Complete, last update: Apr 4 at 2pm

Math 432: Set Theory and Topology HOMEWORK 9 Due date: Apr 6 (**Thu**)

Exercises from Kaplansky's book.
Sec 4.3: 2

1. Define a metric d on $\mathbb{N}$ such that for every $n \in \mathbb{N}$, the set $[n, \infty)_{\mathbb{N}} := \{k \in \mathbb{N} : k \geq n\}$ is an open, as well as a closed, ball centered at n, i.e. there is $r_n > 0$ such that

$$B_d(n, r_n) = \bar{B}_d(n, r_n) = [n, \infty)_{\mathbb{N}}.$$

2. Show that in any metric space, every open set is a union of open balls of rational radius.

3. Consider $\mathbb{R}$ with its usual metric.

(a) Show that every open set is a union of open intervals with rational endpoints.

(b) What is the cardinality of the set $\mathcal{U}$ of all open intervals with rational endpoints?

(c)* (Optional) How many open sets are there in $\mathbb{R}$? More precisely, letting $\mathcal{T}$ denote the set of all open subsets of $\mathbb{R}$, is $\mathcal{T} \equiv \mathbb{R}$?

HINT: Define a surjection $\mathscr{P}(\mathcal{U}) \twoheadrightarrow \mathcal{T}$.

4. Let (X, d) be a metric space. Define the following two metrics on X^2:

$$d_\infty^{(2)}\big((x_1, y_1), (x_2, y_2)\big) := \max\{d(x_1, x_2), d(y_1, y_2)\}$$
$$d_1^{(2)}\big((x_1, y_1), (x_2, y_2)\big) := d(x_1, x_2) + d(y_1, y_2).$$

Show that regardless which of these two metrics we take as $d^{(2)}$, $(x_n, y_n) \to (x, y)$ in the metric space $(X^2, d^{(2)})$ if and only if $x_n \to x$ and $y_n \to y$ in (X, d).

5. Let $(x_n)_n$ be a sequence in the Cantor space $2^{\mathbb{N}}$ (with the usual metric). Prove that $x_n \to x$ if and only if for each fixed index $i \in \mathbb{N}$, $\forall^\infty n \; x_n(i) = x(i)$.

6. Let (X, d) be a metric space. For a set $A \subseteq X$, define its *diameter* by

$$\operatorname{diam}(A) := \sup_{x,y \in A} d(x, y).$$

Show that for any convergent[1] sequence $(x_n)_n \subseteq X$ has finite diameter, i.e.

$$\operatorname{diam}\big(\{x_n : n \in \mathbb{N}\}\big) < \infty.$$

[1]Call a sequence $(x_n)_n \subseteq X$ *convergent* if it has a limit, i.e. there is $x \in X$ such that $x_n \to x$.