Measure theory with ergodic horizons | Homework 8 | **Due: Apr 22**

1. Let μ be a locally finite Borel measure on $\mathbb{R}$. Recall that we call $f : \mathbb{R} \to \mathbb{R}$ a **distribution** of μ if $\mu\big((a,b]\big) = f(b) - f(a)$ for all reals $a \leqslant b$. Let f be a distribution of μ.
   (a) Prove that $\lim_{x \nearrow a^-} f(x) + \mu(\{a\}) = f(a)$.
   (b) Conclude that if μ is atomless, then f is continuous.

2. Let (X, μ) be a measure space. Prove that the integral of simple functions is well-defined, i.e. for all μ-measurable sets $A_i, B_j \subseteq X$ and $a_i, b_j \in \mathbb{R}$,
$$\sum_{i<n} a_i \mathbb{1}_{A_i} = \sum_{j<m} b_j \mathbb{1}_{B_j} \text{ implies } \sum_{i<n} a_i \mu(A_i) = \sum_{j<m} b_j \mu(B_j).$$

3. Let (X, μ) be a measure space. Prove that a μ-measurable function $f : X \to \mathbb{R}$ is simple if and only if its image $f(X)$ is finite.

4. Let $(X, \mathcal{B}, \mu)$ be a probability space and let $T : X \to X$ be a $(\mathcal{B}, \mathcal{B})$-measurable transformation (not necessarily one-to-one). Suppose T preserves μ, i.e. $T_* \mu = \mu$; in other words, $\mu(T^{(B)}) = \mu(B)$ for each $B \in \mathcal{B}$.
   (a) A set $W \subseteq X$ is called T-**wandering** if the sets $T^{-n}(W)$ are pairwise disjoint for $n \in \mathbb{N}$, i.e. $T^{-n}(W) \cap T^{-m}(W) = \emptyset$ for all distinct $n, m \in \mathbb{N}$. Prove that every T-wandering set in $\mathcal{B}$ is null.
   (b) Call a set $B \subseteq X$ is called T-**recurrent** if every $x \in B$ admits an $n \in \mathbb{N}^+$ with $T^n(x) \in B$. Prove the following:

   Theorem (Poincaré recurrence)**.** *Every set $B \in \mathcal{B}$ is T-recurrent a.e., more precisely, there is a T-recurrent set $B' \subseteq B$ with $B' =_\mu B$.*

   Hint: B' is the set of points of B that "see" infinitely many points of B in front of them.