Math 432: Set Theory and Topology HOMEWORK 10 **Due: April 25/26**

1. Recall that a topology $\mathcal{T}$ on a set X is a subset $\mathcal{T} \subseteq \mathscr{P}(X)$ that contains $\emptyset, X$ and is closed under finite intersections and arbitrary unions. For a set X and $\mathcal{S} \subseteq \mathscr{P}(X)$, prove that the collection $\langle \mathcal{S} \rangle \subseteq \mathscr{P}(X)$ consisting of $\emptyset, X$ and arbitrary unions of finite intersections of sets in $\mathcal{S}$ is the $\subseteq$-least topology containing $\mathcal{S}$.

HINT: Need to show that $\langle \mathcal{S} \rangle$ is a topology and it is contained in any topology containing $\mathcal{S}$.

2. For a topological space X, a *base* $\mathcal{B}$ is collection of open sets such that every open sets $U \subseteq X$ is a union of sets in $\mathcal{B}$. We say that X is *second-countable* if it admits a countable base.

(a) Show that in any metric space, the collection of all open balls of radii of the form $\frac{1}{n}$, $n \geqslant 1$, is a base.

(b) Show that $\mathbb{R}$ and $\mathbb{N}^{\mathbb{N}}$ are second-countable.

3. A topological space is called *separable*, if it admits a countable dense subset.

(a) Prove that the following spaces are separable: $\mathbb{R}^n$ $(n \geqslant 1)$, $\mathbb{N}^{\mathbb{N}}$, $\mathbb{R}^{\mathbb{N}}$.

(b) Prove that if a topological space is second-countable then it is separable.

(c) The converse is not true in general, however show that it holds for metric spaces: If a metric space is separable, then it is second countable.

4. Prove that a topological space is Hausdorff if and only if the diagonal $\Delta_X := \{(x,y) \in X \times X : x = y\}$ is a closed subset of $X \times X$ (in the product topology).

5. Let X, Y be topological spaces, where Y is Hausdorff. Let $D \subseteq X$ be a dense subset of X. Let $C(X,Y)$ denote the set of all continuous functions from X to Y.

(a) Prove that the restriction map $C(X,Y) \to C(D,Y)$ given by $f \mapsto f|_D$ is one-to-one.

HINT: Let $f, g \in C(X,Y)$ be such that $f|_D = g|_D$ and yet, $f(x_0) \neq g(x_0)$ for some $x_0 \in X$. Use Hausdorffness of Y and recall that continuous means that preimages of open sets are open.

(b) Conclude that there are exactly continuum-many continuous functions $\mathbb{R} \to \mathbb{R}$.

HINT: To show that $\mathbb{R}^{\mathbb{N}} \equiv \mathbb{R}$, recall that $\mathbb{R} \equiv 2^{\mathbb{N}}$ and $(2^{\mathbb{N}})^{\mathbb{N}} \equiv 2^{\mathbb{N} \times \mathbb{N}}$.

6. For a topological space X, call a set $A \subseteq X$ *connected* if the induced topology on A is connected, i.e., if $A = (A \cap U) \sqcup (A \cap V)$ for two disjoint open sets U, V in X, then either $A \cap U = \emptyset$ or $A \cap V = \emptyset$.

(a) Show that the connected subsets of $\mathbb{R}$ are precisely the convex sets. Recall that a set $A \subseteq \mathbb{R}$ is called *convex*, if for each $a, b \in A$ with $a < b$, $(a,b) \subseteq A$.

(b) For topological spaces X, Y, prove that any continuous function $f \colon X \to Y$ maps connected subsets of X to connected subsets of Y, i.e., if $A \subseteq X$ is connected, then $f(A)$ is connected.

(c) Deduce the **Intermediate Value Theorem**: If X is a connected topological space (e.g., $\mathbb{R}$, $[0,1)$) and $f \colon X \to \mathbb{R}$ is continuous, then $f(X)$ is convex.