Math 347H: Fundamental Math (H.) HOMEWORK 1 **Due date: Sept 14 (Thu)**

Exercises from Sally's book. 1.2.3, 1.3.9(iv, v, xi, xii), 1.4.6

Other (mandatory) **exercises.**

1. For each pair of sets A, B below, determine whether $A \in B$, or $A \subseteq B$, or both.

(a) $A := \emptyset$ and $B := \{\{\emptyset\}\}$

(b) $A := \{\emptyset\}$ and $B := \{\{\emptyset\}, \emptyset, \{\{\emptyset\}\}\}$

(c) $A := \{\emptyset, 1, 2\}$ and $B := \{\{\emptyset\}, 1, 2, \{\emptyset, 1, 2\}\}$

2. Write down how you would start the proofs of the following statements or what you'd need to do to prove them. Do not prove the statements, just write how you would start the proof without even knowing what the involved terms mean.

(a) If a matrix A has nonzero determinant, then it is invertible.

(b) If an integer n is divisible by 4 or $n + 1$ is prime, then n is even.

(c) Every bounded sequence of reals has a convergent subsequence.

(d) There is a 2×2 matrix with real coefficients whose characteristic polynomial splits, yet it is not diagonalizable.

(e) For every continuous function f on $\mathbb{R}$ and every compact set $K \subseteq \mathbb{R}$, $f(K)$ is compact.

(f) For all reals $a_1, a_2, \ldots, a_n$, if $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$, then $a_i = 0$ for every $i \in \{1, 2, \ldots, n\}$.

3. Let X, Y be sets and let $f \subseteq X \times Y$. For each of the following examples, determine whether f is a function from X to Y. Justify your answer: if YES, then show that all of the conditions in the definition of a function hold; if NO, then provide an example witnessing the failure of one of the conditions.

(a) $X := \mathbb{R}, Y := \mathbb{R}$, $f := \{(x, y) \in \mathbb{R}^2 : x^2 = y\}$.

(b) $X := [0, +\infty), Y := \mathbb{R}$, $f := \{(x, y) \in [0, +\infty) \times \mathbb{R} : x = y^2\}$.

(c) $X := [0, +\infty), Y := [0, +\infty)$, $f := \{(x, y) \in [0, +\infty) \times [0, +\infty) : x = y^2\}$.

(d) $X := \mathbb{N}, B := \mathbb{N}$, $f := \{(x, y) \in \mathbb{N}^2 : x = y^2\}$.

4. A binary relation R on a set A is called

- *reflexive* if for all $a \in A$, $(a, a) \in R$.
- *symmetric* if for all $a, b \in A$, $(a, b) \in R \Rightarrow (b, a) \in R$.
- *transitive* if for all $a, b, c \in A$, $\big[(a, b) \in R$ and $(b, c) \in R\big] \Rightarrow (a, c) \in R$.

For each of the following examples of R and each of the properties above (reflexive, symmetric, and transitive), determine whether R has that property. Justify your answer.

(a) $A :=$ the set of Facebook subscribers and

$$R := \left\{(a,b) \in A^2 : a \text{ and } b \text{ are Facebook-friends}\right\}.$$

(b) $A :=$ the set of all webpages on internet and

$$R := \left\{(a,b) \in A^2 : a \text{ has a link to } b\right\}.$$

(c) $A := \mathbb{Z}$ and R is $<$.

(d) $A := \mathbb{Z}$ and R is $\leq$.

(e) $A := \mathbb{Z}$ and R is the relation *divide*, i.e.

$$R := \left\{(a,b) \in A^2 : \exists c \in \mathbb{Z} \text{ such that } ac = b\right\}.$$

TRICK QUESTION: Does 0 divide 0, i.e. $(0,0) \in R$?

(f) $A := \mathbb{R}$ and $R := \{(x,y) \in \mathbb{R}^2 : |x-y| = 1\}$.