Last update: 10:40pm, Feb 20.

Measure theory with ergodic horizons | Homework 2 | **Due: Feb 25**

1. Let $\mathcal{A}$ denote the algebra of all finite unions of boxes in $\mathbb{R}^d$.

   (a) Finish the proof of Claim (b) for boxes in $\mathbb{R}^d$, namely: For any finite partitions $\mathcal{P}$ and $\mathcal{Q}$ of a set $A \in \mathcal{A}$ into boxes, we have
   $$\sum_{P \in \mathcal{P}} \tilde{\lambda}(P) = \sum_{Q \in \mathcal{Q}} \tilde{\lambda}(Q).$$

   (b) Deduce that λ is finitely additive on $\mathcal{A}$.

2. Let $\mathcal{A}$ denote the algebra of all finite unions of boxes in $\mathbb{R}^d$. Using the statement that λ is countably additive on bounded boxes, i.e. $\lambda(B) = \sum_{n \in \mathbb{N}} \lambda(B_n)$ for a bounded box $B \subseteq \mathbb{R}^d$ and a partition $\{B_n\}_{n \in \mathbb{N}}$ of B into boxes, finish the proof of countable additivity of λ on $\mathcal{A}$. More precisely:

   (i) Prove that $\lambda(B) = \sum_{n \in \mathbb{N}} \lambda(B_n)$ for an unbounded box $B \subseteq \mathbb{R}^d$ and a partition $\{B_n\}_{n \in \mathbb{N}}$ of B into boxes.

   Caution: An unbounded box has measure ∞ or 0.

   (ii) Finally conclude that $\lambda(A) = \sum_{n \in \mathbb{N}} \lambda(A_n)$ for a set $A \in \mathcal{A}$ and a partition $\{A_n\}_{n \in \mathbb{N}}$ of A into sets in $\mathcal{A}$.

3. Let X be a set and $\mathcal{A} \subseteq \mathscr{P}(X)$ be a collection of sets containing $\emptyset$ and covering X. Let $m : \mathcal{A} \to [0, \infty]$. Prove that the induced outer measure $m^* : \mathscr{P}(X) \to [0, \infty]$ is

   (a) monotone: $A \subseteq B$ implies $m^*(A) \leqslant m^*(B)$ for all $A, B \in \mathscr{P}(X)$.

   (b) countably subadditive*: $m^*(\bigcup_{n \in \mathbb{N}} B_n) \leqslant \sum_{n \in \mathbb{N}} m^*(B_n)$ for all $B_0, B_1, \ldots \in \mathscr{P}(X)$.

4. Prove the existence part of Carathéodory's theorem via Carathéodory's proof. Let μ be a premeasure on an algebra $\mathcal{A}$ and let μ^* be its outer measure. Let $\mathcal{M}$ be the collection of conservative sets, i.e. sets M such that $\mu^*(S) = \mu^*(M \cap S) + \mu^*(M^c \cap S)$ for all $S \subseteq X$. Prove:

   (a) $\mathcal{M} \supseteq \mathcal{A}$.

   (b) $\mathcal{M}$ is an algebra.

   (c) μ^* is countably additive on $\mathcal{M}$.

   Hint: It's enough to observe finite additivity.

   (d) Countable disjoint unions of sets in $\mathcal{M}$ don't butcher the sets in $\mathcal{A}$, i.e. if $M := \bigsqcup_{n \in \mathbb{N}} M_n$ with $M_n \in \mathcal{M}$ and $A \in \mathcal{A}$, then $\mu^*(A) = \mu^*(M \cap A) + \mu^*(M^c \cap A)$.

   Hint: First show that $\mu^*(A) \geqslant \mu^*((\bigsqcup_{n<N} M_n) \cap A) + \mu^*(M^c \cap A)$ then let $N \to \infty$.

(e) $\mathcal{M}$ is closed under countable disjoint unions, and is hence a σ-algebra.

5. A **translation-invariant** Borel measure on $\mathbb{R}^d$ is a measure μ defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ such that $\mu(\vec{x}+B) = \mu(B)$ for each point $\vec{x} \in \mathbb{R}^d$ and each Borel set $B \subseteq \mathbb{R}^d$. For example the Lebesgue measure is translation-invariant.

Prove that $\mathbb{R}^d$ does not admit a translation-invariant Borel *probability* measure μ. In fact, show that for any nonzero vector $\vec{x} \in \mathbb{R}^d$, there does not exist a Borel probability measure that is invariant under translation by $\vec{x}$.

Hint: Find a Borel set $W \subseteq \mathbb{R}^d$ such that $\mathbb{R}^d = \bigsqcup_{n\in\mathbb{Z}}(n\vec{x}+W)$.