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Measure theory with ergodic horizons | HOMEWORK 7 | **Due: Apr 15**

1. Let (X, μ) be a measure space[1] and let $f_n : X \to \mathbb{R}$ be a sequence of measurable functions. Prove that $\limsup_{n\to\infty} f_n$ and $\liminf_{n\to\infty} f_n$ are both μ-measurable. In particular, if $\lim_{n\to\infty} f_n$ exists, then it is μ-measurable. (We have proved this last statement in class for an arbitrary separable metric space in place of $\mathbb{R}$.)

2. Prove that universally measurable functions are closed under compositions. More precisely, if X, Y, Z are topological spaces, and $f : X \to Y$ and $g : Y \to Z$ are universally measurable functions, then $g \circ f : X \to Z$ is universally measurable.

3. Learn the proof of the Cantor–Schröder–Bernstein theorem from this note.

4. Prove the (following) measure isomorphism theorem for standard infinite measure spaces using the measure isomorphism theorem for standard probability spaces.

   Theorem (Measure Isomorphism)**.** *Every atomless standard infinite measure space* [2] $(X, \mathcal{B}, \mu)$ *is measure-isomorphic to* $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$*. Moreover, there is a Borel isomorphism* $f : X \to \mathbb{R}$ *such that* $f_* \mu = \lambda$*.*

   REMARK: Because I stated measure isomorphism for standard probability spaces with the closed interval $[0, 1]$, you may need to fiddle with a countable set to get the "moreover" part of the theorem. It's not important, so you may ignore the "moreover" part.

[1] We omit the σ-algebra from the notation when we only use the σ-algebra of measurable sets.

[2] Recall that $(X, \mathcal{B}, \mu)$ being a **standard measure space** means that X is a Polish space, $\mathcal{B}$ is its Borel σ-algebra, and μ is a σ-*finite* Borel measure.