Math 432: Set Theory and Topology HOMEWORK 5 Due date: Mar 2 (Thu)

Notation. For a set A, denote by $A^{<\mathbb{N}}$ the set of all *finite sequences of elements of* A, i.e.

$$A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n,$$

where $A^0 := \{\emptyset\}$ thinking of the empty set as the empty sequence. Furthermore, denote by $A^{\mathbb{N}}$ the set of all *infinite sequences of elements of* A, by which we simply mean functions $\mathbb{N} \to A$.

1. Prove that, for any set A, the following three definitions of *countable* are equivalent:

(1) $\exists$ a surjection $\omega \twoheadrightarrow A$.

(2) $A \sqsubseteq \omega$.

(3) A is finite or $A \equiv \omega$.

You may not use Axiom of Choice in your proofs, so be careful when proving (1)⇒(2).

HINT: For (2)⇒(3), we may assume that $A \subseteq \omega$ and your task is to define a new injection $f : A \hookrightarrow \omega$ such that $f(A)$ is an initial segment of ω. Because $A \subseteq \mathbb{N}$, you can define f by recursion.

IMPORTANT REMARK: One should think of (1) as the statement that A can be enumerated, possibly with repetitions, i.e. there is an infinite sequence $(a_n)_{n\in\mathbb{N}}$ of elements of A such that $A = \{a_n : n \in \mathbb{N}\}$. This is used in proofs to **avoid considering the finite and infinite cases separately** because, in either case, one would be dealing with an infinite sequence.

2. Prove the following statements.

(a) If sets A, B are countable, then $A \times B$ is also countable.

(b) Countable union of countable sets is countable. More precisely, for a sequence of countable sets $(A_n)_{n\in\mathbb{N}}$, the set $\bigcup_{n\in\mathbb{N}} A_n$ is countable.

3. Prove that the following sets are countable. You may use the Schröder–Bernstein theorem, as well as Problem 2.

(a) $\mathbb{Q}$

(b) The set $A^{<\mathbb{N}}$ for any countable A

(c) The set $P(\mathbb{Q})$ of polynomials with rational coefficients

(d) (Optional) The set of all algebraic numbers[1]

4. Prove that the following sets are equinumerous with $\mathbb{R}$. You may use the Schröder–Bernstein theorem.

(a) $(0, 1)$

(b) $[0, 1]$

HINT: $[0, 1] \subseteq (-1, 2)$.

(c) $\mathbb{R} \cup A$ for any countable set A

[1] A real $r \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with rational coefficients.

(d) The set $2^{\mathbb{N}}$ of all infinite sequences of 0-s and 1-s.

(e) $\mathbb{R}^2$

HINT: Intertwine the decimal expansions.

(f) (Optional) The set $\mathbb{R}^{\mathbb{N}}$ of all infinite sequences of reals

HINT: Intertwine the decimal expansions diagonally, just like in the proof of $\mathbb{N}^2 \equiv \mathbb{N}$.

5. (a) Prove $(0,1] \equiv (0,1)$ without using the Schröder–Bernstein theorem.

HINT: Isolate a Hilbert hotel inside of $(0,1]$ and push 1 into it.

(b) (Optional) More generally, for any set A, if $\omega \sqsubseteq A$, then $A \cup \{x\} \equiv A$ for any element $x \notin A$.

6. Prove the Claim in the proof of the Schröder–Bernstein theorem (Theorem 6.5 in the notes).