Math 432: Set Theory and Topology HOMEWORK 12 Due date: Apr 27 (Thu)

Reflections. Write an essay about the whole course in general. Treat this as part of the review and try to recall the highlights of the entire material.

Exercises from Kaplansky's book.

Sec 5.1: 14, 15

1. Let $(X, d_X), (Y, d_Y)$ be metric spaces and let (f_n) be a sequence of bounded functions $X \to Y$. Suppose that (f_n) converges uniformly to a function $f : X \to Y$ and prove that f is also bounded.

2. (a) Give an example of a bounded complete metric space that contains a sequence that does not have any convergent subsequence.

(b) However, prove the following:

Theorem (Bolzano–Weierstrass)**.** *Every bounded sequence $(x_n) \subseteq \mathbb{R}$ has a convergent subsequence.*

HINT: Say $(x_n) \subseteq [a, b]$. Divide (the interval $[a, b]$) and conquer.

Definition. Call a metric space *separable*[1] if it admits a countable dense subset.

Definition. Let X be a metric space. Call a colletion $\mathcal{B}$ of open sets a *base* (or an *open base*) for X if every nonempty open set in X is a (possibly uncountable) union of sets from $\mathcal{B}$.

Example. We proved in class that, in any metric space, the collection of open balls is a base.

3. Let X be a metric space. Prove:

(a) For any dense set $D \subseteq X$, the collection $\mathcal{B}_D$ of all open balls of rational radius centered at the points of D is a base for X.

(b) Conversely, for any base $\mathcal{B}$ for X, if a set $A \subseteq X$ intersects every nonempty set $U \in \mathcal{B}$, then A is dense.

(c) Conclude that a metric space is separable if and only if it admits a countable base. Does this use Axiom of Choice?

4. Prove that (sequentially) compact metric spaces are bounded.

5. (Optional) Let X be a metric space and prove that the following are equivalent:

(1) Every decreasing sequence (C_n) of nonempty closed sets has a nonempty intersection, i.e. $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$.

(2) Every countable collection $\mathcal{C}$ of closed sets with the FIP[2] has a nonempty intersection, i.e. $\bigcap \mathcal{C} \neq \emptyset$.

(3) Every countable open cover of X has a finite subcover.

[1] Daniel Shteynberg suggests that the term comes from any two reals being *separated* by a rational, which is a very special case and it's unfortunate that the term *separable* was extrapolated from $\mathbb{R}$ to general metric spaces.

[2] FIP stands for *Finite Intersection Property*, which means that **any** finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ has a nonempty intersection.