Mathematical Logic HOMEWORK 11 **Due: May 24 (Fri)**

1. Let $\sigma_{\text{arthm}} := (0, S, +, \cdot)$ be the signature of arithmetic and let $\ulcorner\urcorner : \text{Formulas}(\sigma) \to \mathbb{N}$ be the coding function for σ_{arthm}-formulas informally defined in class. Recall that a σ-theory T is called **computable** (resp. **arithmetical**), if the set

$$\ulcorner T \urcorner := \{\ulcorner\varphi\urcorner : \varphi \in T\} \subseteq \mathbb{N}$$

is computable (resp. arithmetical). Define $\text{Proof}_T \subseteq \mathbb{N}^2$ by

$$\text{Proof}(a,b) :\Leftrightarrow b = \ulcorner\varphi\urcorner \text{ and } a \text{ is a code of a proof of } \varphi \text{ from } T.$$

It was proven in class that if a σ-theory T is computable, then the relation Proof_T is computable. Prove the same for arithmetical, i.e. if a σ-theory T is arithmetical, then so is the relation Proof_T.

Definition. A function $f : \mathbb{N}^k \to \mathbb{N}$ is called **primitive recursive** if it is one of the basic functions below or is obtained from the latter by finitely many applications of the operations of composition and primitive recursion. Basic functions:

- Successor function $S : \mathbb{N} \to \mathbb{N}$ given by $n \mapsto n+1$;
- Constant functions $C^k_m : \mathbb{N}^k \to \mathbb{N}$ given by $\vec{a} \mapsto m$ for each $k, m \in \mathbb{N}$;
- Projection functions $P^k_i : \mathbb{N}^k \to \mathbb{N}$ given by $P^k_i(x_1, \ldots, x_k) := x_i$ for each $k \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$.

2. Prove that the following functions are primitive recursive.

(a) Predecessor function $PD : \mathbb{N} \to \mathbb{N}$ defined by $n \mapsto n-1$ if $n \geqslant 1$ and 0 otherwise.

(b) Secure subtraction $\dot{-} : \mathbb{N}^2 \to \mathbb{N}$ defined by $(n, m) \mapsto n - m$ if $n \geqslant m$ and 0 otherwise.

(c) Addition $N^2 \to \mathbb{N}$ defined by $(x, y) \mapsto x + y$.

(d) Multiplication $N^2 \to \mathbb{N}$ defined by $(x, y) \mapsto x \cdot y$.

(e) Exponentiation $N^2 \to \mathbb{N}$ defined by $(x, y) \mapsto x^y$ if $x \neq 0$ and 0 otherwise.

3. **Intertwined primitive recursion.** For $i \in \{0, 1\}$, let $g_i : \mathbb{N}^k \to \mathbb{N}$ and $h_i : \mathbb{N}^{k+2} \to \mathbb{N}$ be computable functions. Let $f_0, f_1 : \mathbb{N}^{k+1} \to \mathbb{N}$ be such that for each $\vec{a} \in \mathbb{N}^k$ and $n \in \mathbb{N}$,

$$\begin{cases} f_0(\vec{a}, 0) = g_0(\vec{a}) \\ f_1(\vec{a}, 0) = g_1(\vec{a}) \\ f_0(\vec{a}, n+1) = h_0(\vec{a}, n, f_1(\vec{a}, n)) \\ f_1(\vec{a}, n+1) = h_1(\vec{a}, n, f_0(\vec{a}, n)). \end{cases}$$

Prove that both f_0 and f_1 are computable.

HINT: Consider the function $f(\vec{a}, n) := \text{Pair}(f_0(\vec{a}, n), f_1(\vec{a}, n))$.