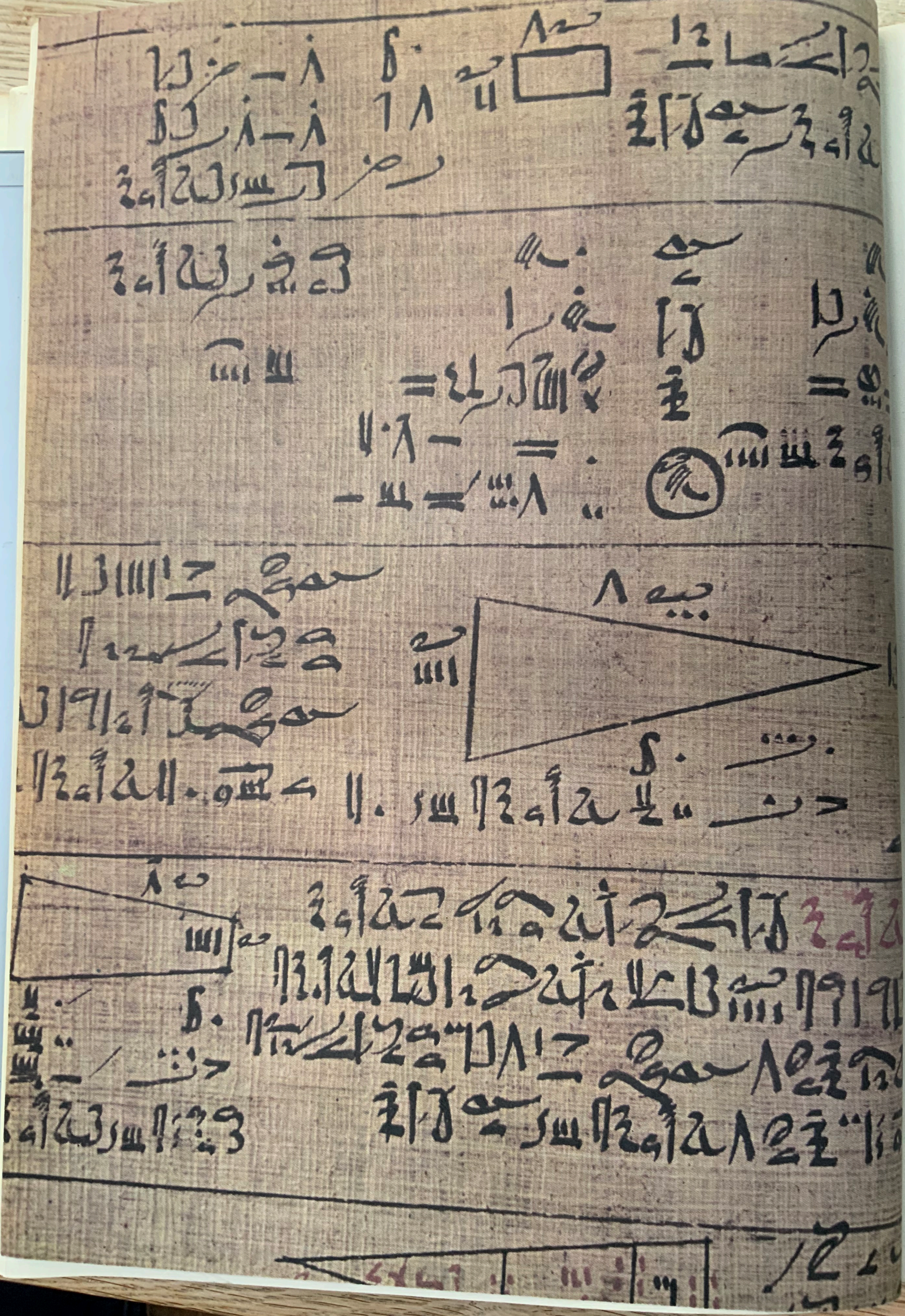

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3 MATHEMATICS IN THE MODERN WORLD

RICHARD COURANT • September 1964

The expanding role of mathematics in the modern world is vividly reflected in the proliferation of mathematicians. Since 1900 memberships in the several professional mathematical organizations in the U.S. have multiplied by an estimated 30 times. Today the number qualified by the doctorate stands at 4,800. During the past 25 years the number of mathematicians at work outside the universities in industry and Government has increased twelvefold. Activities of a more or less mathematical character now employ tens of thousands of workers at all levels of competence. In the colleges three times as many undergraduates were majoring in mathematics in 1962 as in 1956. Mathematics is no longer the preoccupation of an academic elite; it is a broad profession attracting talented men and women in increasing numbers. The scope of mathematical research and teaching has been greatly extended in the present period, and mathematical techniques have penetrated deep into fields outside the mathematical sciences such as physics, into new realms of technology, into the biological sciences and even into economics and the other social sciences. Computing machines and computing techniques have stimulated areas of research with obviously enormous and as yet only partly understood importance for mathematics itself and for all the sciences with inherent mathematical elements.

The contemporary role of mathematics is best appraised, however, by comparison with previous stages in its development. As recently as three centuries ago the main fabric of mathematical thought was supplied by geometry, inherited from the ancients and only meagerly augmented during the intervening 20 centuries. Then began a radical and rapid transformation of mathematics. The rigorous, axiomatic, deductive style of geometry yielded to inductive, intuitive insights, and purely geometric notions gave way to concepts of number and algebraic operations embodied in analytic geometry, the calculus and mechanics. It was the small intellectual aristocracy of the new mathematics that now spearheaded the forward thrust of science. By the time of the French Revolution the accumulated wealth of results and the demonstrated power of the mathematical sciences brought a widening of the narrow human basis of scientific activity, with the writing of textbooks to make the new mathematics more widely accessible, the systematic training of scientists and mathematicians in the universities and the opening up of new careers in the expansion of human knowledge.

The "classical" mathematics that had its beginnings in the 17th century retains its power and central position today. Some of the most fruitful work has come from the clarification and generalization of the two basic concepts of the calculus: that of function, which is concerned with the interdependence of two or more variables, and that of limit, which brings the intuitive notion of continuity within rigorous scrutiny. The concepts of mathematical analysis, including the theory of differential equations for one or more variables, which is an essential tool for dealing with rates of change, pervade the vastly extended territory of modern mathematics. That territory is surveyed in three articles in this volume (see "Number," page 89, "Algebra," page 102, and "Geometry," page 112) from three points of vantage—number, geometry and algebra—

SECTION OF ANCIENT EGYPTIAN PAPYRUS on the opposite page reflects one of the earliest applications of mathematics: the measurement of land. Called the Rhind Papyrus after its modern discoverer, A. Henry Rhind, the entire work is a handbook of practical problems compiled about 1550 B.C. by a scribe named A'h-mosě. The horizontal lines separate five of the problems, which read from right to left. At the top is a section of "an example of reckoning area" of "a rectangle of land khet 10 by khet 2." Second from the top is a calculation of the area of a "round field" with a perimeter of khet 9. Other examples shown in the section calculate the area of triangular and trapezoidal fields. The title page of the papyrus describes it as a guide to "accurate reckoning of entering into things, knowledge of existing things all." The main portion of the papyrus is now in the British Museum.

that offer perspectives familiar to the nonmathematician. As will be seen, geometry has had a most fruitful growth, liberated by the concepts of function and of the number continuum; its youngest offspring, topology and differential geometry, rank among the most active and "modern" branches of mathematics. The special field of probability deserves a chapter in itself because it has found such wide application in science and technology and because it gives mathematical expression to some of the deep unsolved problems in the philosophy of science.

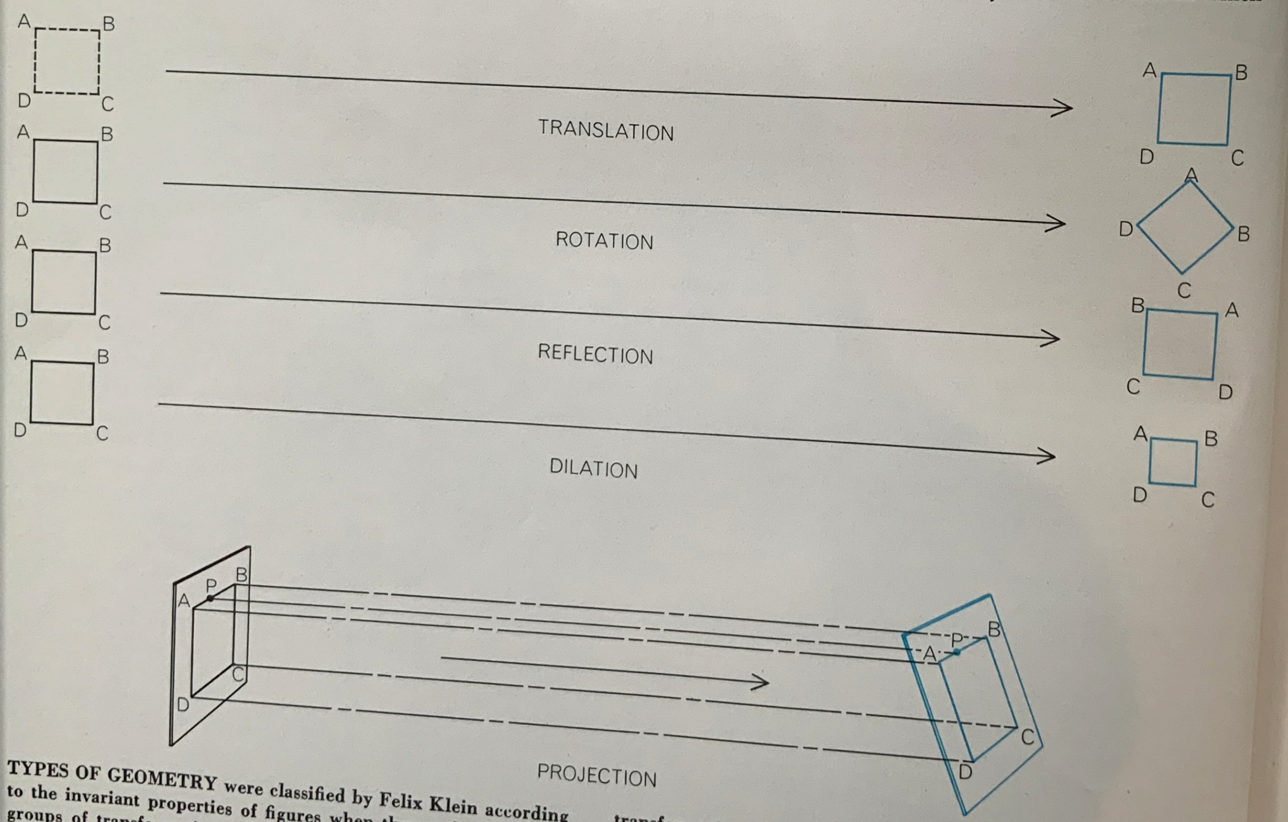
Mathematics today also reflects a vigorous trend, started early in the 19th century, toward solidification of the new conquests in the spirit of the mathematical rigor practiced by the ancients. This effort has inspired intense work on the foundations of mathematics, directed at clarifying the structure of mathematics and the meaning of "existence" for the objects of mathematical thought.

Inevitably the expansion of mathematics has enhanced immanent tendencies toward specialization and isolation; mathematics is threatened with a loss of unity and cohesion. Mutual understanding among representatives of different fields of mathematics has become difficult, and contact of mathematics with other sciences has been weakened. Yet remarkable advances continue to be won, mostly by young talent amply supported by a society that recognizes the increasing importance of mathematics. At the same time the growing volume of mathematical activity has led to a bewildering avalanche of publications, a multiplicity of meetings, administrative tangles and pressures of commercialism. It becomes the urgent duty of mathematicians, therefore, to meditate about the essence of mathematics, its motivations and goals and the ideas that must bind divergent interests together. For this purpose they can find no better occasion than the opportunity to explain their work to a wider public.

The question "What is mathematics?" cannot be answered meaningfully by philosophical generalities, semantic definitions or journalistic circumlocutions. Such characterizations also fail to do justice to music or painting. No one can form an appreciation of these arts without some experience with rhythm, harmony and structure, or with form, color and composition. For the appreciation of mathematics actual contact with its substance is even more necessary.

With this caution, some remarks of a general nature can nevertheless be made. As is so often said, mathematics aims at progressive abstraction, logically rigorous axiomatic deduction and ever wider generalization. Such a characterization states the truth but not the whole truth; it is one-sided, almost a caricature of the live reality. Mathematics, in the first place, has no monopoly on abstraction. The concepts of mass, velocity, force, voltage and current are all abstract idealizations of physical reality. Mathematical concepts such as point, space, number and function are only somewhat more strikingly abstract.

The model of rigorous axiomatic deduction for so long impressed on mathematics by Euclid's *Elements* constitutes the remarkably attractive form in which



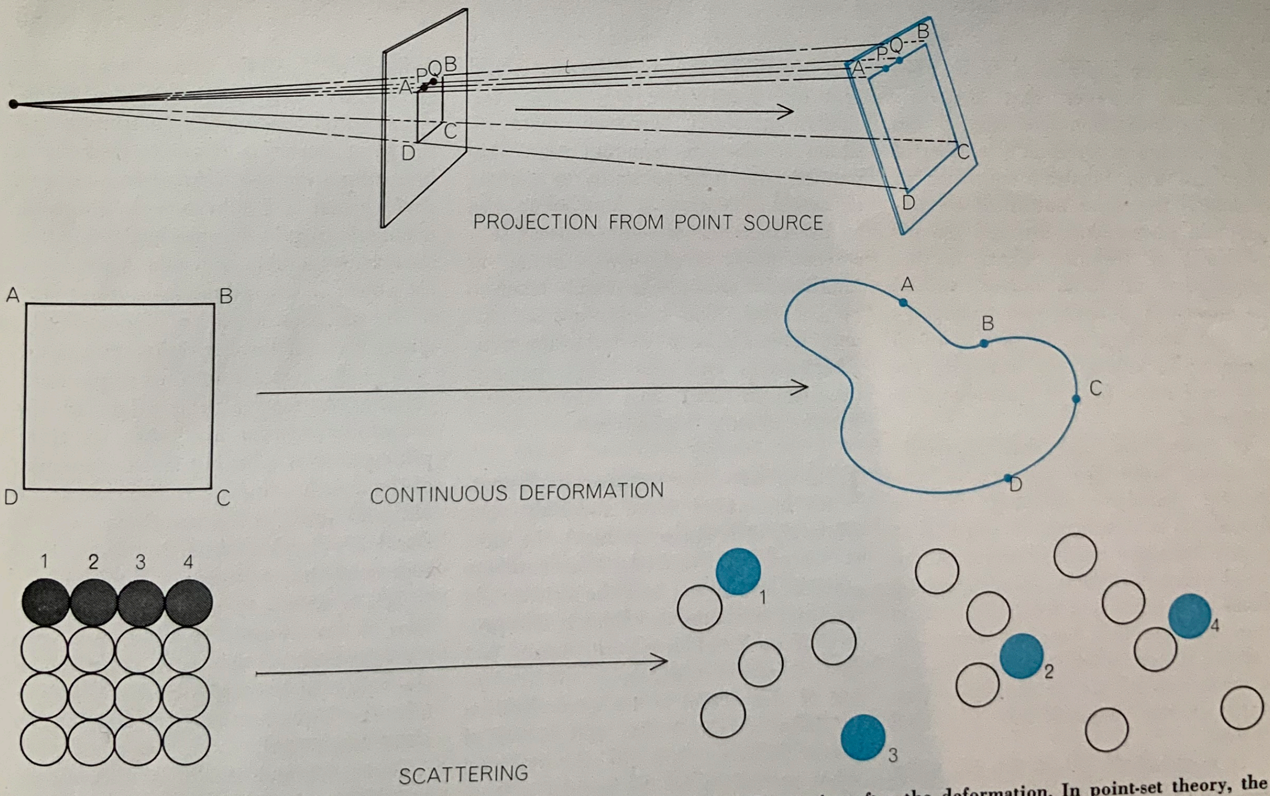
TYPES OF GEOMETRY were classified by Felix Klein according to the invariant properties of figures when they undergo various groups of transformations. Euclidean geometry is represented at top left as the study of properties such as "angle" that are retained when the square *ABCD* is translated, rotated, reflected or dilated. Affine geometry, represented at bottom left, permits all these transformations, and projection by parallel rays to a plane that can be tilted. In this instance the ratio of collinear points is constant. (If *P* is a point on the line *AB*, then the ratio of *AP* to *PB* does not change when the figure is transformed.) At top right is a representation of projective geometry, which permits point-source projection to a randomly tilted screen. An invariant propthe end product of mathematical thought can often be crystallized. It signifies ultimate success in penetrating and ordering mathematical substance and laying bare its skeletal structure. But emphasis on this aspect of mathematics is totally misleading if it suggests that construction, imaginative induction and combination and the elusive mental process called intuition play a secondary role in productive mathematical activity or genuine understanding. In mathematical education, it is true, the deductive method starting from seemingly dogmatic axioms provides a shortcut for covering a large territory. But the constructive Socratic method that proceeds from the particular to the general and eschews dogmatic compulsion leads the way more surely to independent productive thinking.

Just as deduction should be supplemented by intuition, so the impulse to progressive generalization must be tempered and balanced by respect and love for colorful detail. The individual problem should not be degraded to the rank of special illustration of lofty general theories. In fact, general theories emerge from consideration of the specific, and they are meaningless if they do not serve to clarify and order the more particularized substance below.

The interplay between generality and individuality, deduction and construction, logic and imagination—this is the profound essence of live mathematics. Any one or another of these aspects of mathematics can be at the center of a given achievement. In a far-reaching development all of them will be involved. Generally speaking, such a development will start from the "concrete" ground, then discard ballast by abstraction and rise to the lofty layers of thin air where navigation and observation are easy; after this flight comes the crucial test of landing and reaching specific goals in the newly surveyed low plains of individual "reality." In brief, the flight into abstract generality must start from and return again to the concrete and specific.

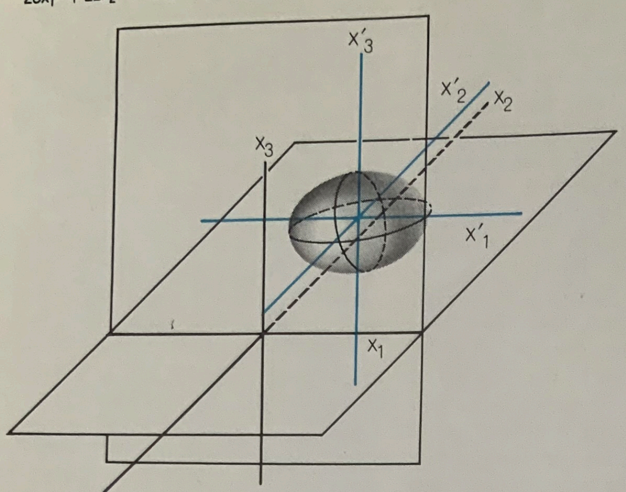
These principles are dramatically and convincingly illustrated in the evolution of the mathematical sciences. Johannes Kepler, with the genius of the true diagnostician, abstracted from the wealth of Tycho Brahe's observations the elliptical shape of the planetary orbits. Isaac Newton, by further abstraction, derived from these models the universal law of gravitation and the differential equations of mechanics. On this elevated level of unencumbered mathematical abstraction mechanics gained an enormous mobility. On descent to concrete and specific earthbound problems it has won success after success in enormous regions outside its original province of celestial dynamics.

Similarly, in electromagnetism Michael Faraday established a body of experimental findings linked by his own ingenious interpretations. From these some mathematical qualitative laws of electromagnetism were soon abstracted. Then, behind the formulations for specific, simple configurations, the genius of James Clerk Maxwell divined a very general quantitative law that combines in a system of differential equations the magnetic and electric forces and their rates of change. These equations, abstracted and cut loose from specific, tangible cases, may at first have seemed



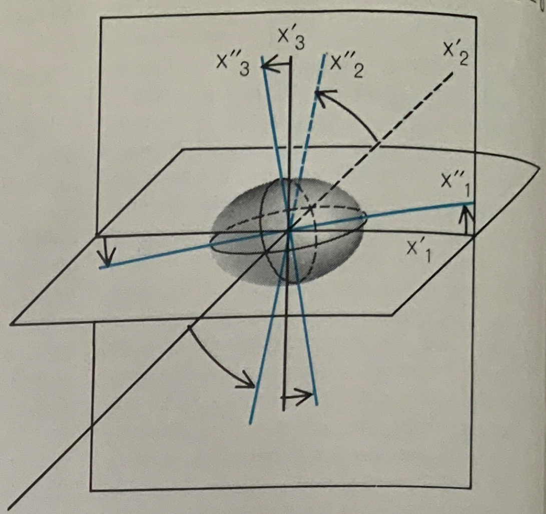
erty of figures projected this way is the cross ratio of collinear points. (If *P* and *Q* are points on the line *AB*, the ratio of *AP*/*PQ*: *AD*/*PD* is unchanged by the transformation.) Topology, a fourth type of geometry that is represented at middle right, studies properties preserved during the bending, stretching and twisting operations called continuous deformation. The order of four points *A*, *B*, *C* and *D* remains after the deformation. In point-set theory, the type of geometry shown at bottom right, the order of points is not retained during the kind of transformation called "scattering." The scattered points *do* remain conumerous with the points in the original figure. Thus point-set theory can be described as the study of the properties preserved under all one-to-one correspondences.

$$25x_1^2+22x_2^2+16x_3^2+20x_1x_2-4x_1x_3-16x_2x_3-62x_1-32x_2-44x_3+55=0$$



$$x_1 = x_1' + 1$$
$$x_2 = x_2' + 1$$
$$x_3 = x_3' + 2$$

$$25x_1'^2 + 22x_2'^2 + 16x_3'^2 + 20x_1'x_2' - 4x_1'x_3' - 16x_2'x_3' - 36 = 0$$



$$x_1' = \frac{1}{3}(-x_1'' + 2x_2'' + 2x_3'')$$
$$x_2' = \frac{1}{3}(2x_1'' - x_2'' + 2x_3'')$$
$$x_3' = \frac{1}{3}(2x_1'' + 2x_2'' - x_3'')$$

ALGEBRA AND GEOMETRY of bringing a quadratic surface into normal form is shown for the case of an ellipsoid with center at point (1, 1, 2) of the coordinate system in which it is considered. By parallel translation the coordinate system can be moved to a new position (*colored axes at left*) so the center of the ellipse is at its origin (*0, 0, 0*). The algebra of this translation calls for the substitutions yielding the equation above the middle diagram. The principal axes of the ellipsoid can be made coincident with the translated coordinate system by rotation of its axes to the position given by the colored lines in the middle diagram. Further

too esoteric for application. It soon became clear, however, that Maxwell's ascent to abstraction had opened the way to further progress in a number of directions. The Maxwell equations illuminated the wave nature of electromagnetic phenomena, inspired the experiments of Heinrich Hertz on the propagation of radio waves, started the growth of an entire new technology and led investigators to new lines of research, including, for example, the now very active field of magnetohydrodynamics.

It cannot be said that Maxwell's equations were the product of systematic deductive thinking. Neither should his achievement be ascribed to purely inductive Socratic processes. Instead Maxwell must be counted among those rare minds that recognize similarities and parallels between seemingly remote, disconnected facts and arrive at a major new insight by combining patently diverse elements into a unified system.

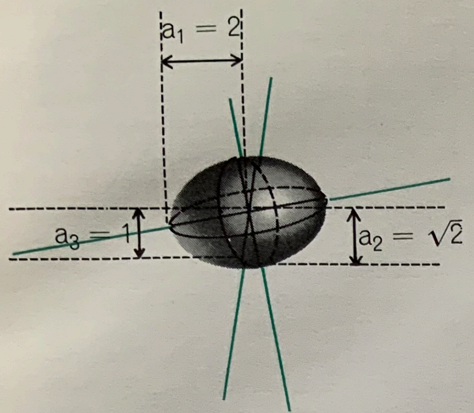
In mathematics proper a corresponding arc of development—from concrete individual substance through abstraction and back again to the concrete and individual—endows a theory with its meaning and significance. To appreciate this basic fact one must bear in mind that the terms "concrete," "abstract," "individual" and "general" have no stable or absolute meaning in mathematics. They refer primarily to a frame of mind, to a state of knowledge and to the character of mathematical substance. What is already absorbed as familiar, for example, is readily taken to be concrete. The words "abstraction" and "generalization" describe not static situations or end results but dynamic processes directed from some concrete stratum to some "higher" one.

Fruitful new discoveries in mathematics sometimes come suddenly with relatively little apparent effort: the view is cleared by abstracting from concrete material and laying bare the structurally essential elements. Axiomatics, irrespective of its Euclidean form, means just that. A recent instance of the fruitful use of abstraction is the generalization by John von Neumann and others of David Hilbert's "spectral" theory, from what proved to be the special case of "bounded" linear operators to "unbounded" ones.

This far-reaching development can be traced in a succession of abstractions upward from the familiar concrete ground of analytic geometry. In the elementary analytic geometry of a three-dimensional space with coordinates x_1, x_2, x_3 a plane is characterized by a linear equation, and a quadratic surface, such as that of a sphere or an ellipsoid, is characterized by a quadratic equation (that is, an equation in which the highest power of an unknown is its square) in the variables x_1, x_2, x_3. For example, an equation of the general form $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 = 1$ describes a quadratic surface centered at the origin of the coordinate system and with its three principal axes pointing in the direction of the coordinate axes. In the case of the ellipsoid the "coefficients" λ_1, λ_2, λ_3 stand for fixed positive numbers; they represent the expressions $1/a_1^2$, $1/a_2^2$, $1/a_3^2$, in which a_1, a_2, a_3 are the semi-axes of the ellipsoid. The ellipsoid consists precisely of those points for which the values of the variables x_1, x_2, x_3 satisfy the equation [*see illustration on these two pages*].

Now, without much ado, the algebraization of geometry permits one to speak of a space of more than three dimensions, say n dimensions, with coordinates x_1, x_2, $x_3 \ldots x_n$. In this space planes are again defined by linear equations and quadratic surfaces by quad-

$$\frac{1}{4}x_1''^2 + \frac{1}{2}x_2''^2 + x_3''^2 = 1$$



$$\lambda_1 = \frac{1}{a_1^2} = \frac{1}{4}$$
$$\lambda_2 = \frac{1}{a_2^2} = \frac{1}{2}$$
$$\lambda_3 = \frac{1}{a_3^2} = 1$$

substitution yields the equation in normal form, shown at the right above the ellipsoid it describes. The lengths of the semiaxes (a_1, a_2, a_3) are related to the coefficients of the terms in the equation as indicated.

ratic equations in the variables x_1, x_2, $x_3 \ldots x_n$. It is one of the most important results of "linear algebra" that quadratic surfaces can be brought into the algebraic normal form $\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 \ldots \lambda_n x_n^2 = 1$ after a suitable transformation of the coordinate system (or a rigid motion of the figure) that centers the figure at the origin with its principal axes along the coordinate axes. This theorem is the key to many applications; for instance, to the theory of mechanical or electrical systems involving the vibrations of a finite number n of mass points or circuit elements about a state of equilibrium.

Physicists such as Lord Rayleigh did not hesitate to apply this result, without mathematical justification, in a much more general way by letting the number of dimensions n tend to infinity. This step toward greater generalization and abstraction of the underlying mathematics has served well in the study of vibrating systems consisting not of a finite number of mass points or circuit elements but rather of a continuum of matter, such as a string, a membrane or a transmission line.

Hilbert, one of the truly great mathematicians of the past generation, recognized that such quadratic forms of infinitely many variables ought to be secured in a complete mathematical theory. In this endeavor he found it necessary first to restrict the domain of the variables by requiring that the sum of their squares should "converge," that is, have a finite value. Another way of stating this, with the help of a "generalized" Pythagorean theorem, is to say that a point in a "Hilbert space" of infinitely many dimensions must have a finite distance $r = \sqrt{x_1^2 + x_2^2 + \ldots}$ from the origin. Next, Hilbert defined the quadratic form in infinitely many variables—the bounded form—as a double infinite sum of the form

$$\begin{aligned} a_{11}x_1^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + \ldots \\ + a_{22}x_2^2 + a_{23}x_2x_3 + \ldots \\ + \ldots, \end{aligned}$$

where the first index (that is, x_1 in the first row, x_2 in the second row and so on) goes to infinity from row to row and the second index (that is, x_2 in the first row, x_3 in the second row and so on) goes to infinity along the row. This double infinite sum is under the crucial restriction that it must converge at every point in a Hilbert space.

In such a space many concepts relating to the properties of planes and the quadratic surfaces in finite-dimensional geometry remain meaningful. This is true in particular of the theory of the transformation of quadratic forms to their principal axes. Hilbert showed that every quadratic form in this class can be brought into a normal form by a rotation of the coordinate system. By analogy with the finite-dimensional case Hilbert called the set of values λ_1, λ_2, λ_3 appearing in this normal form the "spectrum" of the quadratic form.

In his generalization of the principal-axis theory from ordinary quadratic forms in n variables to forms in infinitely many variables Hilbert discovered many new phenomena, such as the occurrence of continuous mathematical spectra. Moreover, Hilbert's work served well in the emergence of quantum mechanics. His term "mathematical spectra" found a prophetic relevance to the spectra of energy states in atoms and their constituent particles. But Hilbert's theory of quadratic forms was not quite equal to the task of handling quantum mechanics; the forms occurring there turned out to be "unbounded."

At this point von Neumann, inspired by Erhard Schmidt and more inclined toward abstraction than his elders, carried the process of abstraction another crucial stratum upward. By discarding Hilbert's concept of a quadratic form as something that can be expressed concretely as an infinite algebraic expression and instead formulating the concept abstractly, he was able to avoid its earlier limitations. Thus extended, Hilbert's spectral theory was made to answer the tangibly concrete needs of contemporary physics.

The theory of groups, a central concern of contemporary mathematics, has evolved through an analogous progression of abstractions. Group theory traces its origins back to a problem that has fascinated mathematicians since the Middle Ages: the solving of algebraic equations of degree greater than two by algebraic processes, that is, by addition, subtraction, multiplication, division and extraction of roots. The theory of quadratic equations was known to the Babylonians, and the solution of equations of the third and the fourth degree was accomplished by the Renaissance mathematicians Girolamo Cardano and Niccolò Tartaglia. The solution of equations of the fifth degree and higher degrees, however, encountered insurmountable obstacles.

Early in the 19th century a novel and profound attack on these old problems was launched by Joseph Louis Lagrange, P. Ruffini and Niels Henrik Abel and, in a most original way, by Évariste Galois. These new approaches started from the known facts that an algebraic equation of degree n of the form $x^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$ has n roots r_1, $r_2 \ldots r_n$, and that this set of n roots determines the equation uniquely. (For example, if 1 and 3 are roots of a quadratic equation, then $(x-1)(x-3) = x^2 - 4x + 3 = 0$ is the equation determined by the roots 1 and 3.) The coefficients of the equation are symmetric functions of the roots; that is, they depend on the set of roots regardless of the order. (For example, in a cubic equation $x^3 + ax^2 + bx + c = 0$ with roots r_1, r_2, r_3 the coefficients can be written $-a = r_1 + r_2 + r_3$, $b = r_1r_2 + r_2r_3 + r_3r_1$, $c = r_1r_2r_3$, and if r_1, r_2, r_3 are permuted, a, b and c are not changed.)

Over the years work with such equations revealed that the key to the problem of expressing the roots of the equations in terms of the coefficients lies not only in the study of symmetric expressions but also more decisively in the study of not completely symmetric expressions and in the analysis of whatever symmetries they possess. The expression $E = r_1r_2 + r_3r_4$ does not, for example, remain unchanged for all arbitrary permutations of the four symbols r_1, r_2, r_3, r_4. If the indices 1 and 2 or 3 and 4 are interchanged, E is invariant, that is, remains unchanged. If 1 and 3 are interchanged, however, the resulting expression is not E. On the other hand, the succession of two permutations that changes and then restores E amounts to a permutation that clearly leaves E invariant. The set of these permutations, called a "group" by Galois, represents the intrinsic symmetries of the expression E. The understanding of permutation groups was recognized by the ingenious Galois as the key to a deeper theory of algebraic equations.

Soon afterward mathematicians were discovering permutation groups in other fields. The set of six motions that carry an equilateral triangle into itself, for example, forms a group [*see illustration on page 103*]. Other groups have been uncovered as fundamental structural elements in most of the branches of mathematics.

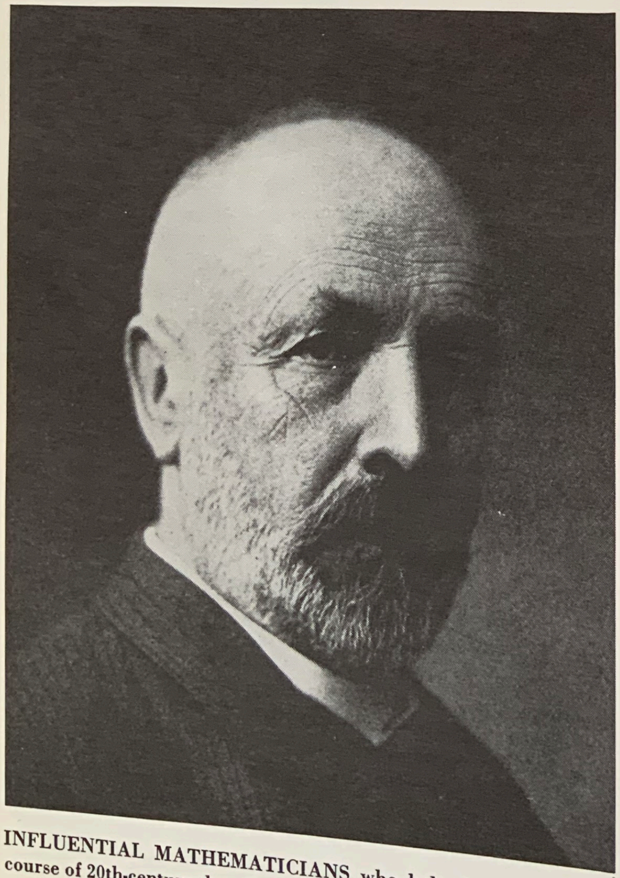
To embrace such groups, in all their different guises and manifestations, in a single concept and to anticipate the even wider scope of undiscovered possibilities required formulation of the underlying group concept in the most abstract terms. This has been done by calling a set of mathematical objects a group if a rule is given for "combining" two elements so as to obtain again an element S of the set; this rule is required to be associative, that is, $(ST)U = S(TU) = S$. Furthermore, the set must include a "unit" element I that, when combined with any other element S of the set, yields S, that is, $IS = SI = S$. Finally, for every element S in the set there must be an "inverse" element S^{-1} such that the combination SS^{-1} yields the unit element, that is, $SS^{-1} = I$.

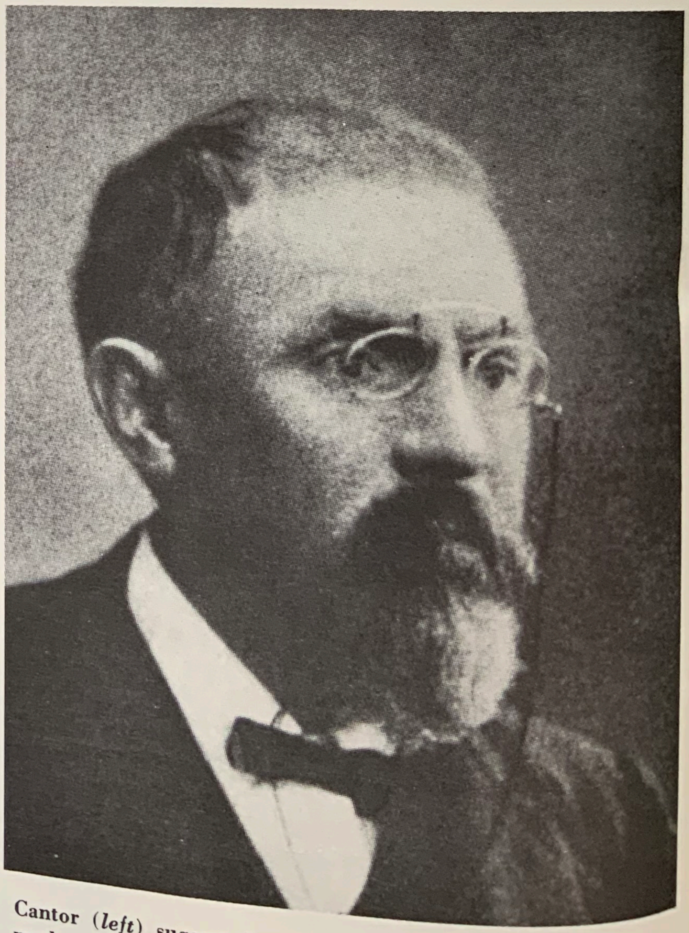
The specific "substantial" nature of the group is left wide open, of course, by this abstract definition. The elements may be numbers, rotations of geometric bodies, deformations of space (such deformations may be defined by linear or other transformation of the coordinates) or, as above, the permutations of n objects.

Altogether the group concept and the clarification and unification it brought to the diverse branches of mathematics must be reckoned a major achievement of the past 150 years. Much of the effort has been expended on the intermediate, lofty sector of the arc of development, that is, on the structural analysis of the abstracted concepts. The work has contributed all along, however, to illuminating more specific concrete areas, such as number theory and algebra. One of the remarkable successes along this line was Felix Klein's famous classification in the 1870's of the various branches of geometry according to groups of transformations under which certain geometrical properties remain invariant [*see illustration on pages 20 and 21*].

Abstract group theory has found significant application in the still more concrete problems of particle physics. Here the occasion is provided by the intricate group of open and hidden symmetries that prevail in the configuration and interaction of the nuclear particles. The success of group theory in bringing order to a great mass of data and predicting the existence of new particles [see "Mathematics in the Physical Sciences," page 249] shows convincingly how abstraction can help in the search for hard facts.

Intuition, that elusive vital agent, is always at work in creative mathematics, motivating and guiding even the most





INFLUENTIAL MATHEMATICIANS who helped to direct the course of 20th-century thought are shown on these two pages. Georg Cantor (*left*) suggested an order for infinite sets, thus focusing mathematical speculation abstract thinking. In its most familiar manifestation, geometrical intuition, it has figured in the many major recent advances in mathematics that have occurred in or flowed from work in geometry. Yet there is a powerful compulsion in mathematics to reduce the visible role of intuition, or perhaps one may better say to buttress it, by precise and rigorous reasoning.

Topology, the youngest and most vigorous branch of geometry, illustrates in a spectacular way the fruitful working of this tension between intuition and reason. With a few isolated but important earlier discoveries—for example the one-sided Möbius band—as its stock-in-trade, topology emerged as a field of serious study in the 19th century. For a long period it was almost entirely a matter of geometrical intuition, of cutting and pasting together surfaces in an effort to visualize the mathematical substance of topology, that is, the properties of surfaces that do not change under arbitrary continuous deformation. Early in the evolution of the new discipline, however, Georg Friedrich Bernhard Riemann brought it to the center of attention. In his sensational work on the theory of algebraic functions of a complex variable (a variable incorporating the imaginary number $\sqrt{-1}$) he showed that the topological facts concerning what are now called Riemann surfaces are essential to a real understanding of these functions.

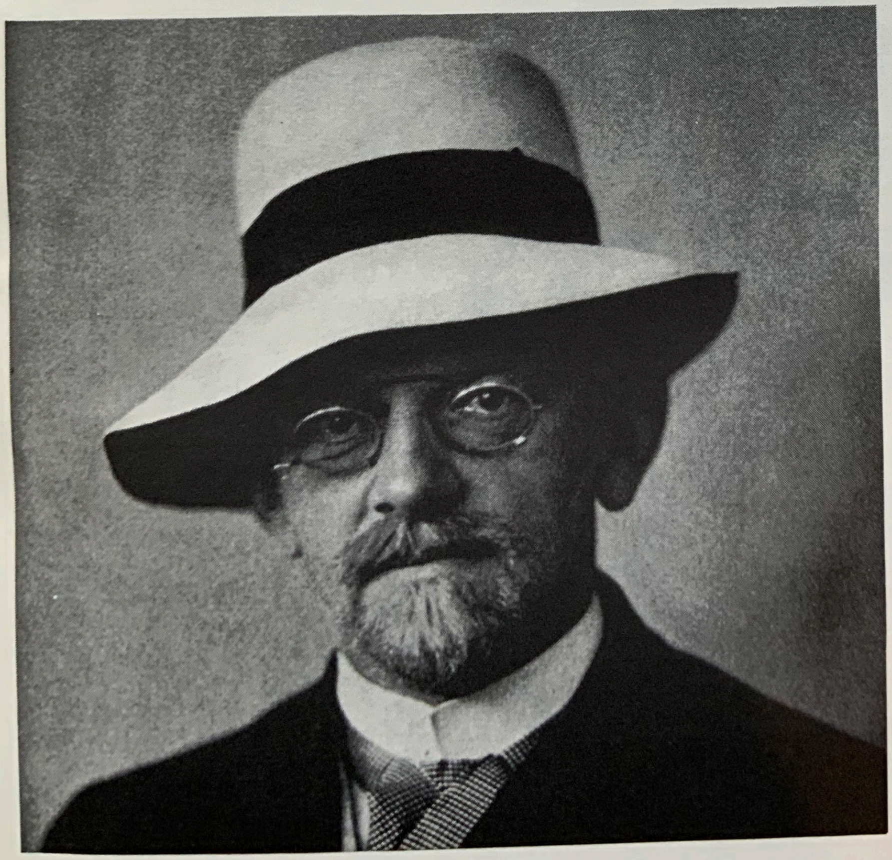
During the 19th century investigators discovered and systematically explored a wide range of topological properties of surfaces of two, three and then of n dimensions. Still on a more or less intuitive basis, early in this century, the great Henri Poincaré and others built a fascinating edifice of topological theory. This work proceeded in close relation to the development of group theory and found uses in other fields of mathematics and in the evolution of the mathematical sciences to higher levels of sophistication. It was put to work, for example, in celestial mechanics, specifically in the construction of planetary orbits in space curved by gravitational fields.

Topologists soon began to feel with urgency the need to sharpen their tools in order to catch the products of geometrical intuition in the vise of modern mathematical precision—without destroying their convincing beauty. This task was accomplished almost single-handed in the first decades of this century by the Dutch mathematician L. E. J. Brouwer. Thanks to his gigantic effort, topology is now as amenable to rigorous treatment as the geometry of Euclid, and advances in the field proceed on the solid ground of logically impeccable reasoning.

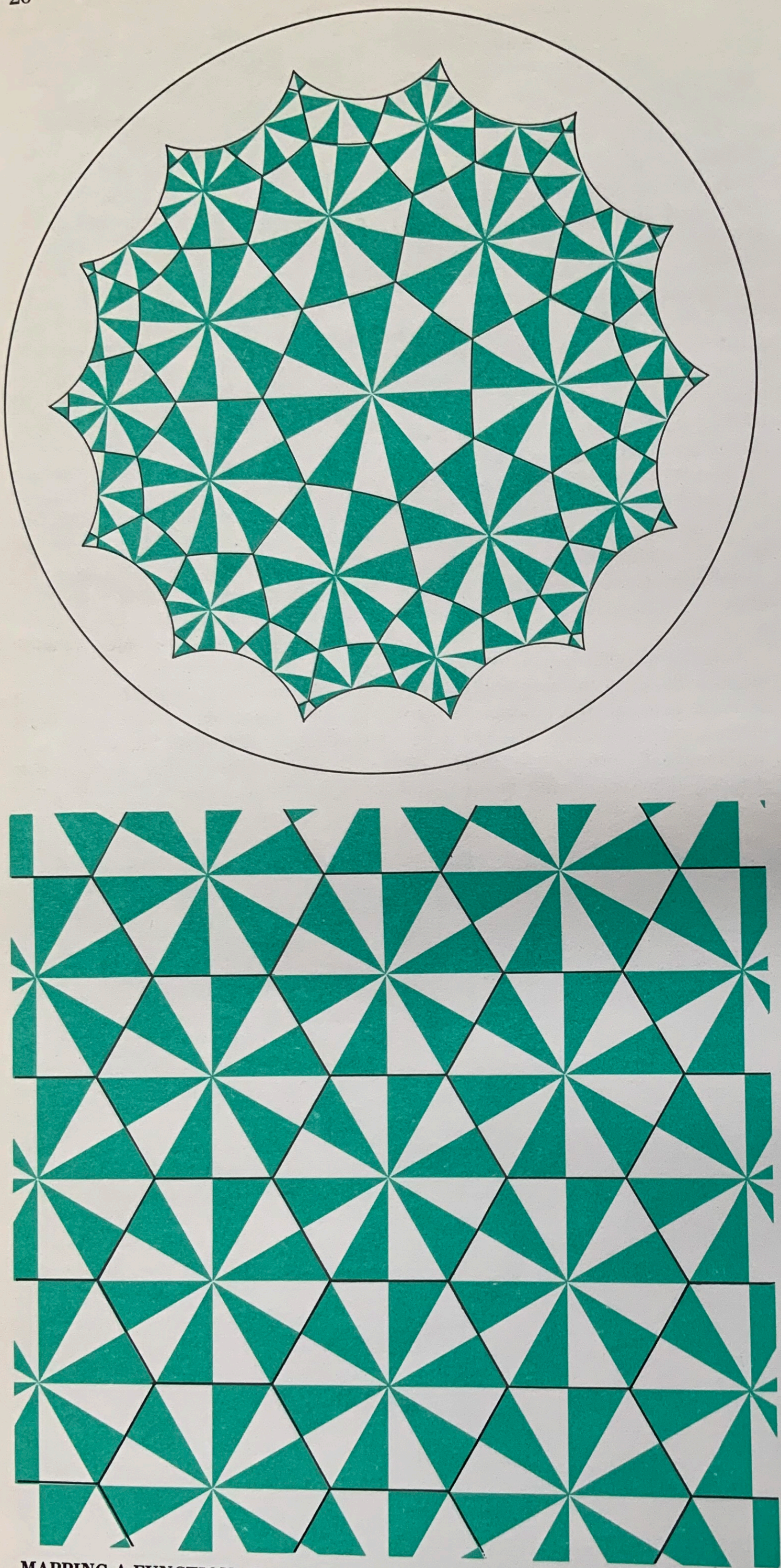
At the center of the difficulties confronting Brouwer was the dilemma presented by the concept of continuity. Everyone has a sure intuitive idea of what continuity is, for example the smoothness of a curve. But the beginning student of calculus loses his assurance at the very outset as he attempts to capture continuity in a precise mathematical formulation. Difficulty is inherent in the task because the geometrical intuition of continuity and the mathematical logical concept do not perfectly match. Rigorous definition brings to the surface whole areas of cases, perhaps marginal, that confound the intuition with paradox. It is easy to construct, for instance, continuous curves (in the exact sense of the definition) that do not have a length [*see lower illustration on page 27*], that have nowhere a direction or that wind around, without any self-intersection, within a square so that they come arbitrarily close to any given point in the square. Such bizarre constructions highlight the need for careful reasoning in proofs of the topological properties of surfaces or other objects subjected to complex continuous deformations.

That need is not at once intuitively apparent to the nontopologist. Consider, for example, C. Jordan's famous theorem stating that any nonintersecting continuous closed curve in a plane bounds two separate domains—the interior and the exterior [*see upper illustration on page 27*]. Every scientist, engineer and student in his naïve right mind will regard the effort to prove such a theorem as an unnecessary, self-imposed, almost masochistic exercise. Yet in writing his classical textbook on analysis Jordan felt strongly the need for a proof and presented one. It is a measure of the subtlety of the problem that Jordan's proof turned out to be not completely correct! Similarly, no one will doubt that the dimensionality of a two-dimensional or three-dimensional geometric figure remains unchanged under any continuous deformation. Yet the precise proof of this fact, under the general assumption of mere abstract continuity, stands as one of Brouwer's major achievements.

It is possible, of course, to evade some



warned against preoccupation with set theory. David Hilbert (*right*) generalized the principal-axis theory. In 1900 he proposed 23 projects for 20th-century mathematicians.



MAPPING A FUNCTION of a complex variable from an infinitely many-sheeted Riemann surface produces the figure shown at top of this illustration. The circle-arc polygons that grow infinitely small toward the outer circle correspond to straight-line polygons (*bottom*) that extend infinitely without changing size throughout the plane on which they are shown.

of the difficulties in the notion of continuity by restricting the group of continuous deformations—by demanding, for example, "smoothness" or differentiability instead of pure continuity. This has been done with great success. Differential topology, as it is called, has recently achieved outstanding results. Investigation of deformations conducted under the requirement of "reasonable" smoothness has produced significantly different classifications of topological structures than would be yielded under a regime of completely general continuity.

These developments may also be welcomed as indicating a healthy deflection of the trend toward boundless generality. Ever since Georg Cantor's achievements in the theory of sets, in the last decades of the 19th century, that trend has occupied many mathematical minds. Some great mathematicians, notably Poincaré, have fought it bitterly as a menace to mathematics, in particular because it leads to unresolved paradoxes. If Poincaré's militant criticism has proved to be overly restrictive and even reactionary, it was nonetheless salutary because it encouraged constructive mathematicians concerned with specific and graspable matters.

Various motivations, in the same individual or in different people, inspire mathematical activity. Certainly the roots in physical reality of large parts of mathematics—especially analysis—supply powerful motivation and inspiration. The situation with respect to other realms of reality is not much different. In number theory and algebra it is the intriguing reality of the world of numbers, so deeply inherent in the human mind. Still more removed from physical reality, one might think, is the reality of the logical processes involved in mathematical thinking. Yet basic ideas from esoteric work in mathematical logic have proved useful for the understanding and even for the design of automatic computing machines.

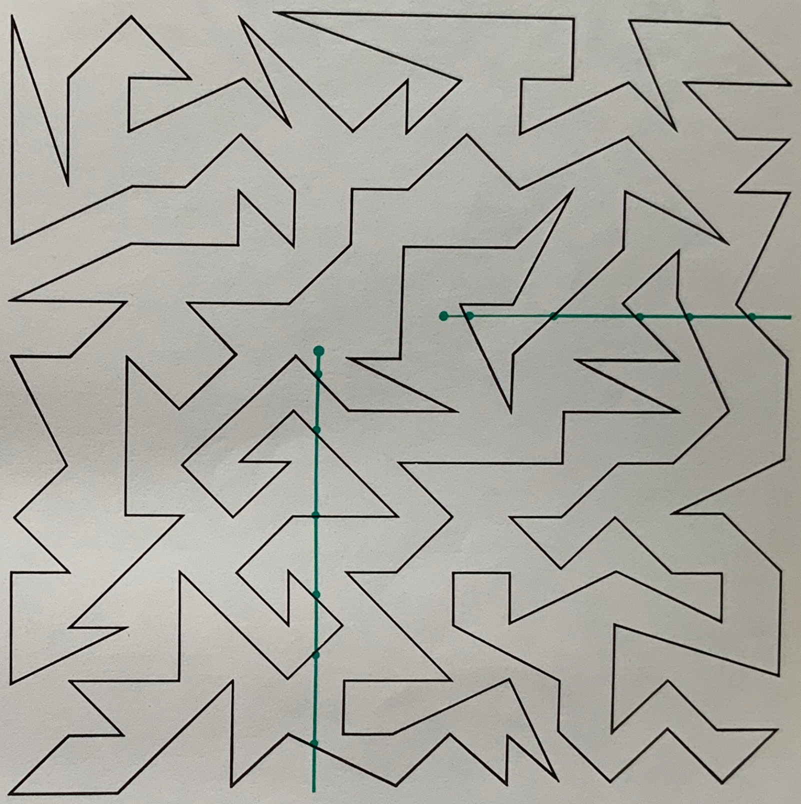
In brief, mathematics must take its motivation from concrete specific substance and aim again at some layer of "reality." The flight into abstraction must be something more than a mere escape; start from the ground and reentry are both indispensable, even if the same pilot cannot always handle all phases of the trajectory. The substance of the purest mathematical enterprise may often be provided by tangible physical reality. That mathematics, an emanation of the human mind, should serve so effectively for the description and understanding of the physical world is a challenging fact that has rightly attracted the concern of philosophers. Leaving philosophical questions aside, however, the engagement in physical questions or the apparent absence of such engagement must not be taken as a criterion for distinguishing between the kinds of mathematics and mathematicians.

No sharp dividing line can, in fact, be drawn between "pure" and "applied" mathematics. There should not be a class of high priests of unadulterated mathematical beauty, exclusively responsible to their own inclinations, and a class of workers who serve other masters. Class distinctions of this kind are at best the symptom of human limitations that keep most individuals from roaming at will over broad fields of interest.

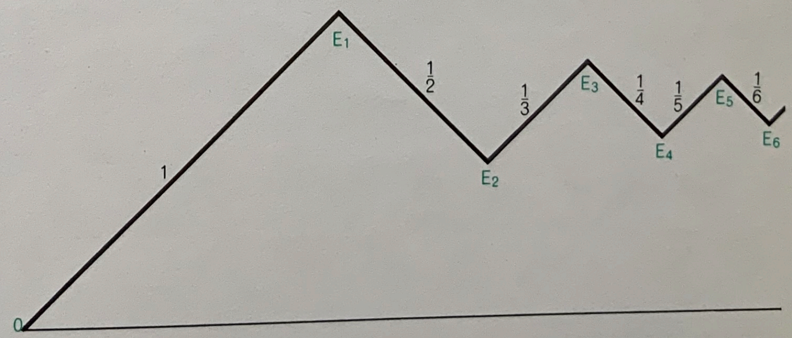
Although the substance of mathematics is indivisible, distinct differences must be acknowledged in the attitudes that the same scientist or different scientists may bring to a problem. The attitude of the purist, which every scientifically inclined mind will at least sometimes assume, demands uncompromising perfection. No gaps or rough spots can be tolerated in the solution of a problem and the result must flow from an unbroken chain of flawless reasoning. If the attempt encounters insurmountable obstacles, then the purist is inclined to restate his problem or replace it with another in which the difficulty is capable of being managed. He may even "solve" his problem by redefining what he means by a solution; this is, in fact, a not uncommon preliminary step toward a true solution of the original problem.

In the case of applied research the situation is different. The problem, in the first place, cannot be as freely modified or avoided; what is wanted is a believable, humanly reliable answer. If necessary, therefore, the mathematician must accept compromise; he must be willing to interpolate guesswork into the train of reasoning and to make allowance for the uncertainty of numerical evidence. But even the most practically motivated study—the analysis, for instance, of flow involving shock discontinuities—may demand fundamental mathematical investigation to discover how to frame the question. Pure existence proofs may also be significant in applied research; ascertaining that a solution exists may give the needed assurance of the suitability of the mathematical model. Finally, applied mathematics is dominated by approximations; these are inescapable in the attempt to mirror physical processes in mathematical models.

To handle the translation of reality into the abstract models of mathematics and to appraise the degree of accuracy thereby attainable calls for intuitive feeling sharpened by experience. It may also often involve the framing of genuine mathematical problems that are far too difficult to be solved by the available techniques of the science. Such, in part, is the nature of the intellectual adventure and the satisfaction experienced by the mathematician who works with engineers and natural scientists on the mastering of the "real" problems that arise in so many places as man extends his understanding and control of nature.



JORDAN CURVE THEOREM states that any closed curve such as the one shown here bounds interior and exterior domains. A line drawn from inside the curve to the outside will make an odd number of intersections; a line drawn from outside it, an even number.



INFINITE ZIGZAG is composed of successive segments with lengths 1, ½, ⅓, ¼, ⅕, ⅙.... The sequence of unit fractions has no finite sum and the curve itself has no finite length.