Complete, last update: Apr 18 at 6:40pm

Math 432: Set Theory and Topology HOMEWORK 11 Due date: Apr 20 (Thu)

Exercises from Kaplansky's book.

Sec 4.4: 2, 9

Sec 5.1: 2, 6, 7

1. For each $n \in \mathbb{N}$, define $f_n : (0, \infty) \to \mathbb{R}$ by $f_n(x) := \frac{nx}{1+n^2x^2}$.

(a) Show that $f_n \to 0$ *pointwise*, i.e., for every $x \in (0, \infty)$, $f_n(x) \to 0$.

(b) Show that $(f_n)_{n \in \mathbb{N}}$ does not converge uniformly to the constant 0 function.

(c) However, for each $\alpha > 0$, prove that the sequence of restrictions $(f_n|_{(\alpha,\infty)})_{n \in \mathbb{N}}$ uniformly converges to the constant 0 function.

2. Let $(x_n) \subseteq \mathbb{R}$ be bounded. Say a real $r \in \mathbb{R}$ is *left of* (x_n) if $\forall^\infty n \; r \leq x_n$. Let L denote the set of reals left of (x_n). Put $l := \sup L$ and prove the following.

(a) L is a nonempty initial segment in $(\mathbb{R}, <)$ and it is bounded above (hence $l < +\infty$).

(b) If, for some $\varepsilon > 0$, $\forall^\infty n \; x_n \notin (l - \varepsilon, l)$, then $l \in L$.

(c) If $l \in L$, then, for every $\varepsilon > 0$, $\exists^\infty n \; x_n \in [l, l + \varepsilon)$.

(d) There is a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ converging to l.

3. Prove that the Cantor space $2^{\mathbb{N}}$ with its usual metric is a complete metric space.