for any $\lambda < L$, there is $N_\lambda \in \mathbb{N}$ such that $\forall n \geq N$, $x_{n+1} > \lambda x_n$.

(ii) Fix a real $\lambda \in (1, L)$ and let $N_\lambda$ be as in part (i). Using induction, prove that for all $n \geq N_\lambda$,

$$x_n \geq \lambda^{n-N_\lambda} x_{N_\lambda}.$$

(iii) Conclude that $(x_n)_n$ is unbounded, and hence divergent.

**5.** (Tricky) Prove that any set $A \subseteq \mathbb{R}$ that is bounded above contains a sequence $(x_n)_n \subseteq A$ whose limit is $\sup A$.

**6.** Let $(a_n)_n$ be increasing and $(b_n)_n$ decreasing. Suppose that $\lim_{n\to\infty} b_n - a_n = 0$.

(a) Prove that for all $n, m \in \mathbb{N}$, $a_n \leq b_m$.

(b) Conclude that both $(a_n)_n$ and $(b_n)_n$ converge.

(c) Carefully prove that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Remark: Although parts (b) and (c) have appeared on Midterm 3, I still think it would be instructive if you carefully wrote up your own understanding of the proofs one more time.