Math 432: Set Theory and Topology HOMEWORK 5 **Due: March 7/8**

1. Prove that $[0,1] \equiv [0,1)$.

HINT: Find a Hilbert hotel in $[0,1]$.

2. For sets X, Y, let Y^X denote the set of all functions from X to Y; in particular, 2^X is the set of all 0-1 valued functions on X. For $A \subseteq X$, let $\mathbb{1}_A : X \to 2$ denote the *characteristic/indicator function of* A, that is: for $x \in X$,

$$\mathbb{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

Prove that the function $\pi : \mathscr{P}(X) \to 2^X$ that takes every $A \subseteq X$ to its characteristic function $\mathbb{1}_A$ is a bijection.

3. Let A, B be sets.

(a) Prove that if A and B are countable, then $A \times B$ is countable.

(b) Prove that if A is countable and every $a \in A$ is also countable, then $\bigcup A$ is countable. Did you use Axiom of Choice?

HINT: Use part (a).

(c) Denote $A^0 := \{\emptyset\}$ and prove that $\{A^n : n \in \mathbb{N}\}$ is a set without using Replacement.

(d) Prove that if A is countable, then the set $A^{<\omega} := \bigcup_{n\in\mathbb{N}} A^n$ is countable, where $\bigcup_{n\in\mathbb{N}} A^n := \bigcup\{A^n : n \in \mathbb{N}\}$.

4. (a) Prove that addition is *well-defined* on $\mathbb{Q}$, that is: although the result $\frac{n_0 m_1 + n_1 m_0}{m_0 m_1}$ of the addition $\frac{n_0}{m_0} + \frac{n_1}{m_1}$ is defined using the particular representatives (n_0, m_0) and (n_1, m_1) of the equivalence classes $\frac{n_0}{m_0}$ and $\frac{n_1}{m_1}$, respectively, the result itself does not depend on the representatives, i.e., $\frac{n_0 m_1 + n_1 m_0}{m_0 m_1} = \frac{n_0' m_1' + n_1' m_0'}{m_0' m_1'}$ whenever $\frac{n_0}{m_0} = \frac{n_0'}{m_0'}$ and $\frac{n_1}{m_1} = \frac{n_1'}{m_1'}$.

(b) Prove that multiplication is well-defined on $\mathbb{Q}$.

(c) Without using Axiom of Choice, define a transversal for the equivalence relation in the definition of $\mathbb{Q}$. This just means finding a subset $S \subseteq \mathbb{Z} \times \mathbb{N}^+$ that intersects every $\sim$-class in exactly one point.

5. For sets A, B, we write $A \twoheadrightarrow B$ to mean that there is a surjection $\pi : A \to B$.

(a) Prove without using Axiom of Choice that for any set X and an ordinal α, $X \sqsubseteq \alpha$ if and only if $\alpha \twoheadrightarrow X$.

(b) Use the Cantor–Schröder–Bernstein theorem to deduce that

$$(\alpha \twoheadrightarrow X \text{ and } \alpha \sqsubseteq X) \iff \alpha \equiv X.$$

(c) Conclude that $\mathbb{N} \equiv \mathbb{Q}$.

6. **Cantor's diagonalization.** Let R be a binary relation on X. For $x \in X$, let $R_x := \{y \in X : (x,y) \in R\}$ and call these sets *sections of* R. Prove that the antidiagonal $\nabla(R) := \{x \in X : (x,x) \notin R\}$ of R is not equal to any of the sections of R.

7. Prove that if $A \subseteq \mathbb{N}$, then there is $\alpha \leqslant \omega$ such that $\alpha \equiv A$.

HINT: Intuitively, you should try to enumerate the elements of A. This is formally done by recursively defining a function $\pi : \omega \to A$ (transfinite, or in this case, finite, recursion) such that for some $\alpha \leqslant \omega$, $\pi|_\alpha : \alpha \to A$ is an order-preserving bijection.