Last update: 6:38pm, Feb 13.

Measure theory with ergodic horizons | HOMEWORK 1 | **Due: Feb 18**

1. Let (X, d) be a metric space. Prove:

   (a) Separability is hereditary for metric spaces, i.e. if X is separable, then every subspace $Y \subseteq X$ is also separable.

   CAUTION: This is not true for general topological spaces. Think of an example.

   (b) For any $Y \subseteq X$, its closure $\overline{Y}$ is equal to $\bigcap_{n \geqslant 1} B_{1/n}(Y)$, where

   $$B_r(Y) := \{x \in X : d(x, Y) < r\}$$

   and $d(x, Y) := \inf_{y \in Y} d(x, y)$. Conclude that every closed set is G_δ[1]; equivalently, every open set is F_σ[1].

2. Let A be a nonempty set (an alphabet) and consider the space $A^{\mathbb{N}}$ of infinite A-valued sequences, equipped with the metric d defined in class.

   (a) Prove that d is in fact an **ultrametric**, i.e. $d(x,z) \leqslant \max\{d(x,y), d(y,z)\}$ for each $x, y, z \in A^{\mathbb{N}}$.

   (b) Prove that the metric space $(A^{\mathbb{N}}, d)$ is complete.

   (c) Prove that $A^{\mathbb{N}}$ is compact if and only if A is finite. I encourage you to prove this using the open covers definition of compactness. (If you'd like a hint, please ask me.)

3. (a) Observe that in every metric space, the clopen sets form an algebra.

   (b) Prove that in $2^{\mathbb{N}}$, the clopen sets are exactly the finite disjoint unions of cylinders.

4. Prove that for metric spaces, separability is equivalent to second countability.

5. Let X be a set and $\mathcal{C} \subseteq \mathscr{P}(X)$. Prove:

   (a) $\langle \mathcal{C} \rangle = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$, where $\mathcal{C}_0 := \mathcal{C}$ and

   $$\mathcal{C}_{n+1} := \{\text{complements and finite unions of sets in } \mathcal{C}_n\}.$$

   (b) [*Optional*] $\langle \mathcal{C} \rangle_\sigma = \bigcup_{\alpha \in \omega_1} \mathcal{C}_\alpha$, where $\mathcal{C}_0 := \mathcal{C}$ and for $\alpha > 0$,

   $$\mathcal{C}_\alpha := \left\{\text{complements and finite unions of sets in } \bigcup_{\beta < \alpha} \mathcal{C}_\beta\right\}.$$

[1] A set is G_δ (resp. F_σ) if it is a countable intersection (resp. countable union) of open (resp. closed) sets.

6. Let X be a set and $\mathcal{C} \subseteq \mathscr{P}(X)$. Put $\neg\mathcal{C} := \{S^c \in \mathcal{C} : S \in \mathcal{C}\}$. Let $\mathcal{S} \subseteq \mathscr{P}(X)$ be the smallest collection of sets containing $\mathcal{C} \cup \neg\mathcal{C}$ and closed under countable unions and countable intersections. Prove that $\mathcal{S} = \langle \mathcal{C} \rangle_\sigma$.

Hint: To show $\mathcal{S} \supseteq \langle \mathcal{C} \rangle_\sigma$, we do something counter-intuitive: we define an even smaller collection $\mathcal{S}' := \{S \in \mathcal{S} : S \text{ and } S^c \text{ are in } \mathcal{S}\}$ and show that $\mathcal{S}'$ is already a σ-algebra containing $\mathcal{C}$.

7. (a) Prove that if $\mathcal{U}$ is a countable basis for a metric (more generally, topological) space X, then $\mathcal{B}(X) = \langle \mathcal{U} \rangle_\sigma$.

Remark: In fact, one can show that in a second countable metric (more generally, topological) space, every basis contains a countable subcollection that is still a basis, whence *every* basis generates the Borel σ-algebra.

(b) Prove that the following collections generate the Borel σ-algebra of $\mathbb{R}^d$:

(i) Balls with rational centers (i.e. in $\mathbb{Q}^d$) and rational radii.

(ii) Open boxes.

(iii) Closed boxes.