Math 432: Set Theory and Topology HOMEWORK 6 Due date: Mar 9 (Thu)

Reflections. Write a short essay surveying well-orderings, ordinals, natural numbers, equinumerosity, cardinals, and Axiom of Choice. Pick a definition, a statement (e.g. theorem), and a proof that you thought were most crucial for the development of the theory; explain your choices.

1. Let $(A, \leq)$ be a partial ordering and let $\mathcal{C}$ be a set of chains in A, i.e. $\mathcal{C} \subseteq \mathscr{P}(A)$ and each $C \in \mathcal{C}$ is a chain. Suppose that any two $C, C' \in \mathcal{C}$ are $\subseteq$-comparable, i.e. $C \subseteq C'$ or $C' \subseteq C$. Prove that $\bigcup \mathcal{C}$ is a chain.

HINT: It is enough to show that for any $a, b \in \bigcup \mathcal{C}$, there is $C \in \mathcal{C}$ with $a, b \in C$.

2. Denote by $<$ the relation $\in$ on ordinals and let $\kappa \geq \omega$ be a cardinal.

(a) For any well-ordering $(A, <)$, if $|\operatorname{pred}(a, A, <)| < \kappa$ for each $a \in A$, then $(A, <) \preceq (\kappa, \in)$.

HINT: Take the unique ordinal α such that $(A, <) \simeq (\alpha, \in)$.

(b) Define a binary relation $<_2$ on $\kappa \times \kappa$ as follows: for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \kappa \times \kappa$, put $(\alpha_1, \beta_1) <_2 (\alpha_2, \beta_2)$ if and only if

$$\max\{\alpha_1, \beta_1\} < \max\{\alpha_2, \beta_2\}$$

or

$$\big[\max\{\alpha_1, \beta_1\} = \max\{\alpha_2, \beta_2\} \text{ and } (\alpha_1, \beta_1) <_{\text{lex}} (\alpha_2, \beta_2)\big].$$

Prove that $<_2$ is a well-ordering.

(c) (Optional) (Doesn't use AC) Prove by transfinite induction that for any cardinal $\kappa \geq \omega$, $|\kappa \times \kappa| = \kappa$.

HINT: Use the induction hypothesis to deduce that for each $(\alpha, \beta) \in \kappa \times \kappa$, $|\operatorname{pred}\big((\alpha, \beta), \kappa \times \kappa, <_2\big)| < \kappa$. Apply (a).

(d) (Uses AC) Conclude that if $(A_\alpha)_{\alpha<\kappa}$ is a sequence of sets of cardinality at most κ, then $|\bigcup_{\alpha<\kappa} A_\alpha| \leq \kappa$. Pinpoint exactly where you use AC.

3. (Uses AC) Let $(A, \leq)$ be a partially ordered set. Call a chain $C \subseteq A$ *maximal* if it cannot be extended to a bigger chain, i.e. there is no $a \in A \setminus C$ such that $C \cup \{a\}$ is a chain. Prove that any chain $C \subseteq A$ is contained in a maximal chain.

4. (Optional) (Uses AC) Prove that every vector space admits a basis.

5. (Uses AC) **Prisoners and hats.** ω-many prisoners were sentenced to death, but they could get out under one condition: on the day of the execution they will be lined up, i.e. enumerated $(p_n)_{n\in\omega}$, so that everybody can the people in front of them (with higher index), i.e. p_n sees p_m if and only if $n < m$. Each of the prisoners will have a red or blue hat put on him/her, but he/she won't be told which color it is. On command, all the prisoners (at once) shout a guess for the color of their hat. If all but finitely many prisoners guess correctly, all prisoners go home free; otherwise they are all executed. The good news is that the prisoners developed a strategy the day before the execution, and indeed, all but finitely many prisoners guessed correctly the next day, so everyone was saved. How did they do it?

HINT: Thinking of the sequence of hats as a binary sequence, call two binary sequences $x, y \in 2^{\mathbb{N}}$ E_0-*equivalent* if all but finitely many of their entries are equal, i.e.

$$xE_0y :\Leftrightarrow \exists N \in \mathbb{N}\, \forall n \geq N\ x(n) = y(n).$$