THE CANTOR–SCHRÖDER–BERNSTEIN THEOREM

ANUSH TSERUNYAN

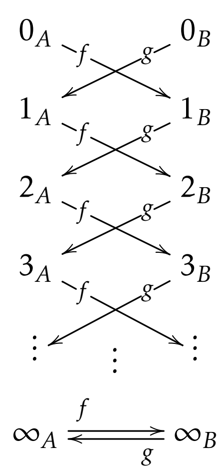
The following is called the Cantor–Schröder–Bernstein or just Schöder–Bernstein theorem [Wikipedia]. It was first announced by Cantor without proof in 1987, then proved by Dedekind in the same year without Axiom of Choice) but not published, then Schröder announced a proof in 1896, but a mistake was later found. In 1897 a 19-year old student Bernstein presented a correct proof. In the same year Dedekind gives another proof of this theorem. Thus in all fairness, this should be called the Cantor–Dedekind–Bernstein theorem, but c'est la vie mathématique.

Theorem (Cantor–Dedekind–Bernstein)**.** *For sets A, B, if $A \hookrightarrow B$ and $B \hookrightarrow A$, then $A \xrightarrow{\sim} B$.*

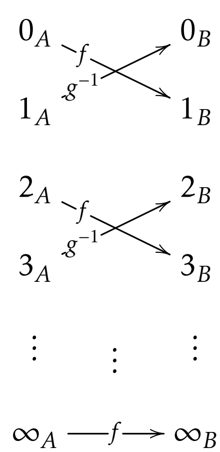
Prototypical example. Let $A := B := \mathbb{N} \cup \{\infty\}$, $f(n) := g(n) := n+1$, for $n \in \mathbb{N}$, and $f(\infty) := g(\infty) := \infty$. To distinguish the two copies of $\mathbb{N} \cup \{\infty\}$, denote the elements of A and B by n_A and n_B, respectively. We define a bijection $h : A \to B$ as follows: for $a \in A$,

$$h(a) := \begin{cases} f(a) & \text{if } a = 2n \text{ or } a = \infty \\ g^{-1}(a) & \text{if } a = 2n+1. \end{cases}$$

Schematically, from



we obtain



Proof. Let $f : A \to B$ and $g : B \to A$ be injections. We prove by reducing this to the prototypical example as follows: we will obtain partitions

$$A = \bigsqcup_{n \in \mathbb{N}} A_n \sqcup A_\infty \text{ and } B = \bigsqcup_{n \in \mathbb{N}} B_n \sqcup B_\infty$$

such that $f[A_n] = B_{n+1}$ and $g[B_n] = A_{n+1}$ for each $n \in \mathbb{N}$, as well as $f[A_\infty] = B_\infty$, so we define a bijection $h : A \to B$ just like in the prototypical example, namely, for $a \in A$,

$$h(a) := \begin{cases} f(a) & \text{if } a \in A_{2n} \text{ or } a \in A_\infty \\ g^{-1}(a) & \text{if } a \in A_{2n+1} \end{cases}$$

Date: April 8, 2025.

and have basically the same picture:

$$\text{from} \quad \begin{tikzcd}
A_0 \arrow[dr, Rightarrow, "f"] & & B_0 \arrow[dl, Rightarrow, "g"'] \\
A_1 \arrow[dr, Rightarrow, "f"] & & B_1 \arrow[dl, Rightarrow, "g"'] \\
A_2 \arrow[dr, Rightarrow, "f"] & & B_2 \arrow[dl, Rightarrow, "g"'] \\
A_3 \arrow[dr, Rightarrow, "f"] & & B_3 \arrow[dl, Rightarrow, "g"'] \\
\vdots & \vdots & \vdots \\
A_\infty \arrow[rr, Rightarrow, shift left, "f"] & & B_\infty \arrow[ll, Rightarrow, shift left, "g"]
\end{tikzcd}
\quad \text{obtain} \quad
\begin{tikzcd}
A_0 \arrow[dr, Rightarrow, "f"] & & B_0 \\
A_1 \arrow[ur, Rightarrow, "g^{-1}"] & & B_1 \\
A_2 \arrow[dr, Rightarrow, "f"] & & B_2 \\
A_3 \arrow[ur, Rightarrow, "g^{-1}"] & & B_3 \\
\vdots & \vdots & \vdots \\
A_\infty \arrow[rr, Rightarrow, "f"] & & B_\infty
\end{tikzcd}$$

To carve out such partitions, we define decreasing sequences $(\tilde{A}_n)_{n\in\mathbb{N}}$ and $(\tilde{B}_n)_{n\in\mathbb{N}}$ by

$$\tilde{A}_0 \coloneqq A \text{ and } \tilde{B}_0 \coloneqq B$$

$$\tilde{A}_{n+1} \coloneqq g[\tilde{B}_n] \text{ and } \tilde{B}_{n+1} \coloneqq f[\tilde{A}_n].$$

Now take $A_n \coloneqq \tilde{A}_n \setminus \tilde{A}_{n+1}$, $B_n \coloneqq \tilde{B}_n \setminus \tilde{B}_{n+1}$, as well as

$$A_\infty \coloneqq \bigcap_{n\in\mathbb{N}} \tilde{A}_n \text{ and } B_\infty \coloneqq \bigcap_{n\in\mathbb{N}} \tilde{B}_n.$$

It remains to verify that the partitions $A = \bigsqcup_{n\in\mathbb{N}} A_n \sqcup A_\infty$ and $B = \bigsqcup_{n\in\mathbb{N}} B_n \sqcup B_\infty$ are as desired. The injectivity of f implies that f-image commutes with set-subtraction and intersections, so we have $f[A_n] = f\left[\tilde{A}_n \setminus \tilde{A}_{n+1}\right] = f\left[\tilde{A}_n\right] \setminus f\left[\tilde{A}_{n+1}\right] = \tilde{B}_{n+1} \setminus \tilde{B}_{n+2} = B_{n+1}$ and

$$f[A_\infty] = f\left[\bigcap_{n\in\mathbb{N}} \tilde{A}_n\right] = \bigcap_{n\in\mathbb{N}} f\left[\tilde{A}_n\right] = \bigcap_{n\in\mathbb{N}} \tilde{B}_{n+1} = \bigcap_{n\in\mathbb{N}} \tilde{B}_n = B_\infty.$$

Similarly, we have the analogous statements for g, which finishes the proof. $\square$

Department of Mathematics and Statistics, McGill University, Montréal, QC, Canada
Email address: anush.tserunyan@mcgill.ca