Math 432: Set Theory and Topology HOMEWORK 8 **Due: April 11/12**

Part of this homework is to redo the theory of Dedekind cuts and completion of total orderings using the better definition of a Dedekind cut than the one initially given in class[1].

Definition. Let $(X,<)$ be a total ordering.

- For $x \in X$, denote $(-\infty,x]_X := \{y \in X : y \leqslant x\}$ and $[x,+\infty)_X := \{y \in X : y \geqslant x\}$.
- Call a set $Y \subseteq X$ *initial* (resp., *terminal*) in $(X,<)$ if for each $y \in Y$, $(-\infty,y]_X \subseteq Y$ (resp., $[y,+\infty)_X \subseteq Y$). Call Y *proper* if $Y \neq \emptyset$ and $Y \neq X$.
- For an initial (resp., terminal) set Y in $(X,<)$, put

$$\overline{Y} := \begin{cases} Y \cup \{\sup Y\} \text{ (resp., } Y \cup \{\inf Y\}) & \text{if } \sup Y \text{ (resp., } \inf Y) \text{ exists in } (X,<) \\ Y & \text{otherwise} \end{cases}$$

  and call it the *closure* of Y in $(X,<)$.
- Call an initial or terminal set Y in $(X,<)$ *closed* if $Y = \overline{Y}$.
- Call a proper closed initial set Y a *Dedekind cut* and let $\mathcal{C}X$ denote the set of all Dedekind cuts of $(X,<)$.
- Here and below, we write $\subset$ to mean the proper subset relation $\subsetneq$. Call the ordering $(\mathcal{C}(X),\subset)$ the *completion of* $(X,<)$.
- Define $\mathbb{R}$ as the completion of $(\mathbb{Q},<)$ and denote $(\mathbb{R},<) := (\mathcal{C}(\mathbb{Q}),\subset)$. Call the elements of $\mathbb{R}$ *reals* or *real numbers*.

1. Let $(X,<)$ be a total ordering.

(a) Prove that $(\mathcal{C}(X),\subset)$ is also a total ordering.

(b) Define $\pi : X \to \mathcal{C}(X)$ by $x \mapsto (-\infty,x]_X$ and show that π is an order-embedding, i.e., for any $x,y \in X$,

$$x < y \iff \pi(x) \subset \pi(y).$$

We call this π the *natural embedding* of $(X,<)$ into $(\mathcal{C}(X),\subset)$. We usually identify X with its image $\pi(X)$ and treat X as a subset of $\mathcal{C}(X)$, just like we treat $\mathbb{Z}$ as a subset of $\mathbb{Q}$.

2. Let $(X,<)$ be a total ordering.

(a) For any $\mathcal{S} \subseteq \mathcal{C}(X)$, prove that $\bigcap \mathcal{S} := \{x \in X : \forall C \in \mathcal{S}\ x \in C\}$ is a closed initial set.

(b) Prove that any nonempty $\mathcal{S} \subseteq \mathcal{C}(X)$ that has a lower bound in $(\mathcal{C}(X),\subset)$ has a greatest lower bound.

(c) Conclude that $(\mathcal{C}(X),\subset)$ is a complete total ordering.

3. Prove that if a total ordering is complete, then the natural embedding into its completion is an order-isomorphism.

[1] The definition of a Dedekind cut given in class is more aligned with model-theoretic (area of logic) philosophy, but the one in this homework is better suited for the purposes of this course.

4. The set $(-\infty, \sqrt{2})_{\mathbb{Q}} := \left\{q \in \mathbb{Q} : q < 0 \text{ or } q^2 < 2\right\}$ is an initial set bounded above, yet does not have a least upper bound in $(\mathbb{Q}, <)$. Conclude that it is closed in $(\mathbb{Q}, <)$.

5. For a total order $(A, <)$, we say that a subset $B \subseteq A$ is *dense* in $(A, <)$ if for every pair a_1, a_2 in A with $a_1 < a_2$, there is $b \in B$ with $a_1 < b < a_2$. Call a total order $(A, <)$ *dense* if A is dense in $(A, <)$.

(a) Show that $(\mathbb{Q}, <)$ is a dense total order.

(b) Prove that if a total ordering $(A, <)$ is dense then A (more precisely, the image of A under the natural embedding) is dense in the completion $(\mathcal{C}(A), \subset)$.

(c) Conclude that $\mathbb{Q}$ is dense in $(\mathbb{R}, <) := (\mathcal{C}(Q), \subset)$.

6. (Optional) Let A, B be Dedekind cuts in $(\mathbb{Q}, <)$, i.e., $A, B \in \mathbb{R}$. Recall that we identify $\mathbb{Q}$ with its image inside $\mathcal{C}(\mathbb{Q})$, so we write $A \geqslant 0$ instead of $A \supseteq (-\infty, 0]_{\mathbb{Q}}$. Denote $A^c := \mathbb{Q} \setminus A$.

Define the operations $+^{\mathbb{R}}, -^{\mathbb{R}}, \cdot^{\mathbb{R}}$ on $\mathbb{R}$ by showing that the following are Dedekind cuts.

(a) $A +^{\mathbb{R}} B := A + B := \{a + b : a \in A \text{ and } b \in B\}$.

(b) $-^{\mathbb{R}} A := -A := \{-a : a \in A\}$.

(c) For $A, B \geqslant 0$, $A \cdot^{\mathbb{R}} B := (A^c \cdot B^c)^c$, where for any $X, Y \subseteq \mathbb{Q}$,

$$X \cdot Y := \{x \cdot y : x \in X \text{ and } y \in Y\}.$$