Last update: 5:45pm, Apr 27.

Measure theory with ergodic horizons

Homework 9

Due: Apr 27

1. (a) Let $(X,\mathcal{B})$ and $(Y,\mathcal{C})$ be measurable spaces and let $T : X \to Y$ be a $(\mathcal{B},\mathcal{C})$-measurable map. Prove the **change of variable formula**: for each measure μ on $\mathcal{B}$ and a measurable $f \in L^1(Y,\mathcal{C},T_*\mu)$,

$$\int_X (f \circ T)\, d\mu = \int_Y f\, d(T_*\mu).$$

(b) As an application, let $T_A : \mathbb{R}^d \to \mathbb{R}^d$ be the linear transformation given by a $d \times d$ invertible matrix A, i.e. $Tx := Ax$. Let λ denote the Lebesgue measure on $\mathbb{R}^d$ and prove that $T_*\lambda = |\det A|^{-1}\lambda$, to conclude that

$$\int (f \circ T_A)\, d\lambda = \int f |\det A|^{-1}\, d\lambda.$$

Remark: You may use without proof that when $e_1, \dots, e_d$ is the standard basis for $\mathbb{R}^d$, the value $|\det A|$ is the "volume" (i.e. the Lebesgue measure) of the paralellepiped on $Ae_1, \dots, Ae_d$.

(c) As another application, prove the following simple statement, which I call the local-global bridge lemma.

Lemma (Local-global bridge). *Let $T : X \to X$ be a measure-preserving $(\mathcal{B},\mathcal{B})$-measurable transformation (not necessarily injective) on a measure space $(X,\mathcal{B},\mu)$. Then for each $f \in L^1(X,\mu)$ and $n \in \mathbb{N}$,*

$$\int f\, d\mu = \int A_n f\, d\mu,$$

where $A_n f(x)$ is the average of f over the set $\{x, Tx, T^2x, \dots, T^n x\}$, i.e.

$$A_n f := \tfrac{1}{n+1} \sum_{i=0}^{n} f \circ T^i.$$

2. Let (X,μ) be a σ-finite measure space.

(a) Prove that the simple functions are dense in $L^1(X,\mu)$.

(b) Suppose that Meas_μ is countably generated mod μ; this means that there is a countable family $\mathcal{F}$ of μ-measurable sets such that for each μ-measurable set $M \subseteq X$ there is a set $B \in \langle \mathcal{F} \rangle_\sigma$ with $M =_\mu B$. Then $L^1(X,\mu)$ is separable, i.e. admits a countable dense subset.