Measure theory with ergodic horizons | Homework 6 | **Due: Apr 8**

1. Follow the steps below to prove the **Steinhaus theorem**: For every Lebesgue measurable non-null set $A \subseteq \mathbb{R}^d$, the difference set $A - A := \{a_0 - a_1 : a_0, a_1 \in A\}$ contains an open neighbourhood of $\vec{0}$.

Tip: For simplicity of thought and pictures, only think of $d = 1$ and replace boxes with intervals.

(i) Check that for all sets $U, V \subseteq \mathbb{R}^d$, we have $U \subseteq V - V$ if and only if $(V + u) \cap V \neq \emptyset$ for each $u \in U$.

(ii) Let B be a nonempty bounded open box whose at least $(1 - 1/2^{d+2})100\%$ is A. Let b_0 be the midpoint of B and put $U := B - b_0$, so U is an open box centered at $\vec{0}$. Show that for each $u \in U$, the intersection $B_u := B + u \cap B$ is a box whose each dimension is at least half of that of B, so B_u occupies at least $(1/2^d)100\%$ of B and of $B + u$.

(iii) Conclude that at least 75% of B_u is A while at least 75% of B_u is $A + u$, so $A + u \cap A \neq \emptyset$ for each $u \in U$, hence $U \subseteq A - A$.

Remark: The exact same theorem holds for the Bernoulli(1/2) measure on $2^{\mathbb{N}}$ identified with the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$. As usual, the proof is easier than for the Lebesgue measure.

2. Use the 99% lemma to prove that the equivalence relation $\mathbb{E}_0$ on $2^{\mathbb{N}}$ is $\mu_{1/2}$-ergodic.

Remark: $\mathbb{E}_0$ is actually μ_p-ergodic for all $p \in (0,1)$, although the action of the group $\oplus_{n\in\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$ on $2^{\mathbb{N}}$ that induces $\mathbb{E}_0$ does not preserve μ_p for $p \neq 1/2$. Try proving this, thinking about whether it is *really* used in your argument for $\mu_{1/2}$ that the action of the group $\oplus_{n\in\mathbb{N}}\mathbb{Z}/2\mathbb{Z}$ preserves $\mu_{1/2}$.

3. Let Γ be a countable group and $(X, \mathcal{B}, \mu)$ be nonzero atomless measure space. Let $\alpha : \Gamma \curvearrowright X$ be an action of Γ on X mapping sets in $\mathcal{B}$ to sets in $\mathcal{B}$, i.e. $\gamma \cdot B \in \mathcal{B}$ for each $\gamma \in \Gamma$ and $B \in \mathcal{B}$. Suppose that this action is **null-preserving**, i.e. $\gamma \cdot B$ is null if and only if B is null for each $\gamma \in \Gamma$ and $B \in \mathcal{B}$. Prove that if this action is *ergodic*, then the orbit equivalence relation E_α does not admit any μ-measurable transversal.

Hint: For any subset $B \subseteq X$, the set $[B]_{E_\alpha} := \bigcup_{\gamma\in\Gamma} \gamma \cdot B$ is the least E_α-invariant set containing B.

4. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_d$ be σ-algebras on sets $X_1, X_2, \ldots, X_d$, and denote by $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \cdots \otimes \mathcal{A}_d$ the σ-algebra on $X_1 \times X_2 \times \cdots \times X_d$ generated by the sets of the form $A_1 \times A_2 \times \cdots \times A_d$.

(a) Prove that for all second countable topological spaces $X_1, X_2, \ldots, X_d$,

$$\mathcal{B}(\textstyle\prod_{i=1}^d X_i) = \bigotimes_{i=1}^d \mathcal{B}(X_i).$$

(Here $\prod_{i=1}^d X_i$ is the product topology.) In particular, $\mathcal{B}(\mathbb{R}^d) = \bigotimes_{i=1}^d \mathcal{B}(\mathbb{R})$.

(b) Let $(X, \mathcal{A})$ and $(Y_i, \mathcal{B}_i)$, $i = 1, 2$, be measurable spaces, i.e., sets equipped with σ-algebras. Prove that for $(\mathcal{A}, \mathcal{B}_i)$-measurable functions $f_i : X \to Y_i$, the function $(f_1, f_2) : X \to Y_1 \times Y_2$ defined by $x \mapsto (f_1(x), f_2(x))$ is $(\mathcal{A}, \mathcal{B}_1 \otimes \mathcal{B}_2)$-measurable.

(c) Now let (X, μ) be a measure space[1] and conclude that if $f_1, f_2 : X \to \mathbb{R}$ are μ-measurable and $g : \mathbb{R}^2 \to \mathbb{R}$ is Borel, then $g(f_1, f_2) : X \to \mathbb{R}$ is μ-measurable. In particular, $f_1 + f_2$ and $f_1 \cdot f_2$ are μ-measurable.

5. For an equivalence relation E on a set X, a **selector** is a function $s : X \to X$ that picks a point from each E-class, more precisely, $s(x) E x$ and $x E y \Leftrightarrow s(x) = s(y)$ for all $x, y \in X$. By a selector/transversal for a group action, we mean that for its orbit equivalence relation.

Let Γ be a countable group and (X, μ) be a second countable metric space equipped with a Borel measure μ. Let $\alpha : \Gamma \curvearrowright X$ be a **Borel action** of Γ, i.e. each group element $\gamma \in \Gamma$ acts as a Borel function from X to X.

Prove that α admits a Borel transversal if and only if it admits a Borel selector.

Hint: For $\Leftarrow$, note that a selector is an idempotent (i.e. $s^2 = s$), so $s(X) = \{x \in X : s(x) = x\}$ and note that the diagonal $\{(x, x) : x \in X\}$ is in $\mathcal{B}(X) \otimes \mathcal{B}(X)$ by Question 4(a).

6. Follow the steps below to build an example of a Borel function $f : [0,1] \to [0,1]$ (in fact, a homeomorphism[2]) and a Lebesgue measurable function $g : [0,1] \to [0,1]$ such that the composition $g \circ f$ is **not** Lebesgue measurable.

(i) Prove that every Lebesgue measurable set A of positive measure contains a non-measurable subset.

Hint: Any transversal of $E_V|_A$ is non-measurable, and the proof is the same as for $A := [0,1]$ done in class after restricting A to a set of finite positive measure.

(ii) Let C_0 and C_+ be Cantor sets contained in $(0,1)$ where C_0 is Lebesgue null, while C_+ has positive Lebesgue measure. For convenience, construct C_0 and C_+ by removing middle open intervals at every step. Then there is a homeomorphism $f : [0,1] \to [0,1]$ such that $f(C_+) = C_0$ and $f([0,1] \setminus C_+^c) \subseteq [0,1] \setminus C_0^c$.

(iii) Let $Y \subseteq C_+$ be a non-measurable set and put $g := \mathbb{1}_{f(Y)}$. Observe that g is Lebesgue measurable, however $g \circ f$ is not.

[1] We suppress the σ-algebra from the notation if it is not important.

[2] A **homeomorphism** is a continuous bijection so that its inverse is also continuous.