189-456A: Abstract Algebra

Midterm Exam

Wednesday, October 25

The first four questions are worth 25 *points, and the last is worth* 10 *points of extra credit.*
The final grade will be out of 100.

1. Let $G = \{1, r, r^2, r^3, V, H, D_1, D_2\}$ be the dihedral group of order 8, where 1 is the identity transformation, r is the counterclockwise rotation by an angle of 90 degrees, and V, H, D_1 and D_2 are the reflections about the vertical, horizontal and two diagonal axes of symetry of the square, respectively.

Write down the conjugacy classes in G and use this to give a complete list of the normal subgroups of G. (Detailed calculations are not necessary to get full marks if your answer is right, but presenting a sound reasoning will mitigate the impact of a wrong answer.)

2. Write down the class equation for a finite group G (in the form that was used in the second proof given in class of the Sylow theorem, which involved the cardinality of center of G). Use this class equation to show that any group of cardinality p^n with p a prime and $n \geq 1$ has a non-trivial center.

3. Show that any finite group of cardinality 77 is abelian.

4. If n is an odd integer, show that the permutation groups S_n and S_{n-1} have the same Sylow 2-subgroups, and that the number of Sylow 2-subgroups of S_n is exactly n times the number of Sylow 2-subgroup of S_{n-1}.

The following question will count as extra credit (10 points) for those who do it successfully.

5. Let p be a prime number and let $n \geq 1$ be a positive integer. Let $G = \mathbf{Z}/p\mathbf{Z}$ be the cyclic group of cardinality p, and let X be the set of all functions from G to $\{1, 2, \ldots, n\}$, equipped with the G-action given by

$$(g \cdot f)(x) = f(g + x), \qquad \text{for all } g \in G, f \in X, x \in G.$$

What is the cardinality of X, and how many fixed points does G acting on X have? Use these two facts to prove Fermat's Little Theorem, which asserts that p always divides $n^p - n$ for any $n \geq 1$.