189-570A: Higher Algebra
Assignment 4

Due: Monday, December 4

1. Let $K_1 \subset K_2 \subset K_3$ be a sequence of field extensions. Suppose that K_3 has finite transcendence degree over K_1. Show that the same is then true for K_2/K_1 and for K_3/K_2, and that

$$tr.deg(K_3/K_1) = tr.deg(K_3/K_2) + tr.deg(K_2/K_1).$$

2. Let S be a subring of a commutative ring R. An element $a \in R$ is said to be *integral* over S if it satisfies a *monic* polynomial with coefficients in S. Show that a is integral over S if and only if the subring $S[a]$ generated by a over S is finitely generated as an S-module.

3. Keeping the notation of the previous exercise, show that the set of elements in R that are integral over S is a subring of R.

4. For each of the following ideals I in $\mathbf{R}[x, y]$, describe the set $V(I) \cap \mathbf{R}^2$, and write $V(I)$ as a union of irredicible algebraic sets not contained in one another. Give the corresponding expression for I as an intersection of prime ideals.

a) $I = (x^2 + y^2 - 1, x^2 - y^2)$;
b) $I = ((y - 1)(y - x^2))$;
c) $I = (y - 1, y - x^2)$;
d) $I = (y(y - x^2))$;
e) $I = (y, y - x^2)$;
f) $I = (y^2 - x^3, x^5 - y^7)$

5. For each of the ideals of question 4, state which ideals are non-reduced, and describe the radical of I for these I.

6. Exercises 2, page 826 of Lang's Algebra (third edition).

7. Exercise 3, page 826 of Lang's Algebra (third edition).

8. Exercise 4, page 826 of Lang's Algebra (third edition).

9. Exercise 5, page 827 of Lang's Algebra (third edition).