Last update: 2:51pm, Mar 7.

Measure theory with ergodic horizons

Homework 4

Due: Mar 11

1. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and let d_μ denote the usual pseudo-metric on Meas_μ defined by $d_\mu(A, B) := \mu(A \triangle B)$.

(a) Prove that d_μ is a complete pseudo-metric on Meas_μ, i.e. every d_μ-Cauchy sequence (M_n) converges in d_μ to a set $M \in \mathrm{Meas}_\mu$.

Hint: Note that it is enough to show that a subsequence of (M_n) converges, and pass to a subsequence (M_{n_k}) so that $d_\mu(M_{n_k}, M_{n_{k+1}}) \leqslant 2^{-k}$. Then the error sets $M_{n_k} \triangle M_{n_{k+1}}$ have summable measures, i.e. $\sum_{k \in \mathbb{N}} \mu(M_{n_k} \triangle M_{n_{k+1}}) < \infty$. Thus, almost every point of X is either eventually in M_{n_k} or eventually outside of M_{n_k}.

(b) If $\mathcal{B}$ admits a countable generating subalgebra $\mathcal{A}$, then the pseudo-metric space $(\mathrm{Meas}_\mu, d_\mu)$ is separable, hence a Polish pseudo-metric space.

Hint: The restriction of μ on $\mathcal{A}$ is a premeasure on $\mathcal{A}$, so Tao's proof applies.

2. Let $\mathbb{E}_0$ be the equivalence relation on $2^{\mathbb{N}}$ of eventual equality, i.e.

$$x \mathbb{E}_0 y :\Leftrightarrow \forall^\infty n \; x(n) = y(n),$$

where $\forall^\infty n$ means for all large enough n. For each n, let $\sigma_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the n^{th} bit flip map, i.e. $\sigma_n(x)$ is the same as x except that its n^{th} coordinate is equal to $1 - x(n)$. Let Γ be the group generated by all the σ_n. (Actually, Γ is isomorphic to $\oplus_{n \in \mathbb{N}} \mathbb{Z}/2\mathbb{Z}$.) Then Γ naturally acts on $2^{\mathbb{N}}$.

(a) Realize that the orbit equivalence relation of this action is exactly $\mathbb{E}_0$.

(b) Note that the Bernoulli(1/2) measure is invariant under this action, i.e. for any $\mu_{1/2}$-measurable set $A \subseteq 2^{\mathbb{N}}$ and $\gamma \in \Gamma$, we have $\mu_{1/2}(\gamma A) = \mu_{1/2}(A)$.

(c) Prove that every transversal for $\mathbb{E}_0$ is not $\mu_{1/2}$-measurable.

(d) [*Optional*] Prove that for every $p \in (0, 1)$, every transversal for $\mathbb{E}_0$ is not μ_p-measurable.

3. Let $(X, \mathcal{B}, \mu)$ be a measure space and recall that for a sequence (A_n) of measurable sets,

$$\limsup_n A_n := \{x \in X : \exists^\infty n \; x \in A_n\}.$$

(a) Prove the following more general (and somehow less useful) version of the Borel–Cantelli lemma: If $\mu(X) < \infty$, then $\mu(\limsup_n A_n) = \lim_N \mu(\bigcup_{n \geqslant N} A_n)$.

(b) From this version of the Borel–Cantelli lemma, deduce the versions (a) and (b) discussed in class.

4. We say that a real $r \in \mathbb{R}$ admits a **sequence of good rational approximations of exponent** $\alpha > 0$ if there are infinitely many pairs $(p,q) \in \mathbb{Z} \times \mathbb{N}^+$ such that

$$|r - \frac{p}{q}| < \frac{1}{q^\alpha}.$$

Dirichlet's approximation theorem (or rather its immediate consequence) states that every real admits a sequence of good rational approximations of exponent 2.

Prove however that for any $\varepsilon > 0$, almost no real admits a sequence of good rational approximations of exponent $2+\varepsilon$, i.e., the set B of all $r \in \mathbb{R}$ that admit a sequence of good rational approximations of exponent $2+\varepsilon$ is null (with respect to Lebesgue measure).

Hint: First note that it is enough to prove the statement for $[0,1)$ instead of $\mathbb{R}$. Next, express B in terms of the sets

$$A_{p,q} := \left\{ r \in \mathbb{R} : |r - \frac{p}{q}| < \frac{1}{q^{2+\varepsilon}} \right\}$$

where $p, q \in \mathbb{N}^+$ and $p < q$. Finally, what is the measure of $A_{p,q}$?