Measure theory with ergodic horizons — HOMEWORK 13 — **Due: Jun 10**

1. **Conditional expectation.** Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space and $\mathcal{C} \subseteq \mathcal{B}$ be a sub-σ-algebra witnessing the σ-finiteness of μ, i.e. $X = \bigcup_{n \in \mathbb{N}} C_n$ where each $C_n \in \mathcal{C}$ and $\mu(C_n) < \infty$. Thus, the restriction $\nu := \mu|_{\mathcal{C}}$ is a σ-finite measure on the measurable space $(X, \mathcal{C})$.

   (a) Show that $\int_X g d\mu = \int_X g d\nu$ for each $\mathcal{C}$-measurable function $g : X \to \overline{\mathbb{R}}$ which is non-negative or μ-integrable. In particular, $\mathcal{C}$-measurable μ-integrable functions are ν-integrable.

   (b) Prove that for each $\mathcal{B}$-measurable $f \in L^1(\mu)$, there is a $\mathcal{C}$-measurable $\tilde{f} \in L^1(\mu)$ such that $\int_C f\, d\mu = \int_C \tilde{f}\, d\mu$ for each $C \in \mathcal{C}$. This function $\tilde{f}$ is unique up to a μ-null set (prove this as well) and it is called the **conditional expectation** of f with respect to the sub-σ-algebra $\mathcal{C}$. In particular, if f is already $\mathcal{C}$-measurable, then $\tilde{f} = f$ a.e.

   HINT: First suppose that $f \geqslant 0$, and consider the measure $\nu_f := \mu_f|_{\mathcal{C}}$ on $\mathcal{C}$, where $\mu_f(B) := \int_B f\, d\mu$. Observe that $\nu_f \ll \nu$ so $\frac{d\nu_f}{d\nu}$ exists.

   (c) Deduce that $\int f g d\mu = \int \tilde{f} g d\mu$ for each f as above and $\mathcal{C}$-measurable function $g : X \to \overline{\mathbb{R}}$ such that fg is μ-integrable.

   (d) To get a handle on conditional expectation, let $\mathcal{C}$ be the σ-algebra generated by a countable partition $\mathcal{P} \subseteq \mathcal{B}$ of X and compute $\tilde{f}$ explicitly in terms of f and $\mathcal{P}$.

   HINT: In this case, $\tilde{f}$ is a countable linear combination of indicator functions.

2. **Measure disintegration.** Let $(X, \mathcal{B})$ and $(Y, \mathcal{C})$ be measurable spaces, μ be a probability measure on $(X, \mathcal{B})$, and $\pi : X \to Y$ be a $(\mathcal{B}, \mathcal{C})$-measurable function. Let $P(X, \mathcal{B})$ denote the set of all probability measures on X.

   (a) Prove that $\pi_* \mu$ admits **measure disintegration**, i.e. a map $y \mapsto \mu_y : Y \to P(X, \mathcal{B})$ such that

      (i) for each $B \in \mathcal{B}$, the function $y \mapsto \mu_y(B)$ is $\mathcal{C}$-measurable and $\mu(B) = \int_Y \mu_y(B) d(\pi_* \mu)$,

      (ii) $\mu_y(C) = \mathbb{1}_C(y)$ for $\pi_* \mu$-a.e. $y \in Y$.

   HINT: Take the conditional expectation $\widetilde{\mathbb{1}_B}$ of $\mathbb{1}_B$ with respect to the sub-σ-algebra $\pi^{-1}(\mathcal{C})$ of $\mathcal{B}$. Set $\mu_y(B) := \widetilde{\mathbb{1}_B}(x)$ for any/some $x \in \pi^{-1}(y)$.

   (b) Show that measure disintegration is unique up to $\pi_* \mu$-null sets.

   (c) Deduce that if $\mathcal{C}$ admits a countable family that separates points (e.g. when $\mathcal{C}$ is standard Borel), then in fact:

      (ii') $\mu_y(\pi^{-1}(y)) = 1$ for $\pi_* \mu$-a.e. $y \in Y$.

3. Consider the space $\mathbb{R}^d$ with Lebesgue measure λ and let $r > 0$. Let A_r be the averaging operator on L^1 defined by $A_r f(x) := \frac{1}{\lambda(B_r(x))} \int_{B_r(x)} f \, d\lambda$, where $B_r(x)$ is the (open) ball of radius r centered at x in the d_∞ metric.

(a) Prove the local-global bridge lemma: $\int f \, d\lambda = \int A_r f \, d\lambda$ for all $f \in L^1$. In particular, A_r is an L^1**-contraction**, i.e. $\|A_r f\|_1 \leqslant \|f\|_1$ for all $f \in L^1$, and hence $A_r : L^1 \to L^1$.

(b) Prove that for each $f \in L^1$, the function $(r, x) \mapsto A_r f(x)$ is continuous as a function $(0, \infty) \times \mathbb{R}^d \to \mathbb{R}$, i.e. it is jointly continuous in (r, x).