Last update: 1:42am, May 26.

Measure theory with ergodic horizons | HOMEWORK 12 | **Due: Jun 3**

1. Let (Y,ν) be a probability space and consider the product space $(X,\mu) := (Y^{\mathbb{N}}, \mu)$. Let $S : X \to X$ be the shift transformation, i.e. $(y_n)_{n\in\mathbb{N}} \mapsto (y_{n+1})_{n\in\mathbb{N}}$. Prove:

(a) S is measure-preserving, i.e. $\mu(S^{-1}(A)) = \mu(A)$ for each measurable $A \subseteq X$.

(b) S is mixing, i.e. for all measurable $A, B \subseteq X$, we have

$$\lim_{n\in\infty} \mu(S^{-1}(A) \cap B) = \mu(A)\mu(B).$$

HINT: Fist prove for cylinders A, B and then approximate.

(c) For $k \geqslant 2$, the k-fold bakers map $b_k : [0,1) \to [0,1)$ with Lebesgue measure λ on $[0,1)$ is measure-isomorphic to the shift transformation on $(k^{\mathbb{N}}, \nu_u^{\mathbb{N}})$, where ν_u is the uniform probability measure on $k := \{0,1,\dots,k-1\}$.

2. Proof of the classical ergodic theorem without the simplifying assumptions. The following steps remove the assumptions that the functions f and $x \mapsto n_x$ are bounded.

(a) Let $\delta > 0$ be small enough so that for each measurable set $B \subseteq X$

$$\mu(B) \leqslant \delta \implies \int_B |f - \delta| d\mu \leqslant \frac{\Delta}{4}.$$

(b) Let $M > 0$ be large enough so that $Y := f^{-1}([-M,\infty))$ has measure $\geqslant 1-\delta$. Thus, $\mathbb{1}_Y(f-\Delta) \geqslant -(M+\Delta)$ and $\int \mathbb{1}_{X\setminus Y}|f-\Delta| d\mu \leqslant \frac{\Delta}{4}$.

(c) Let $L \in \mathbb{N}$ be large enough so that the set

$$Z := \{x \in X : n_x > L\}$$

has measure $\leqslant \varepsilon \cdot \delta$, where $\varepsilon := \frac{1}{2(M+\Delta)} \frac{\Delta}{4}$. Thus, by (b) of the local-global bridge (small^2 measure implies small density), for every $n \in \mathbb{N}$, we have $A_n \mathbb{1}_Z(x) = \frac{|I_n(x) \cap Z|}{|I_n(x)|} \leqslant \varepsilon$ for all x in a set $X_n \subseteq X$ of measure $\geqslant 1-\delta$.

(d) **Tiling.** Let $N \in \mathbb{N}$ be large enough so that $\frac{L}{N} < \varepsilon$. Then for each $x \in X_N$, at least $(1-2\varepsilon)$-fraction of the set $I_N(x)$ is tiled by intervals of the form $I_{n_y}(y)$. Thus, $A_N(\mathbb{1}_Y(f-\Delta))(x) \geqslant -2\varepsilon(M+\Delta) = -\frac{\Delta}{4}$.

HINT: Do not tile the points of Z in $I_N(x)$. They occupy at most ε-fraction of $I_N(x)$.

(e) **Local-global contradiction.** $-\Delta = \int (f-\Delta) d\mu \geqslant \int \mathbb{1}_Y (f-\Delta) d\mu - \frac{\Delta}{4}$, and the local-global bridge (again!) gives:

$$\int \mathbb{1}_Y(f-\Delta) d\mu = \int A_N(\mathbb{1}_Y(f-\Delta)) d\mu \geqslant \int_{X_N} A_N(\mathbb{1}_Y(f-\Delta)) d\mu - \frac{\Delta}{4} \geqslant -\frac{\Delta}{4} - \frac{\Delta}{4},$$

so $-\Delta \geqslant -\frac{\Delta}{4} - \frac{\Delta}{4} - \frac{\Delta}{4} = -\frac{3}{4}\Delta$, a contradiction.

3. **Lebesgue decomposition theorem.** Follow the steps to prove the Lebesgue decomposition theorem: for any σ-finite measures μ, ν on a measurable space $(X, \mathcal{B})$, there are measures μ_0, μ_1 on $(X, \mathcal{B})$ with $\mu = \mu_0 + \mu_1$ such that $\mu_0 \ll \nu$ and $\mu_1 \perp \nu$.
   (a) It is enough to prove that there is a partition $X = X_0 \sqcup X_1$, with $X_i \in \mathcal{B}$, such that $\mu|_{X_0} \ll \nu|_{X_0}$ and $\nu(X_1) = 0$.
   (b) Reduce to the case when both μ and ν are finite measures, and below assume that they are finite.
   (c) By a $\frac{1}{2}$-measure exhaustion argument, collect together "all" sets $B \in \mathcal{B}$, which are ν-null but not μ-null. Let X_1 be their union, so X_1 is ν-null, and verify that $X_0 := X \setminus X_1$ satisfies the desired property.

4. Prove that the function f in the Radon–Nikodym theorem is unique up to ν-null sets, i.e. if g is another function satisfying $\mu(B) = \int_B g \, d\nu$ for all $B \in \mathcal{B}$, then $f = g$ ν-a.e.
   This function f is called the **Radon–Nikodym derivative** of μ over ν, and is denoted by $\frac{d\mu}{d\nu}$.

5. Let μ be a locally finite Borel measure on $\mathbb{R}$. Let $f_\mu : \mathbb{R} \to \mathbb{R}$ be a distribution function of μ, i.e. $\mu((a,b]) = f_\mu(b) - f_\mu(a)$ for all reals $a < b$. Let λ denote the Lebesgue measure on $\mathbb{R}$. Prove: if f_μ is differentiable and f_μ' is continuous, then $\mu \ll \lambda$ and $\frac{d\mu}{d\lambda} = f_\mu'$.

6. Let μ be the pushforward of the Bernoulli($\frac{1}{2}$) measure from $2^{\mathbb{N}}$ to $\mathbb{R}$ via the usual homeomorphism $\phi : 2^{\mathbb{N}} \xrightarrow{\sim} C$ from $2^{\mathbb{N}}$ to the standard Cantor set $C \subseteq [0,1]$. Let f_μ be a distribution function of μ, i.e. $\mu((a,b]) = f_\mu(b) - f_\mu(a)$ for all reals $a < b$. Let λ denote the Lebesgue measure. Prove that f_μ is continuous and f_μ' exists and is equal to 0 λ-a.e. Nevertheless, $f_\mu(1) - f_\mu(0) = 1$. Thus the fundamental theorem of calculus fails for f_μ.
   Hint: Find the graph of f_μ on our course webpage.