Math 432: Set Theory and Topology HOMEWORK 7 **Due: April 4/5**

Notation. For a set A, we denote by $|A|$ the *cardinality* of A, i.e., the unique cardinal κ with $A \equiv \kappa$. Note that Axiom of Choice implies that every set has cardinality (Zermelo's theorem), but this cannot be proven from only the ZF axioms.

1. Let $(A, <)$ be a partial ordering.

(a) Let $\mathcal{C}$ be a set of chains in A, i.e., $\mathcal{C} \subseteq \mathscr{P}(A)$ and each $C \in \mathcal{C}$ is a chain. Suppose that any two $C, C' \in \mathcal{C}$ are $\subseteq$-comparable, i.e., $C \subseteq C'$ or $C' \subseteq C$. Prove that $\bigcup \mathcal{C}$ is a chain.

HINT: It is enough to show that for any $a, b \in \bigcup \mathcal{C}$, there is $C \in \mathcal{C}$ with $a, b \in C$.

(b) Call a chain $C \subseteq A$ *maximal* if it is $\subseteq$-maximal, i.e., there is no chain $C' \subseteq A$ that properly contains C. Prove that any chain $C \subseteq A$ is contained in a maximal chain.

2. Denote by $<$ the relation $\in$ on ordinals and let $\kappa \geqslant \omega$ be a cardinal.

(a) For any well-ordering $(A, <)$, if $|A_{<a}| < \kappa$ for each $a \in A$, then $(A, <) \preceq (\kappa, \in)$.

HINT: Take the unique ordinal α such that $(A, <) \simeq (\alpha, \in)$.

(b) Define a binary relation $<_2$ on $\kappa \times \kappa$ as follows: for $(\alpha_1, \beta_1), (\alpha_2, \beta_2) \in \kappa \times \kappa$, put $(\alpha_1, \beta_1) <_2 (\alpha_2, \beta_2)$ if and only if

$$\max\{\alpha_1, \beta_1\} < \max\{\alpha_2, \beta_2\}$$

or

$$\Big[\max\{\alpha_1, \beta_1\} = \max\{\alpha_2, \beta_2\} \text{ and } (\alpha_1, \beta_1) <_{\text{lex}} (\alpha_2, \beta_2)\Big].$$

Prove that $<_2$ is a well-ordering.

(c) Without Axiom of Choice, prove by transfinite induction that for any cardinal $\kappa \geqslant \omega$, $|\kappa \times \kappa| = \kappa$.

HINT: Use the induction hypothesis to deduce that for each $(\alpha, \beta) \in \kappa \times \kappa$, $\big|\text{pred}\big((\alpha, \beta), \kappa \times \kappa, <_2\big)\big| < \kappa$. Apply (a).

(d) Taking $\kappa := \omega$, this gives a slightly different proof that $\omega^2 \equiv \omega$. The difference is in the ordering on ω^2. What was the ordering we considered in class to prove $\omega^2 \equiv \omega$?

(e) (Optional) Conclude that if $(A_\alpha)_{\alpha < \kappa}$ is a sequence of sets of cardinality at most κ, then $|\bigcup_{\alpha < \kappa} A_\alpha| \leqslant \kappa$. Pinpoint exactly where you use AC.

3. Consider the partial ordering $(\mathscr{P}(\mathbb{N}), \subseteq)$.

(a) Show that the set E of even natural numbers is a chain in both partial orderings $(\omega, \in)$ and $(\mathscr{P}(\omega), \subseteq)$.

(b) Put $C_n := \{k \cdot 2^n : k \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. Show that the set $\mathcal{C} := \{C_n : n \in \mathbb{N}\}$ is a chain in the partial ordering $(\mathscr{P}(\omega), \subseteq)$. Find the $\subseteq$-least and $\subseteq$-largest elements of $\mathcal{C}$, if they exist.

(c) Exhibit an infinite chain in $(\mathscr{P}(\omega), \subseteq)$ consisting of only infinite subsets of ω and admitting an $\subseteq$-least element.

4. Prove that for any set A, its Hartog set $\chi(A)$, as defined in class, is the least ordinal that does not inject into A. Recall that we already proved in class that it does not inject, so all you have to show is the leastness.

5. Let $(A, <)$ be a partial ordering and let $\mathbf{WO}(A)$ be the set as in the proof of Hartog's theorem, i.e.,

$$\mathbf{WO}(A) := \{(B, <_B) : B \subseteq A \text{ and } <_B \text{ is a well-ordering of } B\}.$$

Define an ordering $\preceq'$ on $\mathbf{WO}(A)$ by putting

$$(B_1, <_1) \preceq' (B_2, <_2) :\Longleftrightarrow (B_1, <_1) \text{ is an initial segment in } (B_2, <_2).$$

In other words, $B_1 \subseteq B_2$, $<_1 \subseteq <_2$, and B_1 is an initial segment in $(B_2, <_2)$.

Let $\mathcal{C}$ be a chain in $(\mathbf{WO}(A), \preceq')$, let $U := \bigcup \mathcal{C}$, and let $<_U := \bigcup_{C \in \mathcal{C}} <_C$.

(a) Prove that $<_U$ is a total ordering of U.

(b) Prove that for any $(C, <_C) \in \mathcal{C}$, C is an initial segment of $(U, <_U)$.

(c) Deduce that for any set $S \subseteq U$ and any $(C, <_C) \in \mathcal{C}$, every element of $S \cap C$ is $<_U$ every element of $S \setminus C$.

(d) Conclude that $<_U$ is a well-ordering of U.