Last update: 2:46am, Feb 26.

Measure theory with ergodic horizons | Homework 3 | **Due: Mar 4**

1. Let μ be a finite premeasure on an algebra $\mathcal{A}$ on a set X and denote $\mathcal{B} := \langle A \rangle_\sigma$. Recall the pseudo-metric d on $\mathcal{P}(X)$ defined by $d(A, B) := \mu^*(A, B)$.

(a) Let $\mathcal{M}$ be as in Carathéodory's proof, i.e. it is the collection of all sets $M \subseteq X$ such that $\mu^*(S) = \mu^*(M \cap S) + \mu^*(M^c \cap S)$ for all $S \subseteq X$. Prove that $\mathcal{M}$ is exactly the collection of sets $M \subseteq X$ with $d(M, B) = 0$ for some $B \in \mathcal{B}$.

Hint: To show that $M \in \mathcal{M}$ implies $d(M, B) = 0$, use the definition of $\mu^*(M)$ to define B as a countable intersection of countable unions of sets in $\mathcal{A}$.

Remark: Part (a) is also true for σ-finite premeasures, but the counter-example for uniqueness of extension given in class shows that part (a) can fail for non-σ-finite premeasures.

(b) Let $\mathcal{M}$ be as in Tao's proof, i.e. it is the closure of $\mathcal{A}$ with respect to d. Prove that $\mathcal{M}$ is exactly the collection of sets $M \subseteq X$ with $d(M, B) = 0$ for some $B \in \mathcal{B}$.

Remark: In particular, the σ-algebras $\mathcal{M}$ in both Carathéodory's and Tao's proofs are the same for finite premeasures.

2. In a metric space X, a set $C \subseteq X$ is called a **Cantor set** if it is homeomorphic to the Cantor space $2^{\mathbb{N}}$, i.e. there is a continuous bijection $f : 2^{\mathbb{N}} \to C$ whose inverse is also continuous[1]. In particular, C is a compact subset of X of cardinality continuum. See my short note on Cantor sets [pdf] to learn more about them.

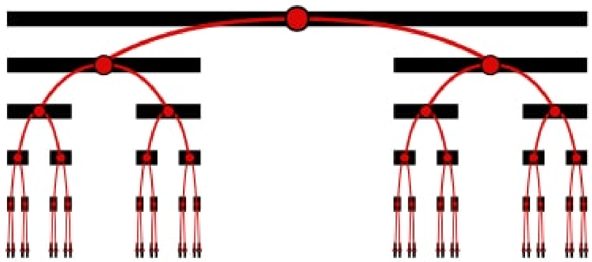
(a) In a connected[2] metric space X (such as $\mathbb{R}^d$), prove every Cantor set has is closed and has empty interior; in particular, it is nowhere dense.

Hint: If a Cantor set C has nonempty interior U, then there is a further set $V \subseteq U$ such that V is clopen relative to C, i.e. there is an open set $O \subseteq X$ and a closed set $K \subseteq X$ such that $O \cap C = V = K \cap C$. Show that V is clopen in X.

(b) The standard Cantor set in $[0, 1]$ is the set $C := \bigcap_{n \in \mathbb{N}} \bigcup_{s \in 2^n} C_s$, where each C_s is a closed interval defined inductively by setting $C_\emptyset := [0, 1]$ and letting C_{s0} and C_{s1} be the bottom third and top third closed subintervals of the closed interval C_s for each $s \in 2^{<\mathbb{N}}$. In particular, $C_0 := [0, \frac{1}{3}]$ and $C_1 := [\frac{2}{3}, 1]$, $C_{00} := [0, \frac{1}{3^2}]$, $C_{01} := [\frac{2}{3^2}, \frac{1}{3}]$, $C_{10} := [\frac{2}{3}, \frac{7}{3^2}]$, and $C_{11} := [\frac{8}{3^2}, 1]$, etc. Prove that C is indeed a Cantor set and that C is Lebesgue null, i.e. has Lebesgue measure 0.

[1] The requirement that f^{-1} is continuous is redundant, it is automatically continuous because $2^{\mathbb{N}}$ is compact and X is Hausdorff.

[2] A metric space is **connected** if it has no clopen sets, other than the whole space and $\emptyset$.



Hint: Let U_s be the middle third open interval in C_s, i.e. $U_s := C_s \setminus (C_{s0} \cup C_{s1})$, and calculate $\sum_{s \in 2^{<\mathbb{N}}} \mathrm{lh}(U_s)$.

(c) Define a Cantor subset of $[0,1]$ of positive Lebesgue measure.

Hint: Note that in the standard Cantor set, the open interval $U_s := C_s \setminus (C_{s0} \sqcup C_{s1})$ that we remove from C_s occupies 1/3 of C_s regardless of s. Change the construction so that U_s has length p_n, where $n := \mathrm{lh}(s)$ and the sequence (p_n) goes to 0 fast enough to guarantee $\sum_{n \in \mathbb{N}} 2^n p_n < 1$.

3. Let A be a countable nonempty set and let ν be a probability measure on $\mathcal{P}(A)$ such that $\nu(a) > 0$ for each $a \in A$; e.g. $A := 3 := \{0,1,2\}$ and $\nu(0) := 1/2$, $\nu(1) := 1/3$, and $\nu(2) := 1/6$. The **Bernoulli measure with base** ν is the measure μ on the Borel σ-algebra of $A^{\mathbb{N}}$ defined via extending the following premeasure μ on the algebra generated by the cylinders:

$$\mu([w]) := \nu(w_0) \cdot \nu(w_1) \cdots \nu(w_{n-1}),$$

where $n := \mathrm{lh}(w)$. The Bernoulli(p) measures on $2^{\mathbb{N}}$ discussed in class are a special case of this with $A = 2 = \{0,1\}$ and $\nu(0) = 1-p$, $\nu(1) = p$, and the constructions are the same. Fix $a \in A$ and prove that $(A \setminus \{a\})^{\mathbb{N}}$ is null with respect to μ, i.e. $\mu((A \setminus \{a\})^{\mathbb{N}}) = 0$.

Remark: If A has at least three elements, then $(A \setminus \{a\})^{\mathbb{N}}$ has cardinality continuum.