1. Let (A, B) and (C, D) be proper[1] Dedekind cuts of $(\mathbb{Q}, <)$. Verify that the following are Dedekind cuts.

(a) $\big(A + C,\ \mathbb{Q} \setminus (A + C)\big)$, where $A + C := \{a + c : a \in A, c \in C\}$.

(b) $\big(\mathbb{Q} \setminus (-\bar{A}),\ -\bar{A}\big)$, where $\bar{A} := A \cup \mathrm{Ends}(A)$ and $-\bar{A} := \{-a : a \in \bar{A}\}$

(c) $\big(\mathbb{Q} \setminus (B \cdot D),\ B \cdot D\big)$ if $B, D \subseteq \mathbb{Q}^{\geq 0}$, where $B \cdot D := \{b \cdot d : b \in B, d \in D\}$ and $\mathbb{Q}^{\geq 0} := \{q \in \mathbb{Q} : q \geq 0\}$.

2. In this problem, we think of $\mathbb{R}$ as the set of all proper Dedekind cuts of $(\mathbb{Q}, <)$.

(a) Define the operations of addition and negation on $\mathbb{R}$ using (a) and (b) of Problem 1 and verify that your definition agree with the usual $+$ and $-$ on $\mathbb{Q}$.

(b) Define the operation of multiplication on $\mathbb{R}$ using (b) and (c) of Problem 1 and verify that your definition agrees with that multiplication on $\mathbb{Q}$.

3. Let d be a metric on a set X and prove that $d' : X^2 \to [0, 1]$ defined by $d'(x, y) := \min\{1, d(x, y)\}$ is also a metric on X.

4. Let (X, d_X) and (Y, d_Y) be metric spaces.

(a) (Very optional) For each positive $p \in \mathbb{N}$, define $d_p : (X \times Y)^2 \to [0, \infty)$ by

$$d_p\big((x_1, y_1), (x_2, y_2)\big) := \sqrt[p]{d(x_1, x_2)^p + d(y_1, y_2)^p}.$$

Show that d_p is a metric on $X \times Y$.

HINT: For $p = 2$, this follows from Cauchy–Schwartz inequality from linear algebra. For other p, one has to use Hölder's inequality, which is really not relevant to this class.

(b) Define $d_\infty : (X \times Y)^2 \to [0, \infty)$ by

$$d_\infty\big((x_1, y_1), (x_2, y_2)\big) := \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Show that d_∞ is a metric on $X \times Y$.

(c) (Optional) Show that for any $x \in X, y \in Y$,

$$\lim_{p \to \infty} d_p\big((x_1, y_1), (x_2, y_2)\big) = d_\infty\big((x_1, y_1), (x_2, y_2)\big).$$

HINT: Suppose that the maximum is achieved by $|x_1 - x_2|$ and take its p^{th} power out of the root.

5. Recalling that $2^{\mathbb{N}}$ is the set of all infinite binary sequences, define $d : 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to [0, 1]$ by $d(x, y) := 2^{-\Delta(x,y)}$, where $\Delta(x, y) :=$ the least index $n \in \mathbb{N}$ such that $x(n) \neq y(n)$. For example, $\Delta(00101..., 00110...) = 3$, so $d(00101..., 00110...) = 2^{-3} = \frac{1}{8}$. Letting $x, y, z \in 2^{\mathbb{N}}$, prove that $\Delta(x, z) \geq \min\{\Delta(x, y), \Delta(y, z)\}$, and hence,

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}. \qquad (*)$$

Conclude that d is a metric on $2^{\mathbb{N}}$.

REMARK: A metric satisfying the stronger condition $(*)$ is called an *ultrametric*.

6. For metric spaces (X, d_X) and (Y, d_Y), a function $f : X \to Y$ is called an *isometry* if for every $x, x' \in X$, $d_X(x, x') = d_Y(f(x), f(x'))$. Do problem 12 of 4.1 of Kaplansky's book.

[1] Call a Dedekind cut (A, B) *proper* if both A and B are nonempty.

7. For a metric space (X, d) and $A \subseteq X$, define $\text{diam}(A) := \sup_{x,y \in A} d(x, y)$. Do problem 20(a) of 4.1 of Kaplansky's book.