Last update: 5:01pm, May 8.

Measure theory with ergodic horizons | HOMEWORK 11 | **Due: May 13**

1. Let (X,μ) be a measure space. Recall that an equivalence relation E on X is called **ergodic** if every E-invariant measurable set is null or conull. Prove that every E-invariant[1] measurable function $f : X \to \mathbb{R}$ is constant a.e. i.e. there is a constant $c \in \mathbb{R}$ such that $f = c$ a.e.

HINT: There are two ways to prove it.

(1) Devide $\mathbb{R}$ into countably many intervals and notice that exactly one of them has a conull preimage. Next define each interval by into two equal pieces, and again, exactly one of them has conull preimage. Continue...

(2) If f isn't constant a.e. then the pushforward $f_* \mu$ isn't a Dirac measure, so there is a set $A \subseteq \mathbb{R}$ such that both A and $\mathbb{R} \setminus A$...

2. Tightness of the Fubini–Tonelli hypothesis.

(a) **Non-σ-finite.** Let $X := [0,1]$, and let μ and ν be, respectively, Lebesgue and counting measures on X. Because ν is not σ-finite, the product measure $\mu \times \nu$ is not unique, but we let $\mu \times \nu$ denote the largest of them, namely, the outer measure $(\mu \times \nu)^*$ induced by the premeasure $\mu \times \nu$ on the algebra generated by $\mathcal{B}(X) \times \mathcal{B}(X)$.

For the diagonal $\Delta := \{(x,x) : x \in X\}$, compute

$$\int \nu(\Delta_x)\, d\mu(x), \int \mu(\Delta^y)\, d\nu(y), \text{ and } \mu \times \nu(\Delta)$$

to verify that no two are equal to each other.

HINT: To compute $\mu \times \nu(\Delta)$, recall the definition of the outer measure.

(b) **Non-integrable.** Let $\mu = \nu$ be the counting measure on $\mathbb{N}$ (defined on $\mathscr{P}(\mathbb{N})$). Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be defined by setting $f(x,y)$ to be 1 if $x = y$, -1 if $x = y+1$, and 0 otherwise. Verify that f is not $\mu \times \nu$-integrable, and although the integrals $\int\int f\, d\mu\, d\nu$ and $\int\int f\, d\nu\, d\mu$ both exist, they are unequal.

(c) [*Optional*] **Non-measurable.** Let $X := \omega_1$ and let $\mathcal{M} = \mathcal{N}$ be the σ-algebra of countable and co-countable subsets of ω_1. Let $\mu = \nu$ be defined to be 0 on countable and 1 on co-countable sets. Let $R :=<$, i.e. $R := \{(x,y) \in \omega_1 \times \omega_1 : x < y\}$. Then R is not in $\mathcal{M} \times \mathcal{N}$. However, the fibers R_x and R^y are in $\mathcal{M} = \mathcal{N}$ for each $x, y \in \omega_1$, and the integrals $\int \nu(R_x)\, d\mu(x)$ and $\int \mu(R^y)\, d\nu(y)$ exist, but they are unequal.

3. $\mu \times \nu$-measurable Fubini–Tonelli. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces and let $f : X \times Y \to \underline{\mathbb{R}}$ be a $\mu \times \nu$-measurable function. Prove:

[1] A function f on X is called E-**invariant** if it is constant on every $\mathbb{E}$-class, i.e. $xEy \implies f(x) = f(y)$ for all $x, y \in X$.

(a) The fibers f_x and f^y are respectively ν and μ measurable for μ-a.e. $x \in X$ and ν-a.e. $y \in Y$.

Caution: Folland claims $\mathcal{M}$ and $\mathcal{N}$ measurability instead, which is wrong.

(b) If $f \geqslant 0$, then functions $g : x \mapsto \int f_x \, d\nu$ and $h : y \mapsto \int f^y \, d\mu$ are defined respectively μ-a.e. and ν-a.e., and they are respectively μ and ν measurable. Moreover,

$$\int \int f \, d\nu \, d\mu = \int f \, d\mu \times \nu = \int \int f \, d\mu \, d\nu. \tag{$*$}$$

(c) If f is $\mu \times \nu$-integrable, then in fact g and h are μ and ν integrable and ($*$) holds.

4. Namioka's trick. Let $f : (0, \infty) \to \mathbb{R}$ be a Lebesgue integrable function. Prove:

(a) $g(x) := \int_x^\infty t^{-1} f(t) d\lambda(t)$ is well-defined for each $x > 0$, i.e. $t \mapsto \mathbb{1}_{(x,\infty)}(t) t^{-1} f(t)$ is a Lebesgue integrable function.

(b) The function $g : (0, \infty) \to \mathbb{R}$ is Lebesgue integrable and

$$\int_0^\infty g \, d\lambda = \int_0^\infty f \, d\lambda.$$