



# The Gross–Kohnen–Zagier theorem via $p$ -adic uniformization

Lea Beneish<sup>1</sup> · Henri Darmon<sup>2</sup> · Lennart Gehrmann<sup>3</sup> · Martí Roset<sup>2</sup> 

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## Abstract

This article gives a new proof of the Gross–Kohnen–Zagier theorem for Shimura curves which exploits the  $p$ -adic uniformization of Cerednik–Drinfeld. The explicit description of CM points via this uniformization leads to an expression relating the Gross–Kohnen–Zagier generating series to the ordinary projection of the first derivative, with respect to a weight variable, of a  $p$ -adic family of positive definite ternary theta series.

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✉ Martí Roset  
marti.rosetjulia@mail.mcgill.ca

Lea Beneish  
lea.beneish@unt.edu

Henri Darmon  
darmon@math.mcgill.ca

Lennart Gehrmann  
gehrmann.math@gmail.com

<sup>1</sup> University of North Texas, Denton, USA

<sup>2</sup> McGill University, Montreal, Canada

<sup>3</sup> Bielefeld University, Bielefeld, Germany

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## 1 Introduction

It is conjectured that generating series of special cycles on orthogonal Shimura varieties are modular forms. (See [27] for example.) One of the first instances of this phenomenon was discovered by Gross, Kohnen and Zagier, who proved in [17] that generating series of Heegner points on the Jacobians of modular curves are modular forms of weight  $3/2$ . The purpose of this article is to study the analogous statement for generating series of Heegner points on Shimura curves. We will present a new proof of the modularity of these generating series – originally established by Borcherds in [5] – using  $p$ -adic methods.

We proceed to explain the theorem in more detail, in a framework that encompasses both modular and Shimura curves. Let  $S$  be a finite set of places of  $\mathbb{Q}$  of odd cardinality containing  $\infty$  and let  $N^+$  be a square-free positive integer which is not divisible by any finite place in  $S$ . This datum gives rise to a *modular* or *Shimura curve*  $X$  defined over  $\mathbb{Q}$ , which is an instance of an orthogonal Shimura variety. Its set  $X(\mathbb{C})$  of complex points can be described in terms of an Eichler  $\mathbb{Z}$ -order  $\mathcal{R}$  of level  $N^+$  in a quaternion algebra  $\mathcal{B}$  over  $\mathbb{Q}$  ramified exactly at  $S - \{\infty\}$ . Namely, the set  $\mathcal{V}$  of trace zero elements in  $\mathcal{B}$  equipped with the quadratic form  $\mathcal{Q}$  induced by the reduced norm is a quadratic space of signature  $(1, 2)$ , and is anisotropic at all the places  $v \in S - \{\infty\}$ . The action of  $\mathcal{B}^\times$  on  $\mathcal{V}$  via conjugation identifies  $\mathcal{B}^\times$  with the group  $\text{GSpin}(\mathcal{V})$  of spinor similitudes of  $\mathcal{V}$ . It naturally acts on the conic  $C_{\mathcal{V}} \subseteq \mathbb{P}(\mathcal{V})$  whose rational points over a field  $E$  of characteristic 0 are given by

$$C_{\mathcal{V}}(E) = \{ \ell \in \mathbb{P}(\mathcal{V}_E) \mid \mathcal{Q}(\ell) = \{0\} \}. \quad (1)$$

Here and from now on, if  $M$  is an abelian group, and  $A$  is a commutative ring, write  $M_A$  for the  $A$ -module  $M \otimes_{\mathbb{Z}} A$ . The group  $\Gamma$  of units of  $\mathcal{R}$  modulo  $\{\pm 1\}$  acts discretely on the symmetric space

$$\mathcal{H} = C_{\mathcal{V}}(\mathbb{C}) - C_{\mathcal{V}}(\mathbb{R})$$

associated to the orthogonal group of  $\mathcal{V}$ . The set  $X(\mathbb{C})$  of complex points of  $X$  is identified with the quotient  $\Gamma \backslash \mathcal{X}$ . A comparison with a more classical description of Shimura curves can be found in the Appendix of [27].

Let  $v \in \mathcal{V}$  be a vector for which  $\mathcal{Q}(v) > 0$ . Its orthogonal complement is thus a two-dimensional negative definite space, whose base change to  $\mathbb{C}$  is a hyperbolic plane. Hence, there are exactly two points in  $\mathcal{X}$  represented by a vector orthogonal to  $v$ . Let  $\Delta(v) \subset \mathcal{X}$  be the divisor consisting of these two points. Each positive integer  $D$  in

$$\mathcal{D}_S := \{D \in \mathbb{Z}_{>0} \mid \exists v \in \mathcal{V} \text{ such that } \mathcal{Q}(v) = D\}$$

gives rise to a zero-cycle on  $X$  by setting

$$\Delta(D) := \sum_{\substack{v \in \Gamma \backslash \mathcal{R}_0, \\ \mathcal{Q}(v) = D}} \frac{1}{\#\text{Stab}_\Gamma(v)} \Delta(v) \in \text{Div}(X(\mathbb{C}))_{\mathbb{Q}}, \tag{2}$$

where  $\mathcal{R}_0 \subseteq \mathcal{V}$  is the  $\mathbb{Z}$ -lattice  $\mathcal{R} \cap \mathcal{V}$ . The divisor  $\Delta(D)$ , which is supported on a finite set of CM points on  $X$ , is a simple instance of a *Heegner divisor* on this Shimura curve.

The Gross–Kohnen–Zagier theorem asserts that the classes of  $\Delta(D)$  in the Jacobian of  $X$  can be packaged into a modular generating series of weight  $3/2$ . Namely, let  $\mathcal{L}$  be the tautological line bundle of isotropic vectors whose spans are points of  $\mathcal{X}$ . This bundle is  $\mathcal{B}^\times$ -equivariant and, therefore, descends to a line bundle on  $X(\mathbb{C})$ , which is identified with the cotangent bundle of  $X$ . In particular, it has a model over  $\mathbb{Q}$ . Denote by  $[\Delta]$  (resp.  $[\mathcal{L}^\vee]$ ) the class in  $\text{Pic}(X)(\mathbb{Q})$  of a divisor  $\Delta$  (resp. of the dual  $\mathcal{L}^\vee$  of the line bundle  $\mathcal{L}$ ) on  $X$ . Then, the formal generating series

$$G(q) := [\mathcal{L}^\vee] + \sum_{D \in \mathcal{D}_S} [\Delta(D)]q^D \in \text{Pic}(X)(\mathbb{Q})_{\mathbb{Q}}[[q]], \tag{3}$$

is a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$ , where  $N$  is the product of  $N^+$  with all finite places in  $S$ . Remember that, given an abelian group  $A$ , a formal  $q$ -series  $f \in A[[q]]$  with coefficients in  $A$  is called a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$  if for every homomorphism  $\varphi: A \rightarrow \mathbb{C}$  the generating series  $\varphi(f) \in \mathbb{C}[[q]]$ , obtained by applying  $\varphi$  to each of the coefficients of  $f$ , is the  $q$ -expansion of a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$ .

The Gross–Kohnen–Zagier theorem was first proved in [17] in the case of modular curves (i.e., where  $S = \{\infty\}$ ) by calculating the Arakelov intersection pairings of the divisors  $\Delta(D)$  with a fixed CM divisor. It was extended by Borcherds [5] to the setting of orthogonal groups of real signature  $(n, 2)$ , encompassing Shimura curves as a special case where the underlying quadratic space is of signature  $(1, 2)$ , as a consequence of his theory of singular theta lifts. The work of Yuan, Zhang, and Zhang [39] proves Theorem 1.1 in much greater generality, for certain orthogonal groups over totally real fields. It should be noted that the theorem also holds if one replaces the lattice  $\mathcal{R}_0$  by a suitable weighted sum of lattice cosets or, equivalently, by a suitable

Schwartz–Bruhat function. In order to keep the exposition as simple as possible we refrain from stating the most general version of the theorem in the introduction.

The goal of this article is to describe a new proof of the Gross–Kohnen–Zagier theorem in the case where  $S \neq \{\infty\}$ , i.e., when  $X$  is not a modular curve. To simplify the exposition we will also assume that  $2 \nmid N$ .

**Theorem 1.1** *The generating series  $G(q) \in \text{Pic}(X)(\mathbb{Q})_{\mathbb{Q}}[[q]]$  of (3) is a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$ .*

Our approach to this theorem rests on the fact that, at a finite place  $p \in S$ , the curve  $X(\mathbb{C}_p)$  admits a  $p$ -adic analytic uniformization. More precisely,  $X(\mathbb{C}_p)$  can be described as the quotient of the  $p$ -adic upper half-plane by the discrete action of the norm one elements of an Eichler  $\mathbb{Z}[1/p]$ -order  $R$  of level  $N^+$  in the (definite) quaternion algebra ramified exactly at  $S - \{p\}$ . Furthermore, the Heegner divisors  $\Delta(D)$  can be described  $p$ -adically in terms of this uniformization. This immediately gives an expression of the generating series of degrees

$$\deg(G)(q) = \deg(\mathcal{L}^\vee) + \sum_{D \in \mathcal{D}_S} \deg(\Delta(D))q^D$$

in terms of definite ternary theta series, recovering a well-known modularity result. (See for example [20, Chapter 2] and [26, Theorem I].) Thus, it is enough to prove modularity of the generating series  $TG(q)$  for Hecke operators of degree 0, for which  $TG(q)$  takes values in the  $\mathbb{Q}$ -rational points of the Jacobian  $J$  of  $X$ . The existence of a basis of modular forms with rational coefficients then reduces the problem to proving modularity of the generating series

$$\log_\omega(TG)(q) := \sum_{D \in \mathcal{D}_S} \log_\omega([T\Delta(D)])q^D \in \mathbb{Q}_p[[q]] \quad (4)$$

for every cotangent vector  $\omega$  of  $J_{\mathbb{Q}_p}$  with associated  $p$ -adic formal logarithm  $\log_\omega : J(\mathbb{Q}_p) \rightarrow \mathbb{Q}_p$ . For appropriate Hecke operators  $T$ , the  $p$ -adic description of the divisors  $T\Delta(D)$  leads to an expression of this series as the ordinary projection of an infinitesimal  $p$ -adic deformation of a positive definite ternary theta series attached to the data  $(\omega, R, T)$ . More precisely, these data give rise to a  $p$ -adic family of weighted theta series  $\Theta_k$  of weight  $k + 3/2$ , with  $k$  in the weight space  $(\mathbb{Z}/(p-1)) \times \mathbb{Z}_p$ , whose specialization at weight  $3/2$  vanishes. (See Sect. 7.3 for its definition.) It then follows that its derivative with respect to  $k$  evaluated at  $k = 0$ , denoted  $\Theta'_0$ , is a  $p$ -adic cusp form of weight  $3/2$ . Let  $e_{\text{ord}}$  be  $p$ -ordinary projector acting on this space. By a classicality result,  $e_{\text{ord}}(\Theta'_0)$  is a cusp form of weight  $3/2$  and level  $\Gamma_0(4N)$ . Let  $\text{pr}_1$  be the projector on the space of cusp forms of weight  $3/2$  and level  $\Gamma_0(4N)$  to the eigenspace of the Hecke operator  $U_{p^2}$  of eigenvalue 1. The main contribution of this article is the following formula.

**Theorem 1.2** *We have*

$$\log_\omega(TG) = \text{pr}_1(e_{\text{ord}}(\Theta'_0)).$$

To summarize, the fact that  $TG$  is a modular form is a consequence of the modularity of definite theta series and classicality of ordinary  $p$ -adic modular forms of half-integral weight.

**Remark 1.3** As the proof of Theorem 1.1 is a purely  $p$ -adic analytic one, it seems likely that it carries over to more general settings, e.g., to Shimura curves over totally real fields which admit a  $p$ -adic uniformization. The assumption that  $N$  is odd and square-free stems from using  $p$ -adic families of *scalar-valued* half-integral modular forms, which seem only well-behaved in that case. Concretely, we study these  $p$ -adic families using the results of [32], [24], and [30], which have a similar condition on the level of the modular forms. Generalizing to arbitrary level likely requires a theory of families of vector-valued modular forms, which so far has only been developed in a few instances. (See [28].)

The strategy sketched above bypasses the global height pairings studied by Gross, Kohnen, and Zagier, or the singular theta lifts that arise in the approach of Borcherds. It can be envisaged as fitting into the broader framework of a “ $p$ -adic Kudla program”, in which  $p$ -adic families of modular forms play much the same role as analytic families of Eisenstein series in the Archimedean setting. Insofar as the generating series  $G$  are among the simplest instances of the modular generating series arising in the Kudla program, it is hoped that the  $p$ -adic techniques described here will be more widely applicable, shedding light on the connection between special cycles on orthogonal and unitary Shimura varieties,  $p$ -adic Borcherds-type lifts, and  $p$ -adic families of theta series. A general framework is laid out in the article [9], which introduces the notion of rigid meromorphic cocycles for orthogonal groups. In *loc.cit.* modularity statements for generating series of special divisors on arithmetic quotients on higher-dimensional  $p$ -adic symmetric spaces are formulated. A crucial input in their proof is the injectivity of the first Chern class when the arithmetic quotient has dimension 3 and higher. Theorem 1.1 complements the main theorem of [9] by extending it to the case of curves, where the kernel of the Chern class map needs to be considered. Relating arithmetic data to Fourier coefficients of ordinary projections of  $p$ -adic modular forms is one of the prevalent themes of the  $p$ -adic Kudla program as demonstrated, for example, in [7] and [10]. The  $p$ -adic modular forms in question arise as first order derivatives of  $p$ -adic families of classical modular forms, although the families considered in [7] and [10] are of a rather different nature than the one considered in Theorem 1.2.

The organization of the article is as follows. Section 2 explains the  $p$ -adic uniformization of  $X$  and states the Gross–Kohnen–Zagier theorem in terms of this uniformization. Theorem 2.7 below describes the main result, which is somewhat more general than Theorem 1.1, since the divisors  $\Delta(D)$  are replaced by linear combinations of Heegner points weighted by Schwartz–Bruhat functions. Section 3 gives a short proof of the modularity of  $\deg(G)$ . Section 4 introduces the  $p$ -adic Abel–Jacobi map, which gives an explicit description of the Jacobian of a Mumford curve. This description is used in Sect. 5 to construct certain functionals on the Jacobian, whose values at Heegner points are computed in Sect. 6. In particular, we give an explicit expression for the quantities  $\log_\omega([T\Delta(D)])$  appearing in (4). In Sect. 7, we define the  $p$ -adic family  $\Theta_k$ , prove a classicality result regarding ordinary  $p$ -adic cusp forms of half-integral weight and prove the main Theorem 1.2, which implies the

Gross–Kohnen–Zagier theorem. Finally, Sect. 8 illustrates the construction of  $\Theta_k$  by presenting a concrete example where  $S = \{7, 13, \infty\}$  and  $p = 7$ . In this case, the ordinary projection of  $\Theta'_0$  is computed numerically modulo  $p$ .

## 2 The Cerednik–Drinfeld theorem

This section recalls the theorem of Cerednik–Drinfeld, which gives a rigid analytic uniformization of  $X$  at a finite prime  $p \in S$  that is fixed once and for all. Moreover, we describe Heegner divisors in terms of this uniformization, which leads to a reformulation of the Gross–Kohnen–Zagier theorem in this setting.

### 2.1 $p$ -adic uniformization of $X$

The rigid analytic uniformization of  $X$  proceeds by replacing the place  $\infty$  in the complex uniformization of the introduction by the prime  $p \in S$ . To describe it, we need to introduce some notation. Let  $B$  be the quaternion algebra over  $\mathbb{Q}$  ramified exactly at the places in  $S - \{p\}$ . Let  $R$  be an Eichler  $\mathbb{Z}[1/p]$ -order in  $B$  of level  $N^+$  and denote by  $\Gamma$  the multiplicative group of elements of reduced norm 1 in  $R$  modulo  $\{\pm 1\}$ . Let  $Q$  be the restriction of the reduced norm to the space

$$V = \{b \in B \mid \text{Tr}(b) = 0\}$$

of elements of reduced trace zero in  $B$ . It endows  $V$  with the structure of a quadratic space of rank 3 over  $\mathbb{Q}$ , which is of real signature  $(3, 0)$ . Denote by  $\langle \cdot, \cdot \rangle$  the symmetric bilinear form attached to  $Q$ , that is,  $\langle v, w \rangle := Q(v + w) - Q(v) - Q(w)$ . As in the case of the quadratic space  $\mathcal{V}$ , the action of  $B^\times$  on  $V$  via conjugation identifies  $B^\times$  with the group of spinor similitudes of  $V$ . The intersection  $R_0 = R \cap V$  is an even  $\mathbb{Z}[1/p]$ -lattice in  $V$ .

A  $p$ -adic symmetric space is associated to the orthogonal group of  $V_{\mathbb{Q}_p}$  as follows. Similarly to (1) denote by  $C_V \subseteq \mathbb{P}(V)$  the conic over  $\mathbb{Q}$  attached to  $V$  whose rational points over a field  $E$  of characteristic 0 are given by

$$C_V(E) = \{\ell \in \mathbb{P}(V_E) \mid Q(\ell) = \{0\}\}.$$

This conic has no rational points, but can be identified with the projective line  $\mathbb{P}$  over  $\mathbb{Q}_p$  as follows: choose an isomorphism of  $B_{\mathbb{Q}_p}$  with the matrix ring  $M_2(\mathbb{Q}_p)$ . The conic is then identified with the space of non-zero nilpotent  $2 \times 2$ -matrices up to scaling. Mapping such a matrix to its kernel yields the desired isomorphism. The action of  $B_{\mathbb{Q}_p}^\times$  is identified with the action of  $GL_2(\mathbb{Q}_p)$  on  $\mathbb{P}_{\mathbb{Q}_p}$  via Möbius transformations.

**Definition 2.1** The Drinfeld  $p$ -adic upper half plane is the  $\mathbb{Q}_p$ -rigid analytic space  $\mathcal{H}_p$ , whose  $E$ -rational points for any complete extension  $E/\mathbb{Q}_p$  is the set

$$\mathcal{H}_p(E) := C_V(E) - C_V(\mathbb{Q}_p) \simeq \mathbb{P}_1(E) - \mathbb{P}_1(\mathbb{Q}_p).$$

We briefly explain the rigid analytic structure on  $\mathcal{H}_p$  in terms of the reduction map to the Bruhat–Tits tree. We refer the reader to [8, Chapter 5] for further details. The Bruhat–Tits tree, denoted  $\mathcal{T}$ , is the graph whose set of vertices is the set of unimodular  $\mathbb{Z}_p$ -lattices in  $V_{\mathbb{Q}_p}$ , that is, lattices which are self-dual with respect to the symmetric bilinear form  $\langle \cdot, \cdot \rangle$ . Two unimodular  $\mathbb{Z}_p$ -lattices  $L_1$  and  $L_2$  are joined by an edge if they are  $p$ -neighbours, that is,

$$[L_1 : L_1 \cap L_2] = [L_2 : L_1 \cap L_2] = p.$$

A choice of a vertex  $L$  gives a smooth  $\mathbb{Z}_{(p)}$ -integral structure  $C_L$  to the conic  $C_V$ , where  $\mathbb{Z}_{(p)}$  denotes the localization of  $\mathbb{Z}$  at the prime ideal generated by  $p$ . If  $L'$  is adjacent to  $L$ , then the image of  $L \cap L'$  in  $L/pL$  is a 2-dimensional non-regular subspace, hence contains a unique isotropic subspace  $\ell_{L'}$ . Mapping the edge  $(L, L')$  to  $\ell_{L'}$  yields a bijection between the set of lattices adjacent to  $L$  and  $C_L(\mathbb{F}_p) \simeq \mathbb{P}_1(\mathbb{F}_p)$ . It follows that  $\mathcal{T}$  is homogeneous of degree  $p + 1$ . The set of vertices and edges of  $\mathcal{T}$  are denoted by  $\mathcal{T}_0$  and  $\mathcal{T}_1$  respectively, and  $\mathcal{T}$  shall be viewed as a disjoint union  $\mathcal{T} = \mathcal{T}_0 \sqcup \mathcal{T}_1$ . Identifying the quadratic space  $V_{\mathbb{Q}_p}$  with the set of trace zero endomorphisms of  $\mathbb{Q}_p^2$  endowed with the norm form gives the more familiar description of the tree in terms of similarity classes of  $\mathbb{Z}_p$ -lattices in  $\mathbb{Q}_p^2$ . Indeed, the assignment  $[\Lambda] \mapsto \text{Hom}_0(\Lambda, \Lambda)$  is a bijection between such similarity classes and unimodular lattices in  $V_{\mathbb{Q}_p}$ . Moreover, two classes  $[\Lambda_1], [\Lambda_2]$  are joined by an edge if they admit representatives  $\Lambda_1$  and  $\Lambda_2$  satisfying  $p\Lambda_1 \subset \Lambda_2 \subset \Lambda_1$ . From this description one easily deduces that  $\mathcal{T}$  is indeed a tree. The identification of the two graphs is compatible with the natural actions of  $B_{\mathbb{Q}_p}^\times$  and  $\text{GL}_2(\mathbb{Q}_p)$ . We define a notion of parity on the vertices of  $\mathcal{T}$  by requiring that every edge connects an even vertex with an odd one. There are exactly two possible choices for this and we choose one of them. The action of the elements of reduced norm one in  $B_{\mathbb{Q}_p}$  on  $\mathcal{T}$  is parity-preserving.

We proceed by describing the well-known *reduction map*

$$\text{red} : \mathcal{H}_p(\mathbb{C}_p) \longrightarrow \mathcal{T}$$

in the language of quadratic forms. For that let  $\mathcal{O}_{\mathbb{C}_p}$  denote the ring of integers of  $\mathbb{C}_p$  and  $\mathfrak{m} \subseteq \mathcal{O}_{\mathbb{C}_p}$  its maximal ideal. Every unimodular  $\mathbb{Z}_p$ -lattice  $L \subseteq V_{\mathbb{Q}_p}$  induces a reduction map

$$C_V(\mathbb{C}_p) = C_L(\mathcal{O}_{\mathbb{C}_p}) \twoheadrightarrow C_L(\overline{\mathbb{F}_p}).$$

- (1) Let  $L \subseteq V_{\mathbb{Q}_p}$  be a unimodular  $\mathbb{Z}_p$ -lattice. Then  $\text{red}^{-1}(L)$  is the complement of the  $p + 1$  residue discs around the points in  $C_L(\overline{\mathbb{F}_p})$ .
- (2) Let  $L, L' \subseteq V_{\mathbb{Q}_p}$  be two unimodular  $\mathbb{Z}_p$ -lattices that are  $p$ -neighbours and  $\ell_{L'} \in C_L(\overline{\mathbb{F}_p})$  the corresponding isotropic line. The preimage of the edge  $(L, L')$  under the reduction map consists of those elements  $z \in C_L(\mathcal{O}_{\mathbb{C}_p})$  that are congruent to  $\ell_{L'}$  modulo  $\mathfrak{m}$  but not modulo  $p$ .

One readily checks that the reduction map is  $B_{\mathbb{Q}_p}^\times$ -equivariant. A finite closed subgraph of  $\mathcal{T}$  is a finite set  $\mathcal{G} \subset \mathcal{T}$  satisfying

$$(v_1, v_2) \in \mathcal{G} \cap \mathcal{T}_1 \Rightarrow v_1, v_2 \in \mathcal{G} \cap \mathcal{T}_0.$$

A standard affinoid subset of  $\mathcal{H}_p$  is a set of the form  $\text{red}^{-1}(\mathcal{G})$ , where  $\mathcal{G}$  is a finite closed subgraph of  $\mathcal{T}$ .

**Definition 2.2** A function on  $\mathcal{H}_p$  is said to be *rigid analytic* if its restriction to any standard affinoid subset  $\mathcal{A} \subset \mathcal{H}_p$  can be written as a uniform limit of rational functions having poles outside of  $\mathcal{A}$ . A function on  $\mathcal{H}_p$  is said to be *rigid meromorphic* if it is the quotient of two rigid analytic functions, where the denominator is non-zero.

The group  $\Gamma$  acts naturally on  $\mathcal{H}_p$  by conjugation. This action is discrete because  $\Gamma$  is a  $p$ -arithmetic subgroup of an algebraic group that is compact at  $\infty$ . It follows from there that the quotient space  $\Gamma \backslash \mathcal{H}_p$  has a natural structure of a rigid analytic variety over  $\mathbb{Q}_p$ . On the other hand, the analytification of  $X$  gives a rigid analytic space over  $\mathbb{Q}_p$ . The Cerednik-Drinfeld theorem states that these two spaces can be identified after base change to the unramified quadratic extension  $\mathbb{Q}_{p^2}$  of  $\mathbb{Q}_p$ . This identification depends on choices. To make this precise, let us introduce the following notation: for a finite set  $\Sigma$  of places of  $\mathbb{Q}$  write  $\mathbb{A}^\Sigma \subset \prod_{v \notin \Sigma} \mathbb{Q}_v$  for the ring of finite adèles away from  $\Sigma$ . Moreover, let  $\hat{\mathbb{Z}}$  (resp.  $\hat{\mathbb{Z}}^{(p)}$ ) be the maximal order of  $\mathbb{A}^\infty$  (resp. of  $\mathbb{A}^{p, \infty}$ ). Given a finitely generated  $\mathbb{Z}[1/p]$ -module  $M$ , we put  $\hat{M} = M \otimes \hat{\mathbb{Z}}^{(p)}$ . Fix an identification

$$\mathcal{V}_{\mathbb{A}^{p, \infty}} \simeq V_{\mathbb{A}^{p, \infty}} \tag{5}$$

sending the  $\hat{\mathbb{Z}}^{(p)}$ -lattice  $\hat{\mathcal{H}}_0$  to  $\hat{R}_0$ .

**Theorem 2.3** (Cerednik–Drinfeld) *The identification (5) induces an isomorphism*

$$X \xrightarrow{\sim} \Gamma \backslash \mathcal{H}_p. \tag{6}$$

of rigid analytic spaces over  $\mathbb{Q}_{p^2}$ .

**Proof** See [37], [12] and [6]. □

## 2.2 $p$ -adic analytic description of Heegner divisors

In analogy with the cycles defined in the introduction, every non-zero element  $v \in V$  yields a cycle  $\Delta(v)$  on  $\mathcal{H}_p$ :  $\Delta(v)$  is the sum of those points in  $\mathcal{H}_p$  that are orthogonal to  $v$ . This cycle has degree 0 or 2 depending on whether the orthogonal complement of  $v$  in  $V_{\mathbb{Q}_p}$  represents 0 or not. Observe that the orthogonal complement of  $v$  in  $V_{\mathbb{Q}_p}$ , being 2-dimensional, represents 0 if and only if the negative of its discriminant is a square in  $\mathbb{Q}_p$ . Since the discriminant of  $V_{\mathbb{Q}_p}$  is a square, the discriminant of the orthogonal complement of  $v$  is equal to the discriminant of  $v$ , which is  $Q(v)$ , modulo squares.



Hence, we deduce  $\Delta(v) \neq 0$  if and only if  $\sqrt{-Q(v)} \notin \mathbb{Q}_p$ . By the Hasse–Minkowski theorem the set  $\mathcal{D}_S$  from the introduction is characterized locally. In particular, one gets the description

$$\mathcal{D}_S = \{D \in \mathbb{Z} \mid \exists v \in V - \{0\} \text{ such that } Q(v) = D \text{ and } \Delta(v) \neq 0\},$$

where we used that  $B$  ramifies exactly at  $S - \{p\}$  while  $\mathcal{B}$  ramifies exactly at  $S - \{\infty\}$ . Indeed, the quadratic spaces  $V$  and  $\mathcal{V}$  are locally isomorphic at all places except at  $\infty$  and  $p$ . This is why the condition  $D > 0$  (or equivalently,  $\sqrt{-D} \notin \mathbb{R}$ ) appearing in the description of  $\mathcal{D}_S$  in the introduction is replaced by the condition  $\Delta(v) \neq 0$  (or equivalently,  $\sqrt{-D} \notin \mathbb{Q}_p$ ). Note that the first condition is automatic for the nonzero lengths of elements of  $V$ , as  $B$  is ramified at  $\infty$  and the second condition is automatic for nonzero lengths of elements of  $\mathcal{V}$ , as  $\mathcal{B}$  is ramified at  $p$ .

**Lemma 2.4** *Let  $D$  be an element of  $\mathcal{D}_S$  and  $v \in V$  with  $Q(v) = D$ .*

- (1) *If  $\text{ord}_p(D) = 0$ , there exists a unique unimodular  $\mathbb{Z}_p$ -lattice  $L$  in  $V_{\mathbb{Q}_p}$  containing  $v$ . The support of  $\Delta(v)$  is contained in  $\text{red}^{-1}(L)$ .*
- (2) *If  $\text{ord}_p(D) = 1$ , there exist exactly two unimodular  $\mathbb{Z}_p$ -lattices  $L_1, L_2$  in  $V_{\mathbb{Q}_p}$  containing  $v$ , which are  $p$ -neighbours. The support of  $\Delta(v)$  is contained in  $\text{red}^{-1}((L_1, L_2))$ .*

**Proof** Let  $W$  be the orthogonal complement of  $v$  in  $V_{\mathbb{Q}_p}$ . As  $W$  is anisotropic, it contains a unique maximal  $\mathbb{Z}_p$ -lattice  $L_W$ , on which  $Q$  takes values in  $\mathbb{Z}_p$  and it is characterized by the property that its discriminant module is an  $\mathbb{F}_p$ -vector space. (See for example [1, Corollary 11].) Remember that the discriminant module of a  $\mathbb{Z}_p$ -lattice  $L \subseteq W$  with  $Q(L) \subseteq \mathbb{Z}_p$  is the quotient  $L^\sharp/L$  where  $L^\sharp \subseteq W$  denotes the dual of  $L$  with respect to the bilinear pairing  $\langle \cdot, \cdot \rangle$ . Suppose that  $L$  is a unimodular lattice containing  $v$ . By [9, Lemma 1.1], the discriminant module of  $L \cap W$  is isomorphic to the discriminant module of  $L \cap \mathbb{Z}_p v$ . Thus, the discriminant module of  $L \cap W$  is an  $\mathbb{F}_p$ -vector space, which implies that  $L \cap W = L_W$ . In particular,  $L$  must contain  $\mathbb{Z}_p v \oplus L_W$ .

If  $\text{ord}_p(D) = 0$ , one easily checks that  $L_W$  is unimodular. Thus,  $\mathbb{Z}_p v \oplus L_W$  is the unique unimodular  $\mathbb{Z}_p$ -lattice containing  $v$ . If  $\text{ord}_p(D) = 1$ , then  $L_W$  is of index  $p$  in its dual. Thus, the discriminant module  $M$  of  $\mathbb{Z}_p v \oplus L_W$  is a 2-dimensional  $\mathbb{F}_p$ -vector space. A quick calculation shows that  $Q$  induces a hyperbolic form on  $M$ . There exist exactly two self-dual lattices containing  $\mathbb{Z}_p v \oplus L_W$  corresponding to the two isotropic lines in  $M$ .

Let  $\sigma_v$  be the unique simplex of  $\mathcal{T}$  corresponding to  $v$  and  $\mathbb{Q}(v)$  the  $\mathbb{Q}$ -subalgebra of  $B$  generated by  $v$ . Since  $\mathbb{Q}(v)^\times$  fixes  $v$ , it follows that it also fixes  $\sigma_v$ . The statements about the support of  $\Delta(v)$  follow from the  $B_{\mathbb{Q}_p}^\times$ -invariance of the reduction map.  $\square$

The space  $\mathcal{S}(V_{\mathbb{A}^p, \infty})$  of  $\mathbb{Z}$ -valued Schwartz–Bruhat functions on  $V_{\mathbb{A}^p, \infty}$  admits an action of  $B_{\mathbb{A}^p, \infty}^\times$  induced by the conjugation action on  $V_{\mathbb{A}^p, \infty}$ . Attached to an  $\hat{R}^\times$ -invariant function  $\Phi \in \mathcal{S}(V_{\mathbb{A}^p, \infty})$  and a non-zero rational number  $D$  is the zero-cycle

$$\Delta_{\Phi, \Gamma}(D) = \sum_{v \in \Gamma \backslash V, Q(v)=D} \frac{1}{\#\text{Stab}_\Gamma(v)} \Phi(v) \Delta(v) \in \text{Div}(\Gamma \backslash \mathcal{H}_p)_\mathbb{Q}, \tag{7}$$

on  $\Gamma \backslash \mathcal{H}_p$ . Note the formal similarities between (2) and (7) when  $\Phi$  is the characteristic function of  $\hat{R}_0$ , that will simply be denoted as  $1_{R_0}$ . We proceed to make them precise. By the theory of complex multiplication, the Heegner points appearing in the divisors  $\Delta(D)$  of the introduction are defined over  $\overline{\mathbb{Q}}$ . Hence, after fixing an embedding  $\overline{\mathbb{Q}} \subset \mathbb{C}_p$ , these divisors can be viewed as elements in  $\text{Div}(X(\mathbb{C}_p))$ .

**Proposition 2.5** *Let  $D \in \mathcal{D}_S$ . The cycle  $\Delta_{1_{R_0}, \Gamma}(D)$  of (7) viewed as an element of  $\text{Div}(X(\mathbb{C}_p))$  via the Cerednik–Drinfeld uniformization theorem is equal to the cycle  $\Delta(D)$  of (2).*

**Proof** Since  $R_0$  is a  $\mathbb{Z}[1/p]$ -lattice,

$$\Delta_{1_{R_0}, \Gamma}(D) = \Delta_{1_{R_0}, \Gamma}(Dp^{2n})$$

for every  $n \geq 0$ . On the other hand,  $\mathcal{B}$  is ramified at  $p$ , and, thus, the unique maximal  $\mathbb{Z}_p$ -order of  $\mathcal{B}_{\mathbb{Q}_p}$  is given by the elements whose reduced norm has non-negative  $p$ -adic valuation. Therefore, multiplication by  $p$  gives a bijection between elements of length  $D$  and elements of length  $Dp^2$  in  $\mathcal{R}_0$ , which yields the equality  $\Delta(D) = \Delta(Dp^{2n})$ , for every  $n \geq 0$ . It is then enough to prove the identification when  $D \in \mathcal{D}_S$  is such that  $\text{ord}_p(D) \in \{0, 1\}$ . The case when  $\text{ord}_p(D) = 0$  is treated in Theorem 5.3 of [3] and the case when  $\text{ord}_p(D) = 1$  follows from [29, Section 3.3] and [31, Proposition 5.12].  $\square$

Let  $\psi: \mathbb{Q} \backslash \mathbb{A} \rightarrow \mathbb{C}^\times$  be the standard character, that is, its local component  $\psi_\ell: \mathbb{Q}_\ell \rightarrow \mathbb{C}^\times$  at a prime  $\ell$  is given by

$$\psi_\ell(x) = e^{-2\pi i q} \quad \text{for } x \in q + \mathbb{Z}_\ell, \quad q \in \mathbb{Q}.$$

The Weil oscillator representation attached to  $V$  and  $\psi$  induces an action of the metaplectic group  $\widetilde{\text{SL}}_2(\mathbb{A}^{p, \infty})$  on  $\mathcal{S}(V_{\mathbb{A}^{p, \infty}})_{\mathbb{C}}$  that commutes with the  $B_{\mathbb{A}^{p, \infty}}^\times$ -action. (See, for example, Section 2.3 of [15] for its construction.) For  $M \geq 1$ , let  $K_0(M)^{(p)}$  be the subgroup of  $\text{SL}_2(\mathbb{A}^{p, \infty})$  consisting of matrices in  $\text{SL}_2(\hat{\mathbb{Z}}^{(p)})$  with left lower entry divisible by  $M$ . Since  $K_0(4M)^{(p)}$  splits the exact sequence

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{\text{SL}}_2(\mathbb{A}^{p, \infty}) \longrightarrow \text{SL}_2(\mathbb{A}^{p, \infty}) \longrightarrow 1$$

defining the metaplectic group, it can be regarded as a subgroup of  $\widetilde{\text{SL}}_2(\mathbb{A}^{p, \infty})$ . (See [15], Proposition 2.14.) We similarly define  $K_0(4M)$  and view it as a subgroup of  $\text{SL}_2(\mathbb{A}^\infty)$  and of  $\widetilde{\text{SL}}_2(\mathbb{A}^\infty)$ .

**Definition 2.6** A Schwartz–Bruhat function  $\Phi \in \mathcal{S}(V_{\mathbb{A}^{p, \infty}})$  is called special if

- (1)  $\Phi$  is  $\hat{R}^\times$ -invariant,
- (2)  $\Phi$  is  $K_0(4N)^{(p)}$ -invariant, and
- (3)  $\Phi(pv) = \Phi(v)$  for all  $v \in V_{\mathbb{A}^{p, \infty}}$ .

The characteristic function  $1_{R_0}$  is the prime example of a special Schwartz–Bruhat function. Let  $\Phi$  be a special Schwartz–Bruhat function. Property (2) implies that for  $D \in \mathbb{Z}_{(p)} - \{0\}$  we have  $\Delta_{\Phi, \Gamma}(D) = 0$  unless  $D \in \mathcal{D}_S$ : indeed, by [15, Theorem 2.22], the matrix

$$u_\ell = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}_\ell)$$

for  $\ell$  a prime different from  $p$  acts via the Weil representation by

$$(u_\ell \Phi)(v) = \psi_\ell(Q(v_\ell)) \cdot \Phi(v) \quad \forall \Phi \in \mathcal{S}(V_{\mathbb{A}^{p,\infty}})_{\mathbb{C}}, v = (v_\ell)_{\ell \neq p} \in V_{\mathbb{A}^{p,\infty}}.$$

Thus, if  $\Phi$  is  $K_0(4N)^{(p)}$ -invariant, it follows that every element  $v$  in the support of  $\Phi$  fulfils  $Q(v) \in \hat{\mathbb{Z}}^{(p)}$ . Furthermore, the equality

$$\Delta_{\Phi, \Gamma}(p^2 D) = \Delta_{\Phi, \Gamma}(D) \tag{8}$$

holds for all  $D \in \mathcal{D}_S$  by Property (3). The remainder of this work will solely be concerned in proving the following  $p$ -adic analytic version of the Gross–Kohnen–Zagier theorem, which implies Theorem 1.1 in view of Proposition 2.5 and the fact that  $1_{R_0}$  is special.

**Theorem 2.7** *Let  $\Phi$  be a special Schwartz–Bruhat function. The generating series*

$$G_{\Phi, \Gamma}(q) := \Phi(0)[\mathcal{L}^\vee] + \sum_{D \in \mathcal{D}_S} [\Delta_{\Phi, \Gamma}(D)] q^D \in \mathrm{Pic}(\Gamma \backslash \mathcal{H}_p)_{\mathbb{Q}}[[q]],$$

*is a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$ .*

**Remark 2.8** The divisors described above are compatible under pullback in the following sense. Let  $\Phi$  be a Schwartz–Bruhat function on  $V_{\mathbb{A}^{p,\infty}}$  invariant under  $\hat{R}^\times$ . Suppose that  $R'$  is an Eichler  $\mathbb{Z}[1/p]$ -order contained in  $R$ , denote by  $\Gamma'$  the group of reduced norm 1 units in  $R'$  modulo  $\{\pm 1\}$  and consider the projection map  $\pi : \Gamma' \backslash \mathcal{H}_p \rightarrow \Gamma \backslash \mathcal{H}_p$ . Then it can be seen in a similar way as in the proof of [25, Proposition 5.10] that, for every  $D \in \mathcal{D}_S$ ,

$$\pi^*(\Delta_{\Phi, \Gamma}(D)) = \Delta_{\Phi, \Gamma'}(D).$$

Using that  $\pi_* \circ \pi^*$  is equal to multiplication by the degree of  $\pi$  on  $\mathrm{Div}(\Gamma \backslash \mathcal{H}_p)$  and the previous identity, we deduce that  $R$  can be replaced by  $R'$  in the proof of Theorem 2.7. In particular, we will assume from now on that  $\Gamma$  is torsion-free by choosing an appropriate level  $N^+$ . This will simplify some calculations as the group  $\Gamma$  will act freely on  $\mathcal{H}_p$  and  $\mathcal{T}$ . In particular, it is a free group on finitely many generators. Moreover, under this assumption the coefficients of the divisors  $\Delta_{\Phi, \Gamma}(D)$  are integral.

### 2.3 Hecke action on divisors

Let  $\mathbb{T}^N$  be the integral Hecke algebra away from  $N$ , which is generated by the standard generators  $\{T_\ell\}_{\ell \nmid N}$ . (See [22, Section 1.2] for its definition.) We conclude the section describing the action of  $\mathbb{T}^N$  on the divisors as well as on the space of  $\hat{R}^\times$ -invariant Schwartz–Bruhat functions. Let  $\ell$  be a prime not dividing  $N$  and fix  $\alpha \in B^\times \cap R$  an element of reduced norm  $\ell$ . Consider the maps

$$\Gamma \backslash \mathcal{H}_p \xleftarrow{\pi_1} (\alpha^{-1}\Gamma\alpha \cap \Gamma) \backslash \mathcal{H}_p \xrightarrow{\alpha} (\Gamma \cap \alpha\Gamma\alpha^{-1}) \backslash \mathcal{H}_p \xrightarrow{\pi_2} \Gamma \backslash \mathcal{H}_p.$$

Then, define the action of the Hecke operator  $T_\ell$  on divisors as

$$T_\ell(\Delta) := (\pi_{2,*} \circ \alpha \circ \pi_1^*)(\Delta) \quad \text{for } \Delta \in \text{Div}(\Gamma \backslash \mathcal{H}_p).$$

On the other hand, the action of  $T_\ell$  on  $\hat{R}^\times$ -invariant Schwartz–Bruhat functions is determined as follows. If  $\Gamma = \sqcup_j (\Gamma \cap \alpha\Gamma\alpha^{-1})\delta_j$  for  $\{\delta_j\}_j \subset \Gamma$  we define

$$T_\ell(\Phi) := \sum_j \Phi \cdot (\alpha^{-1}\delta_j),$$

where if  $\beta \in B^\times$ ,  $\Phi \cdot \beta(v) := \Phi(\beta v \beta^{-1})$ . Note that, since  $\ell \nmid N$ ,  $R \cap \alpha R \alpha^{-1}$  is an Eichler  $\mathbb{Z}[1/p]$ -order. Hence, by strong approximation [38, Ch. III, §4], the double coset space  $(\hat{R} \cap \alpha \hat{R} \alpha^{-1})^\times \backslash \hat{B}^\times / B^\times$  has precisely one element. Using this, together with the fact that  $R \cap \alpha R \alpha^{-1}$  has an element of reduced norm  $p$  [2, Lemma 1.5], we deduce: for the same  $\{\delta_j\}_j \subset \Gamma$  as above  $\hat{R}^\times = \sqcup_j (\hat{R} \cap \alpha \hat{R}^\times \alpha^{-1})\delta_j$ . Hence,  $\hat{R}^\times \alpha^{-1} \hat{R}^\times = \sqcup_j \hat{R}^\times \alpha^{-1} \delta_j$  and it follows from there that  $T_\ell(\Phi)$  is  $\hat{R}^\times$ -invariant. It follows from this description that the Hecke action preserves the subspace of special Schwartz–Bruhat functions.

**Lemma 2.9** *Let  $\Phi$  be a Schwartz–Bruhat function on  $V_{\mathbb{A}^p, \infty}$  invariant under  $\hat{R}^\times$ . The following identity of divisors holds:*

$$T_\ell(\Delta_{\Phi, \Gamma}(D)) = \Delta_{T_\ell(\Phi), \Gamma}(D).$$

**Proof** Using that  $\Gamma$  is torsion-free, the equalities

$$\begin{aligned} T_\ell(\Delta_{\Phi, \Gamma}(D)) &= (\pi_{2,*} \circ \alpha \circ \pi_1^*)(\Delta_{\Phi, \Gamma}(D)) \\ &= (\pi_{2,*} \circ \alpha)(\Delta_{\Phi, \alpha^{-1}\Gamma\alpha \cap \Gamma}(D)) \\ &= \pi_{2,*} \circ \alpha \Delta_{\Phi \cdot \alpha^{-1}, \Gamma \cap \alpha\Gamma\alpha^{-1}}(D) \\ &= \Delta_{T_\ell(\Phi), \Gamma}(D) \end{aligned}$$

can be proven in the same way as [25, Prop. 5.9 and Prop. 5.10]. □

When it is clear from context that we are viewing  $\Phi$  as a  $\Gamma$ -invariant Schwartz–Bruhat function, we simply write  $\Delta_\Phi(D)$  (resp.  $G_\Phi$ ) to denote  $\Delta_{\Phi, \Gamma}(D)$  (resp.  $G_{\Phi, \Gamma}$ ).

### 3 Modularity of degrees of Heegner divisors

Fix a special Schwartz–Bruhat function  $\Phi$ . In this section, we prove that

$$\deg(G_\Phi)(q) = \Phi(0) \deg(\mathcal{L}^\vee) + \sum_{D \in \mathcal{D}_S} \deg(\Delta_\Phi(D))q^D$$

is a modular form by comparing  $\deg(G_\Phi)$  to a genus theta series attached to  $V$ .

Fix  $L_0, \dots, L_r$  unimodular  $\mathbb{Z}_p$ -lattices in  $V_{\mathbb{Q}_p}$  that give a set of representatives of  $\Gamma \backslash \mathcal{T}_0$ . For every  $i$  consider the ternary theta series attached to the Schwartz–Bruhat function  $\Phi \otimes 1_{L_i}$  on  $V_{\mathbb{A}^\infty}$

$$\Theta_i = \sum_{v \in V} \Phi(v) 1_{L_i}(v) q^{Q(v)}.$$

Note that theta series  $\Theta_i$  only depends on the class of  $L_i$  in  $R^\times \backslash \mathcal{T}_0$ . Since  $\Phi$  is invariant under  $K_0(4N/p)^{(p)}$  and  $L_i$  is a unimodular  $\mathbb{Z}_p$ -lattice,  $\Phi \otimes 1_{L_i}$  is invariant under  $K_0(4N/p)$ . It is well known that  $\Theta_i$  is a modular form of weight  $3/2$  and level  $\Gamma_0(4N/p)$  for every  $i$ . (See for example Theorem 4.1 of [4].) Define the modular form

$$E_\Phi := \sum_{i=1}^r \Theta_i.$$

The following lemma relates the degrees of those  $\Delta_\Phi(D)$  with  $\text{ord}_p(D) \in \{0, 1\}$  with the corresponding Fourier coefficients of  $E_\Phi$ .

**Lemma 3.1** *Let  $D \in \mathcal{D}_S$ , then:*

- (1) *if  $\text{ord}_p(D) = 0$ , then  $2a_D(E_\Phi) = \deg(\Delta_\Phi(D))$ , and*
- (2) *if  $\text{ord}_p(D) = 1$  then,  $a_D(E_\Phi) = \deg(\Delta_\Phi(D))$ .*

**Proof** Since  $\Gamma$  is torsion-free, it does not stabilize any vertex of  $\mathcal{T}_0$ . Thus, Lemma 2.4 implies that

$$\bigsqcup_{i=1}^r \{v \in V \cap L_i \mid Q(v) = D\} \xrightarrow{\sim} \{v \in \Gamma \backslash V \mid Q(v) = D\}$$

$$v \longmapsto [v]$$

is bijective if  $\text{ord}_p(D) = 0$  and surjective and two-to-one if  $\text{ord}_p(D) = 1$ , which implies the assertion. □

Let  $M \in \mathbb{Z}_{>0}$  and  $k$  such that  $2k \in \mathbb{Z}_{>0}$ . When  $k$  is a half-integer, we will always assume that the level  $M$  is divisible by 4. Denote by  $M_k(\Gamma_0(M))$  (resp.  $S_k(\Gamma_0(M))$ ) the space of modular forms (resp. cusp forms) forms of weight  $k$  and level  $\Gamma_0(M)$ . Consider

the subspaces  $M_k(\Gamma_0(M), \mathbb{Z})$  (resp.  $S_k(\Gamma_0(M), \mathbb{Z})$ ) of forms whose  $q$ -expansion has integer coefficients and, for any abelian group  $A$ , put

$$M_k(\Gamma_0(M), A) := M_k(\Gamma_0(M), \mathbb{Z}) \otimes_{\mathbb{Z}} A \text{ and } S_k(\Gamma_0(M), A) := S_k(\Gamma_0(M), \mathbb{Z}) \otimes_{\mathbb{Z}} A.$$

We view these as subspaces of the group  $A[[q]]$  of formal  $q$ -series with coefficients in  $A$ . By [33, Lemma 8] there exists a basis of  $M_k(\Gamma_0(M))$  consisting of forms with integer coefficients. Hence, the natural homomorphisms

$$M_k(\Gamma_0(M), \mathbb{C}) \xrightarrow{\sim} M_k(\Gamma_0(M)) \text{ and } S_k(\Gamma_0(M), \mathbb{C}) \xrightarrow{\sim} S_k(\Gamma_0(M))$$

are bijective. We now introduce several operators acting on  $M_{3/2}(\Gamma_0(M), A)$ . For that let

$$f = \sum_{n \geq 0} a_n q^n \in A[[q]]$$

be a formal  $q$ -series with coefficients in  $A$ . Following Shimura ([35, Theorem 1.7]), define

$$T_{p^2}(f) := \sum_{n \geq 0} \left( a_{p^2 n} + \left( \frac{-n}{p} \right) a_n + p a_{n/p^2} \right) q^n,$$

$$U_{p^2}(f) := \sum_{n \geq 0} a_{p^2 n} q^n,$$

and put  $V_{p^2} f := T_{p^2} f - U_{p^2} f$ . Now suppose that  $f \in M_{3/2}(\Gamma_0(M), R)$  is a modular form. Then  $T_{p^2} f \in M_{3/2}(\Gamma_0(M), R)$  if  $p \nmid M$  and, in case  $p \mid M$ , we have  $U_{p^2} f \in M_{3/2}(\Gamma_0(M), R)$ .

**Proposition 3.2** *The modular form  $E_\Phi$  is an Eisenstein series of level  $\Gamma_0(4N/p)$ . In particular, it satisfies  $T_{p^2}(E_\Phi) = (p + 1)E_\Phi$ .*

**Proof** Strong approximation implies that the modular form  $E_\Phi$  is the genus theta function associated to the Schwartz–Bruhat function  $\Phi \otimes 1_{L_i}$ . Thus  $E_\Phi$  is an Eisenstein series by the Siegel–Weil theorem [26, Theorem 4.1 (ii)]. A direct proof of this fact can be found in [23, Corollary 4.3]. Moreover, since  $\Phi$  is special, Theorem 4.2 of *loc.cit.* shows that  $E_\Phi$  is an eigenvector of  $T_{p^2}$  with eigenvalue  $(p + 1)$ .  $\square$

Since  $\Phi$  is special, (8) implies that

$$U_{p^2}(G_\Phi(q)) = G_\Phi(q).$$

We proceed to modify  $E_\Phi$  so that it becomes invariant under  $U_{p^2}$  as well. For that, put  $E_\Phi^1 := E - V_{p^2}(E) \in M_{3/2}(\Gamma_0(4N))$ .

**Corollary 3.3** *We have  $U_{p^2}(E_\Phi^1) = E_\Phi^1$ .*

**Proof** Since  $(U_{p^2} \circ V_{p^2})(E) = pE$  (which can be verified directly from the description of  $U_{p^2}$  and  $V_{p^2}$  given above), we have

$$U_{p^2}(E_\Phi^1) = U_{p^2}(E) - pE.$$

Using that  $U_{p^2} = T_{p^2} - V_{p^2}$  and Proposition 3.2 yields the desired result. □

We can finally prove the main result of this section.

**Proposition 3.4** *The equality  $\deg(G_\Phi)(q) = E_\Phi^1$  holds. In particular,  $\deg(G_\Phi)(q)$  is an Eisenstein series of weight  $3/2$  and level  $\Gamma_0(4N)$ .*

**Proof** By Lemma 3.3, the equality  $U_{p^2}(E_\Phi^1) = E_\Phi^1$  holds. On the other hand, since  $\Phi$  is special we have  $\deg(\Delta_\Phi(Dp^2)) = \deg(\Delta_\Phi(D))$  for all  $D$ . Hence, it is enough to verify that the Fourier coefficients of  $E_\Phi^1$  and of  $\deg(G_\Phi)$  are equal in the following cases:

- If  $\text{ord}_p(D) = 1$ , the second point of Lemma 3.1 implies

$$a_D(E_\Phi^1) = a_D(E_\Phi) = \deg(\Delta_\Phi(D)).$$

- If  $\text{ord}_p(D) = 0$  and  $\left(\frac{-D}{p}\right) = -1$ , the first point of Lemma 3.1 gives

$$a_D(E_\Phi^1) = 2a_D(E_\Phi) = \deg(\Delta_\Phi(D)).$$

- If  $\text{ord}_p(D) = 0$  and  $\left(\frac{-D}{p}\right) = 1$ , one calculates

$$a_D(E_\Phi^1) = a_D(E_\Phi) - a_D(E_\Phi) = 0.$$

On the other hand, we have that  $\Delta_\Phi(D) = 0$ , as  $G_\Phi(q)$  is supported only on non-negative integers that belong to  $\mathcal{D}_S$ .

- If  $D = 0$ , we have  $a_0(E_\Phi^1) = \Phi(0)(1 - p)r$ , where we recall that  $r = \#(\Gamma \backslash \mathcal{T}_0)$ . Now, since  $\Gamma$  is torsion-free, it follows that  $\Gamma \backslash \mathcal{T}$  is a  $(p + 1)$ -regular graph. Thus, we readily compute its first Betti number

$$g(\Gamma \backslash \mathcal{T}) = 1 - \#(\Gamma \backslash \mathcal{T}_0) + \#(\Gamma \backslash \mathcal{T}_1) = 1 - r + \frac{p + 1}{2}r,$$

which by [14, Theorem 5.4.1] equals the genus  $g$  of  $X$ . The degree of the cotangent bundle of  $X$  is equal to  $2g - 2$ . This implies that

$$a_0(E_\Phi^1) = \Phi(0)(1 - p)r = \Phi(0)(2 - 2g) = \Phi(0)\deg(\mathcal{L}^\vee).$$

Therefore, we obtain the desired equality  $\deg(G_\Phi) = E_\Phi^1$ . □

## 4 The Abel–Jacobi map

Because the curve  $X$  is a *Mumford curve* over  $\mathbb{Q}_p^2$ , its Jacobian, denoted by  $J$ , has purely toric reduction and admits a concrete description in terms of equivalence classes of automorphy factors of rigid meromorphic functions in  $\mathcal{H}_p$ . In this section, we explain how the class in  $J$  of a degree zero divisor can be described explicitly in these terms. Then, we introduce the notion of divisors of strong degree 0, for which there exists a preferred choice of automorphy factor describing its class in  $J$ . Finally, we use this notion to reduce Theorem 2.7 to the case where all divisors appearing as coefficients of the generating series  $G_\Phi$  have strong degree 0.

### 4.1 Definition and properties of the Abel–Jacobi map

A *formal divisor* on  $\mathcal{H}_p$  is a formal, possibly infinite  $\mathbb{Z}$ -linear combination of points in  $\mathcal{H}_p$ . A formal divisor

$$\hat{\mathcal{D}} = \sum_{x \in \mathcal{H}_p} m_x(x)$$

is said to be *discrete* if the formal divisor

$$\hat{\mathcal{D}} \cap \mathcal{A} := \sum_{x \in \mathcal{A}} m_x(x)$$

is an actual divisor, i.e., involves a finite sum for all standard affinoid subsets  $\mathcal{A} \subset \mathcal{H}_p$ . The set of all discrete formal divisors on  $\mathcal{H}_p$  is denoted by  $\text{Div}^\dagger(\mathcal{H}_p)$ . Denote by  $\text{Div}(\mathcal{H}_p)$  (resp.  $\text{Div}^0(\mathcal{H}_p)$ ) the subset of finite divisors (resp. finite divisors of degree 0). The quotient map  $\pi : \mathcal{H}_p \rightarrow \Gamma \backslash \mathcal{H}_p$  induces pushforward and pullback maps

$$\pi_* : \text{Div}(\mathcal{H}_p) \longrightarrow \text{Div}(\Gamma \backslash \mathcal{H}_p), \quad \pi^* : \text{Div}(\Gamma \backslash \mathcal{H}_p) \longrightarrow \text{Div}^\dagger(\mathcal{H}_p),$$

since  $\Gamma$  acts on  $\mathcal{H}_p$  with discrete orbits. Given  $\Delta \in \text{Div}(X(\mathbb{C}_p))$ , let  $\mathcal{D} \in \text{Div}(\mathcal{H}_p)$  and  $\hat{\mathcal{D}} \in \text{Div}^\dagger(\mathcal{H}_p)$  be (formal) divisors satisfying

$$\pi_*(\mathcal{D}) = \Delta, \quad \hat{\mathcal{D}} = \pi^*(\Delta). \quad (9)$$

The divisor  $\mathcal{D}$  is not unique, while the formal divisor  $\hat{\mathcal{D}}$  is completely determined by  $\Delta$ .

Given any degree zero divisor  $\mathcal{D}$  on  $C_V(\mathbb{C}_p) \simeq \mathbb{P}_1(\mathbb{C}_p)$ , there is a rational function  $f_{\mathcal{D}}$  on  $C_V(\mathbb{C}_p)$  having  $\mathcal{D}$  as a divisor, which is unique up to a multiplicative constant. A rational function  $f$  is extended multiplicatively to any divisor  $\mathcal{D} = \sum_{x \in C_V(\mathbb{C}_p)} m_x \cdot (x)$  by setting

$$f(\mathcal{D}) := \prod_{x \in C_V(\mathbb{C}_p)} f(x)^{m_x}.$$



**Definition 4.1** The *Weil symbol* attached to two degree zero divisors  $\mathcal{D}_0$  and  $\mathcal{D}_1$  on  $C_V(\mathbb{C}_p)$  with disjoint supports is the quantity

$$[\mathcal{D}_0; \mathcal{D}_1] := f_{\mathcal{D}_0}(\mathcal{D}_1) \in \mathbb{C}_p.$$

The Weil symbol generalises the familiar cross-ratio which one recovers when  $\mathcal{D}_0$  and  $\mathcal{D}_1$  are both differences of two points, and satisfies the following familiar properties:

(1) It is bilinear: for all degree zero divisors  $\mathcal{D}_0, \mathcal{D}_1$  and  $\mathcal{D}_2$

$$[\mathcal{D}_0; \mathcal{D}_1 + \mathcal{D}_2] = [\mathcal{D}_0; \mathcal{D}_1] \times [\mathcal{D}_0; \mathcal{D}_2], \quad [\mathcal{D}_0 + \mathcal{D}_1; \mathcal{D}_2] = [\mathcal{D}_0; \mathcal{D}_2] \times [\mathcal{D}_1; \mathcal{D}_2],$$

(2) It is  $B_{\mathbb{C}_p}^\times$ -equivariant:

$$[\gamma \mathcal{D}_0; \gamma \mathcal{D}_1] = [\mathcal{D}_0; \mathcal{D}_1] \quad \text{for all } \gamma \in B_{\mathbb{C}_p}^\times.$$

(3) It is symmetric (Weil reciprocity):

$$[\mathcal{D}_0; \mathcal{D}_1] = [\mathcal{D}_1; \mathcal{D}_0].$$

(4) Given any pair  $\mathcal{D}_0$  and  $\mathcal{D}_1$  of degree zero divisors on  $\mathcal{H}_p$  for which the support of  $\mathcal{D}_0$  is disjoint from the  $\Gamma$ -orbit of the support of  $\mathcal{D}_1$ , the infinite product

$$\left[ \mathcal{D}_0; \mathcal{D}_1 \right]_\Gamma := \prod_{\gamma \in \Gamma} [\mathcal{D}_0; \gamma \mathcal{D}_1]$$

converges absolutely in  $\mathbb{C}_p^\times$ . (See page 47 of [16].)

The quantity  $[\mathcal{D}_0; \mathcal{D}_1]_\Gamma$  is called the *modular Weil symbol* attached to the divisors  $\mathcal{D}_0$  and  $\mathcal{D}_1$  on  $\mathcal{H}_p$  and to the discrete  $p$ -arithmetic group  $\Gamma$ . It can be used to describe the Jacobian of  $X$  as follows: let  $L$  be a complete extension of  $\mathbb{Q}_{p^2}$ ,  $\mathcal{D} \in \text{Div}^0(\mathcal{H}_p(L))$  a divisor of degree 0 and choose  $\eta \in \mathcal{H}_p(L)$  such that  $(\eta)$  and  $\alpha \mathcal{D}$  have disjoint support for all  $\alpha \in \Gamma$ . Then, define  $\theta_{\mathcal{D}}$  via

$$\theta_{\mathcal{D}}(z) = [(z) - (\eta); \mathcal{D}]_\Gamma \quad \forall z \in \mathcal{H}_p(L).$$

Note that for  $\gamma \in \Gamma$  one gets

$$\frac{\theta_{\mathcal{D}}(\gamma z)}{\theta_{\mathcal{D}}(z)} = [(\gamma z) - (z); \mathcal{D}]_\Gamma = [(\gamma \eta) - (\eta); \mathcal{D}]_\Gamma \in L^\times, \tag{10}$$

where in the second equality we used that the modular Weil symbol is invariant under the action of  $\Gamma$  on any of the two divisors, and therefore  $[(\gamma z) - (\gamma \eta); \mathcal{D}]_\Gamma = [(z) - (\eta); \mathcal{D}]_\Gamma$ , which implies the desired equality by the linearity of the Weil symbol. Thus, the automorphy factor in (10) is independent of  $z$ . We then denote

$$j_{\mathcal{D}}(\gamma) = [(\gamma \eta) - (\eta); \mathcal{D}]_\Gamma. \tag{11}$$

The function  $j_{\mathcal{D}}$  defines an element in  $\text{Hom}(\Gamma, L^\times) = \text{Hom}(\Gamma_{\text{ab}}, L^\times)$ , where  $\Gamma_{\text{ab}}$  is the abelianization of  $\Gamma$ . Note that the group  $\Gamma_{\text{ab}}$  is a finitely generated free abelian group of rank equal to the genus  $g$  of the Shimura curve  $X$ . We need to introduce one more ingredient, the so-called *p-adic period pairing*. Define

$$\langle\langle \cdot, \cdot \rangle\rangle: \Gamma \times \Gamma \rightarrow \mathbb{C}_p^\times,$$

by choosing arbitrary base points  $\tau_1, \tau_2 \in \mathcal{H}_p$  that are not  $\Gamma$ -equivalent and setting

$$\langle\langle \gamma_1, \gamma_2 \rangle\rangle := [(\gamma_1 \tau_1) - (\tau_1); (\gamma_2 \tau_2) - (\tau_2)]_\Gamma.$$

In a similar way as above, it can be seen that this expression does not depend on the choice of  $\tau_1$  and  $\tau_2$ , and is a homomorphism in each argument. Moreover, it descends to a pairing  $\langle\langle \cdot, \cdot \rangle\rangle: \Gamma_{\text{ab}} \times \Gamma_{\text{ab}} \rightarrow \mathbb{Q}_p^\times$ , which gives an embedding (see VI.2 and VIII.3 of [16])

$$j: \Gamma_{\text{ab}} \hookrightarrow \text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_p^\times) \simeq (\mathbb{Q}_p^\times)^g.$$

Now, for a given  $\Delta \in \text{Div}^0(\Gamma \backslash \mathcal{H}_p(L))$ , choose  $\mathcal{D} \in \text{Div}^0(\mathcal{H}_p(L))$  such that  $\pi_* \mathcal{D} = \Delta$  and define

$$\text{AJ}: \text{Div}^0(\Gamma \backslash \mathcal{H}_p(L)) \longrightarrow \text{Hom}(\Gamma_{\text{ab}}, L^\times) / j(\Gamma_{\text{ab}}), \quad \Delta \longmapsto [j_{\mathcal{D}}].$$

It is a calculation to verify that the equivalence class of  $j_{\mathcal{D}}$  is independent of the choice of lift of  $\Delta$ , showing that the map AJ is well-defined. Remember that  $J$  denotes the Jacobian of the curve  $X$ .

**Proposition 4.2** *The map AJ defined above is trivial on the group of principal divisors and, for every complete extension  $L$  of  $\mathbb{Q}_{p^2}$ , it induces an identification*

$$J(L) \simeq \text{Hom}(\Gamma_{\text{ab}}, L^\times) / j(\Gamma_{\text{ab}}).$$

*Moreover, if  $L/\mathbb{Q}_{p^2}$  is a Galois extension, the identification is  $\text{Gal}(L/\mathbb{Q}_{p^2})$ -equivariant.*

**Proof** See VI.2. and VIII.4 of [16]. □

In view of the previous proposition, AJ can be interpreted as a *p*-adic Abel–Jacobi map. We also note that, by the positive definiteness of the pairing  $\text{ord}_p \circ \langle\langle \cdot, \cdot \rangle\rangle$  (see VI.2 and VIII.3 of [16]), the natural homomorphism from  $\text{Hom}(\Gamma_{\text{ab}}, \mathbb{Z}_p^\times)$  to  $\text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_{p^2}^\times) / j(\Gamma_{\text{ab}})$  is an injection, whose image has finite index. This gives the explicit description

$$J(\mathbb{Q}_{p^2})_{\mathbb{Q}} \simeq H^1(\Gamma_{\text{ab}}, \mathbb{Z}_p^\times)_{\mathbb{Q}}.$$

### 4.2 Divisors of strong degree 0

For any vertex  $L \in \mathcal{T}_0$ , consider the affinoid  $\mathcal{A}_L := \text{red}^{-1}(L) \subset \mathcal{H}_p$  and the wide open  $\mathcal{W}_L \subset \mathcal{H}_p$  given as the preimage by  $\text{red}$  of the union of the vertex  $L$  and all the (open) edges of  $\mathcal{T}$  that have  $L$  as one of its endpoints.

**Definition 4.3** Let  $\mathcal{D}$  be a finite divisor on  $\mathcal{H}_p$ .

- (1)  $\mathcal{D}$  is of *strong degree 0 in the even sense* if  $\mathcal{D} \cap \mathcal{W}_L$  is of degree 0 for every even vertex  $L \in \mathcal{T}_0$ , and  $\mathcal{D} \cap \mathcal{A}_L$  is of degree 0 for every odd vertex  $L \in \mathcal{T}_0$ .
- (2)  $\mathcal{D}$  is of *strong degree 0 in the odd sense* if  $\mathcal{D} \cap \mathcal{W}_L$  is of degree 0 for every odd vertex  $L \in \mathcal{T}_0$ , and  $\mathcal{D} \cap \mathcal{A}_L$  is of degree 0 for every even vertex  $L \in \mathcal{T}_0$ .

A divisor  $\Delta \in \text{Div}(\Gamma \backslash \mathcal{H}_p)$  is of *strong degree zero* if the following equivalent conditions hold:

- (1) There exists divisors  $\mathcal{D}_e, \mathcal{D}_o \in \text{Div}(\mathcal{H}_p)$  of strong degree 0 in an even and odd sense respectively such that  $\pi_*(\mathcal{D}_e) = \pi_*(\mathcal{D}_o) = \Delta$ .
- (2) The formal divisor  $\hat{\mathcal{D}} = \pi^* \Delta$  satisfies that, for every  $L \in \mathcal{T}_0$ , the divisors  $\hat{\mathcal{D}} \cap \mathcal{W}_L$  and  $\hat{\mathcal{D}} \cap \mathcal{A}_L$  have degree 0.

We denote by  $\text{Div}_s^0(\Gamma \backslash \mathcal{H}_p)$  the group of divisors of strong degree 0 on  $\Gamma \backslash \mathcal{H}_p$ . We also denote  $\text{Div}_{s,e}^0(\mathcal{H}_p)$  (resp.  $\text{Div}_{s,o}^0(\mathcal{H}_p)$ ) the group of divisors of strong degree 0 on  $\mathcal{H}_p$  in an even (resp. odd) sense. The motivation for these notions is explained in the next lemma.

**Lemma 4.4** Let  $\Delta$  be an element of  $\text{Div}_s^0(\Gamma \backslash \mathcal{H}_p)$ . The homomorphism  $j_{\mathcal{D}_e} \in \text{Hom}(\Gamma_{\text{ab}}, \mathbb{C}_p^\times)$  does not depend on a choice of  $\mathcal{D}_e \in \text{Div}_{s,e}^0(\mathcal{H}_p)$  with  $\pi_*(\mathcal{D}_e) = \Delta$ . In particular, the morphism

$$\text{Div}_s^0(\Gamma \backslash \mathcal{H}_p) \longrightarrow \text{Hom}(\Gamma_{\text{ab}}, \mathbb{C}_p^\times), \Delta \longmapsto j_{\mathcal{D}_e},$$

is a well-defined lift of the restriction of AJ to  $\text{Div}_s^0(\Gamma \backslash \mathcal{H}_p)$ . The same is true if one replaces  $e$  by  $o$  everywhere.

**Proof** Let  $\mathcal{D}, \mathcal{D}' \in \text{Div}_{s,e}^0(\mathcal{H}_p)$  be such that  $\pi_* \mathcal{D} = \pi_* \mathcal{D}' = \Delta$ . By the strong degree 0 assumption there exist vertices  $L_1, \dots, L_r \in \mathcal{T}_0$  and a decomposition

$$\mathcal{D} = \mathcal{D}_1 + \dots + \mathcal{D}_r$$

such that for  $1 \leq i \leq r$  the divisor  $\mathcal{D}_i$  is of degree 0 and supported on

- (1)  $\mathcal{W}_{L_i}$  if  $L_i$  is an even vertex, or
- (2)  $\mathcal{A}_{L_i}$  if  $L_i$  is an odd vertex.

Since  $j_{\mathcal{D}_j} = j_{\gamma \mathcal{D}_j}$ , for every  $\gamma \in \Gamma$ , we can suppose that the vertices  $L_1, \dots, L_r$  are not  $\Gamma$ -equivalent. Proceeding similarly for  $\mathcal{D}'$ , there exist lattices  $L'_1, \dots, L'_{r'} \in \mathcal{T}_0$

and degree 0 divisors  $\mathcal{D}'_1, \dots, \mathcal{D}'_{r'}$   $\in \text{Div}^0(\mathcal{H}_p)$  satisfying the same conditions as above. We have

$$\hat{\mathcal{D}} = \sum_{i=1}^r \sum_{\alpha \in \Gamma} \alpha \mathcal{D}_i = \sum_{i=1}^{r'} \sum_{\alpha \in \Gamma} \alpha \mathcal{D}'_i.$$

For  $\alpha \in \Gamma$ , the divisor  $\alpha \mathcal{D}_i$  has support in  $\mathcal{W}_{\alpha L_i}$  if  $L_i$  is even and has support in  $\mathcal{A}_{\alpha L_i}$  if  $L_i$  is odd. Note that these supports are disjoint when  $i$  varies from 1 to  $r$  and  $\alpha$  varies over  $\Gamma$ , as  $\Gamma$  does not stabilize any vertex because it is torsion-free. Moreover, the same holds for the divisors  $\alpha \mathcal{D}'_i$ . Thus, we conclude that  $r = r'$  and there exist  $\alpha_1, \dots, \alpha_r \in \Gamma$  such that

$$\mathcal{D}_i = \alpha_i \mathcal{D}'_i$$

for every  $i$  (after rearranging terms, if needed). We therefore have that  $j_{\mathcal{D}_i} = j_{\mathcal{D}'_i}$  for all  $i$  and the equality  $j_{\mathcal{D}} = j_{\mathcal{D}'}$  follows.  $\square$

If  $\Delta \in \text{Div}_s^0(\Gamma \backslash \mathcal{H}_p)$  is a divisor supported on preimages of vertices by the reduction map, both lifts  $\mathcal{D}_e$  and  $\mathcal{D}_o$  are divisors of strong degree 0 in an even sense and in an odd sense simultaneously. We will sometimes drop the subindices  $e$  and  $o$  in this case.

### 4.3 Reduction of the main theorem to convenient Schwartz–Bruhat functions

Recall the action of  $\mathbb{T}^N$  on  $\hat{R}^\times$ -invariant Schwartz–Bruhat functions introduced in Sect. 2. We can similarly define an action of  $\mathbb{T}^N$  on the space  $\text{Funct}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$  of  $\Gamma$ -invariant integral functions on  $\mathcal{T}_0$ .

**Definition 4.5** A Schwartz–Bruhat function  $\Phi$  on  $V_{\mathbb{A}^p, \infty}$  is *convenient* if it is special,  $\Phi(0) = 0$ , and for every  $D \in \mathcal{D}_S$  the divisor  $\Delta_\Phi(D)$  is of strong degree 0.

**Lemma 4.6** *Let  $\Phi$  be a special Schwartz–Bruhat function and let  $T \in \mathbb{T}^N$  be a Hecke operator that annihilates the space  $\text{Funct}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$ . Then, the Schwartz–Bruhat function  $T(\Phi)$  is convenient.*

**Proof** For  $L \in \mathcal{T}_0$ , denote by  $\delta_L$  the characteristic function of  $L$ . Define the homomorphism  $\text{deg}_{\mathcal{T}_0} : \text{Div}(\mathcal{H}_p) \rightarrow \text{Funct}(\mathcal{T}_0, \mathbb{Z})$  by

$$\text{deg}_{\mathcal{T}_0}((P)) = \begin{cases} \delta_L, & \text{if } \text{red}(P) = L \in \mathcal{T}_0, \\ \delta_L + \delta_{L'} & \text{if } \text{red}(P) = (L, L') \in \mathcal{T}_1. \end{cases}$$

This morphism is  $B^\times$ -equivariant, hence induces a  $\mathbb{T}^N$ -equivariant morphism

$$\text{deg}_{\mathcal{T}_0} : \text{Div}(\Gamma \backslash \mathcal{H}_p) \longrightarrow \text{Funct}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z}).$$

We proceed to verify that  $\Delta_{T(\Phi)}(D)$  is of strong degree 0 for a fixed  $D \in \mathcal{D}_S$ . From the Hecke equivariance of  $\text{deg}_{\mathcal{T}_0}$ , we have

$$\text{deg}_{\mathcal{T}_0}(\Delta_{T(\Phi)}(D)) = \text{deg}_{\mathcal{T}_0}(T(\Delta_\Phi(D))) = T(\text{deg}_{\mathcal{T}_0}(\Delta_\Phi(D))) = 0,$$

where we used Lemma 2.9 in the first equality. Since  $\Delta_{T(\Phi)}(D)$  is supported on preimages of vertices (resp. edges) if  $\text{ord}_p(D)$  is even (resp. odd), the fact that  $\text{deg}_{\mathcal{T}_0}(\Delta_{T(\Phi)}(D)) = 0$  implies that  $\Delta_{T(\Phi)}(D)$  is of strong degree 0. Finally, from the fact that  $T$  sends the constant functions on  $\text{Func}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$  to 0, it follows that  $(T(\Phi))(0) = 0$ .  $\square$

Let  $G_\Phi(q) \in \text{Pic}(\Gamma \backslash \mathcal{H}_p)[[q]]$  be the generating series introduced in Theorem 2.7. We now use the Jacquet–Langlands correspondence to justify that to prove Theorem 2.7 it is enough to prove it for the particular case where  $\Phi$  is convenient.

**Proposition 4.7** *The following statements are equivalent:*

- (1) *The generating series  $G_\Phi(q)$  is a modular form of weight  $3/2$  and level  $\Gamma_0(4N)$  for every special Schwartz–Bruhat function  $\Phi$ .*
- (2) *The generating series  $G_\Phi(q)$  is a cusp form of weight  $3/2$  and level  $\Gamma_0(4N)$  for every convenient Schwartz–Bruhat function  $\Phi$ .*

**Proof** Clearly (1) implies (2). We justify the reverse implication. By Jacquet–Langlands, we have:

- The action of  $\mathbb{T}^N$  on  $\text{Func}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$  factors through the action of the Hecke algebra (away from  $N$ ) on  $M_2(\Gamma_0(N/p), \mathbb{Q})$ .
- The action of  $\mathbb{T}^N$  on  $J(\mathbb{C}_p)_{\mathbb{Q}}$  factors through the action of the Hecke algebra (away from  $S$ ) on the space of forms in  $S_2(\Gamma_0(N), \mathbb{Q})$  that are new at  $p$ .

Let  $\ell$  be a prime such that  $\ell \notin S$  and denote by  $T_\ell \in \mathbb{T}^N$  the corresponding Hecke operator. From the second point and the fact that the map  $T_\ell - \ell - 1$  is an isomorphism on  $S_2(\Gamma_0(N), \mathbb{Q})$ , as the eigenvalues of  $T_\ell$  acting on  $S_2(\Gamma_0(N), \mathbb{Q})$  have absolute value less than or equal to  $2\sqrt{\ell}$  (see [13], [34], and [21]), we deduce that we have an isomorphism

$$\text{Pic}(X)(\mathbb{C}_p)_{\mathbb{Q}} \xrightarrow{\sim} J(\mathbb{C}_p)_{\mathbb{Q}} \oplus \mathbb{Q}, [\Delta] \longmapsto ((T_\ell - \ell - 1)\Delta, \text{deg}(\Delta)).$$

Let  $\Phi$  be a special Schwartz–Bruhat function. Since we proved that  $\text{deg}(G_\Phi)$  is a modular form in Proposition 3.4, after replacing  $\Phi$  by  $(T_\ell - \ell - 1)(\Phi)$  (and by Lemma 2.9) we may suppose that  $G_\Phi(q) \in J(\mathbb{C}_p)_{\mathbb{Q}}[[q]]$ . Now, choose  $T \in \mathbb{T}^N$  satisfying

- $T$  annihilates  $\text{Func}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$  and
- $T : J(\mathbb{C}_p)_{\mathbb{Q}} \rightarrow J(\mathbb{C}_p)_{\mathbb{Q}}$  is a bijection.

Such a  $T \in \mathbb{T}^N$  exists because there are Hecke operators acting on  $M_2(\Gamma_0(N), \mathbb{Q})$  which are 0 on  $M_2(\Gamma_0(N/p), \mathbb{Q})$  and are isomorphisms on the subspace of cusp forms which are new at  $p$ . By the first property and Lemma 4.6,  $T(\Phi)$  is convenient and therefore  $G_{T(\Phi)}(q) = T(G_\Phi(q))$  is a modular form by hypothesis. Here  $T(G_\Phi(q))$  denotes the  $q$ -expansion obtained by applying  $T$  to each of the coefficients of  $G_\Phi(q)$ . The fact that  $G_\Phi(q) \in J(\mathbb{C}_p)_{\mathbb{Q}}[[q]]$  is a modular form follows from the bijectivity of  $T$  on the Jacobian.  $\square$

Let  $\Phi$  be a convenient Schwartz–Bruhat function. For every  $D \in \mathcal{D}_S$ , fix  $\mathcal{D}_\Phi(D)_e \in \text{Div}_{s,e}^0(\mathcal{H}_p)$  and  $\mathcal{D}_\Phi(D)_o \in \text{Div}_{s,o}^0(\mathcal{H}_p)$  such that  $\pi_* \mathcal{D}_\Phi(D)_e = \pi_* \mathcal{D}_\Phi(D)_o =$

$\Delta_\Phi(D)$ . Note that, since the cycles  $\Delta(v)$  for  $v \in V$  introduced in Section 2.2 are invariant under the action of  $\text{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$ , these lifts of  $\Delta_\Phi(D)$  can be chosen such that they are invariant under the action of  $\text{Aut}(\mathbb{C}_p/\mathbb{Q}_p)$ . It follows from there that the homomorphisms  $j_{\mathcal{D}_\Phi(D)_e}$  and  $j_{\mathcal{D}_\Phi(D)_o}$  take values in  $\mathbb{Q}_p^\times$ . Consider the generating series

$$G_\Phi^+(q) = \sum_{D \in \mathcal{D}_S} j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o} q^D \in \text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_p^\times)[[q]].$$

Note that  $a_D(G_\Phi^+(q)) = a_{Dp^{2n}}(G_\Phi^+(q))$  for all  $n \geq 0$  since  $\Phi$  is special.

Observe that by Lemma 4.4, if we apply the quotient map

$$\text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_p^\times) \longrightarrow \text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_p^\times)/j(\Gamma)$$

to each of the coefficients of  $G_\Phi^+(q)$  we obtain  $2AJ(G_\Phi)(q)$ . Here  $2AJ(G_\Phi)(q)$  is the generating series obtained by applying  $2AJ$  to each of the coefficients of  $G_\Phi(q)$ . Hence, by Proposition 4.2 the modularity of  $G_\Phi^+(q)$  implies the modularity of  $G_\Phi(q)$ . The remainder of this article is dedicated to proving modularity of  $G_\Phi^+(q)$ .

The advantage of working with  $G_\Phi^+(q)$  over  $AJ(G_\Phi)(q)$  is that the group of continuous homomorphisms from  $\text{Hom}(\Gamma_{\text{ab}}, \mathbb{Q}_p^\times)$  to  $\mathbb{Q}_p$  can be described explicitly. Indeed, generators of this space are given by homomorphisms of the form  $j \mapsto \log_p(j(\gamma))$  respectively  $j \mapsto \text{ord}_p(j(\gamma))$  with  $\gamma \in \Gamma$ , where  $\log_p$  is the branch of the  $p$ -adic logarithm for which  $\log_p(p) = 0$ .

### 5 Values of $p$ -adic theta functions

The goal of this section is to give an explicit expression for the quantity  $j_{\mathcal{D}}(\gamma)$  when  $\gamma \in \Gamma$  is hyperbolic at  $p$  and  $\mathcal{D}$  is a divisor on  $\Gamma \setminus \mathcal{H}_p$  of strong degree 0. The formulas we will present have a similar flavor to the ones for toric values of lifting obstructions of rigid meromorphic cocycles given in [11, Section 5.3]. There, the orthogonal group of signature  $(3, 0)$  is replaced by an orthogonal group of signature  $(1, 2)$ .

Fix an element  $\gamma \in \Gamma$  that is hyperbolic at  $p$ . It has two distinct fixed points

$$\xi^+, \xi^- \in C_V(\mathbb{Q}_p)$$

on the boundary of  $\mathcal{H}_p$ . We order them in such a way that  $\xi^+$  and  $\xi^-$  are the attractive and repulsive fixed points of  $\gamma$ , i.e.,

$$\lim_{M \rightarrow +\infty} \gamma^M \tau = \xi^+, \quad \lim_{M \rightarrow -\infty} \gamma^M \tau = \xi^-,$$

for all  $\tau \in \mathcal{H}_p$ .

**Lemma 5.1** For every  $\mathcal{D} \in \text{Div}^0(\mathcal{H}_p)$  the following equality holds:

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}].$$

**Proof** Recall first from (11) that

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \Gamma} [(\alpha \gamma \tau) - (\alpha \tau); \mathcal{D}],$$

where  $\tau$  is an arbitrary base point in  $\mathcal{H}_p$ . Since this infinite product converges absolutely, it can be rearranged by grouping together the factors that belong to the same coset for  $\gamma^{\mathbb{Z}}$  in  $\Gamma$

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \Gamma / \gamma^{\mathbb{Z}}} \left( \prod_{i=-\infty}^{\infty} [(\alpha \gamma^{i+1} \tau) - (\alpha \gamma^i \tau); \mathcal{D}] \right).$$

The innermost product on the right hand side is equal to

$$\begin{aligned} \lim_{M \rightarrow \infty} \prod_{i=-M}^M [(\alpha \gamma^{i+1} \tau) - (\alpha \gamma^i \tau); \mathcal{D}] &= \lim_{M \rightarrow \infty} [(\alpha \gamma^{M+1} \tau) - (\alpha \gamma^{-M} \tau); \mathcal{D}] \\ &= [(\alpha \xi^+) - (\alpha \xi^-); \mathcal{D}]. \end{aligned}$$

It follows that

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \Gamma / \gamma^{\mathbb{Z}}} [(\alpha \xi^+) - (\alpha \xi^-); \mathcal{D}] = \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}],$$

where the last equation was obtained by substituting  $\alpha$  for  $\alpha^{-1}$  and exploiting the fact that the Weil Symbol is  $B_{\mathbb{Q}_p}^{\times}$ -equivariant.  $\square$

### 5.1 The quotient $\gamma^{\mathbb{Z}} \setminus \mathcal{T}$

We will rewrite the infinite product of Lemma 5.1 by making an explicit choice of coset representatives for  $\gamma^{\mathbb{Z}}$  in  $\Gamma$ , well adapted to the calculation at hand. To make this choice, we will exploit the action of  $\gamma^{\mathbb{Z}}$  on the Bruhat-Tits tree  $\mathcal{T}$ . We explain some of the properties of such action.

Since the element  $\gamma \in \Gamma$  is hyperbolic at  $p$ , and therefore its image in  $\text{SL}_2(\mathbb{Q}_p)$  diagonalizes by the fixed isomorphism  $B_{\mathbb{Q}_p} \simeq \text{M}_2(\mathbb{Q}_p)$ , we deduce that it acts by conjugation on  $V_{\mathbb{Q}_p}$  with three distinct eigenvalues  $\varpi$ , 1, and  $\varpi^{-1}$ , where  $\varpi$  is a global  $p$ -unit of norm 1 in the quadratic imaginary field that splits the characteristic polynomial of  $\gamma$  (relative to an embedding of this quadratic imaginary field into  $\mathbb{Q}_p$ ). The

valuation  $\text{ord}_p(\varpi) = 2t > 0$  is an even integer. Letting  $V[\lambda]$  denote the eigenspace in  $V$  on which  $\gamma$  acts as multiplication by  $\lambda$ , one obtains the decomposition

$$V_{\mathbb{Q}_p} = V[\varpi] \oplus V[\varpi^{-1}] \oplus V[1].$$

The first two eigenspaces are isotropic and together generate a hyperbolic plane in  $V_{\mathbb{Q}_p}$  whose orthogonal complement is  $V[1]$ . The fixed points  $\xi^+$  and  $\xi^-$  of  $\gamma$  in  $C_V(\mathbb{Q}_p)$  correspond to the isotropic lines  $V[\varpi]$  and  $V[\varpi^{-1}]$  respectively. Let  $w^+$  be a generator of  $V[\varpi]$ , let  $w^-$  be a generator of  $V[\varpi^{-1}]$ , and  $e$  a generator of  $V[1]$ . It follows from this discussion that we can scale these vectors so that

$$L_0 := \langle w^+, w^-, e \rangle$$

is a unimodular  $\mathbb{Z}_p$ -lattice of  $V_{\mathbb{Q}_p}$ . Then,  $L_0$  admits an eigenspace decomposition under the conjugation action of  $\gamma$  as a module over  $\mathbb{Z}_p$ . The same is true for the lattices

$$L_j := \langle w_j^+ := p^j w^+, w_j^- := p^{-j} w^-, e \rangle, \quad j \in \mathbb{Z}.$$

The unimodular lattices  $L_j$  and  $L_{j+1}$  are  $p$ -neighbours, and the element  $\gamma$  sends  $L_j$  to  $L_{j+2t}$ . The sequence of successive  $p$ -neighbours

$$g_\gamma = \{ \dots, L_{-2}, L_{-1}, L_0, L_1, L_2, L_3, \dots \}$$

determines an infinite geodesic on  $\mathcal{T}$  which is globally preserved by  $\gamma$ . We suppose that the scaling of the eigenvectors  $w^+$ ,  $w^-$  and  $e$  are chosen so that the lattice  $L_0$  is an even vertex of  $\mathcal{T}$ .

**Definition 5.2** Let  $L \subseteq V_{\mathbb{Q}_p}$  be a unimodular  $\mathbb{Z}_p$ -lattice in  $V_{\mathbb{Q}_p}$ . The lattice  $L_i \in g_\gamma$  that is closest to  $L$  is called the *parent* of  $L$ . The distance from  $L$  to its parent  $L_i$  is called the *depth* of  $L$  with respect to  $\gamma$ .

A fundamental region for  $\gamma^{\mathbb{Z}} \backslash \mathcal{T}_0$  can therefore be defined by setting

$$\mathcal{T}_{0,\gamma} := \{ L \in \mathcal{T}_0 \text{ with Parent}(L) \in \{L_0, L_1, L_2, \dots, L_{2t-1}\} \}.$$

The subset  $\mathcal{T}_{0,\gamma} \subset \mathcal{T}$  can be written as an increasing union of finite subsets

$$\mathcal{T}_{0,\gamma} = \bigcup_{n \geq 0} \mathcal{T}_{0,\gamma}^{\leq n}, \quad \mathcal{T}_{0,\gamma}^{\leq n} := \{ L \in \mathcal{T}_{0,\gamma} \text{ with depth}(L) \leq n \}.$$

Let  $\mathcal{A}_\gamma$  respectively  $\mathcal{A}_\gamma^{\leq n}$  be the subsets of  $\mathcal{H}_p$  given as the preimages of  $\mathcal{T}_{0,\gamma}$  respectively  $\mathcal{T}_{0,\gamma}^{\leq n}$  under the reduction map. The set  $\mathcal{A}_\gamma$  can thus be expressed as an increasing union of affinoid subsets,

$$\mathcal{A}_\gamma = \bigcup_{n \geq 0} \mathcal{A}_\gamma^{\leq n}. \tag{12}$$



Using  $\mathcal{T}_{0,\gamma}$ , we proceed to give several fundamental regions for  $\gamma^{\mathbb{Z}} \backslash \mathcal{T}_1$ . Define  $\mathcal{T}_{1,\gamma,e}$  to be the set of edges in  $\mathcal{T}_1$  such that its even vertex is in  $\mathcal{T}_{0,\gamma}$ . For a given vertex  $L \in \mathcal{T}_0$ , let  $W_L \subset \mathcal{T}_1$  be the set of open edges that have  $L$  as one of its endpoints. We then have,

$$\mathcal{T}_{1,\gamma,e} = \bigcup_{n \geq 0} \mathcal{T}_{1,\gamma,e}^{\leq n}, \quad \mathcal{T}_{1,\gamma,e}^{\leq n} := \bigcup_{\substack{L \text{ even} \\ L \in \mathcal{T}_{0,\gamma}^{\leq n}}} W_L.$$

Let  $\mathcal{W}_{\gamma,e}$  and  $\mathcal{W}_{\gamma,e}^{\leq n}$  be the preimage by red of  $\mathcal{T}_{1,\gamma,e}$  and  $\mathcal{T}_{1,\gamma,e}^{\leq n}$ , respectively. We then have

$$\mathcal{W}_{\gamma,e} = \bigcup_{n \geq 0} \mathcal{W}_{\gamma,e}^{\leq n}. \tag{13}$$

Observe that for every  $n$  the set  $\mathcal{W}_{\gamma,e}^{\leq n}$  can be written as the disjoint union of sets of the form  $\mathcal{W}_L - \mathcal{A}_L$ , where  $L$  runs over even vertices in  $\mathcal{T}_{0,\gamma}^{\leq n}$ . Similarly, define  $\mathcal{T}_{1,\gamma,o}$ ,  $\mathcal{T}_{1,\gamma,o}^{\leq n}$ ,  $\mathcal{W}_{\gamma,o}$  and  $\mathcal{W}_{\gamma,o}^{\leq n}$  by replacing even by odd everywhere.

### 5.2 Computation of $j_{\mathcal{D}}(\gamma)$ for divisors of strong degree 0

With the notations given in the previous section in place, we can prove the following formulas.

**Proposition 5.3** *Let  $\mathcal{D}$  be a divisor on  $\mathcal{H}_p$  of strong degree zero supported on preimages of vertices of  $\mathcal{T}$  under the reduction map. Let  $\hat{\mathcal{D}} = \sum_{\alpha \in \Gamma} \alpha \mathcal{D} \in \text{Div}^{\dagger}(\mathcal{H}_p)$ . Then,*

$$j_{\mathcal{D}}(\gamma) = \lim_{n \rightarrow \infty} [(\xi^+) - (\xi^-); \hat{\mathcal{D}} \cap \mathcal{A}_{\gamma}^{\leq n}].$$

**Proof** Since  $\mathcal{D}$  is of strong degree 0 we can write  $\mathcal{D} = \sum_{i=1}^r \mathcal{D}_{L_i}$ , where  $\mathcal{D}_{L_i}$  is a degree 0 divisor supported on  $\mathcal{A}_{L_i}$  and the  $L_i$  are vertices in  $\mathcal{T}_0$ . By Lemma 5.1, we have

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \backslash \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}] = \prod_{i=1}^r \prod_{\alpha \in \gamma^{\mathbb{Z}} \backslash \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}_{L_i}]. \tag{14}$$

Now, it follows from the definition of  $\mathcal{T}_{0,\gamma}$ , that for every  $i \in \{1, \dots, r\}$  and for every class  $[\alpha] \in \gamma^{\mathbb{Z}} \backslash \Gamma$ , there is precisely one representative  $\gamma^{k_i,\alpha}$  such that  $\gamma^{k_i,\alpha} \mathcal{D}_{L_i}$  is supported on  $\mathcal{A}_{\gamma}$ . This implies that if we write

$$\hat{\mathcal{D}} = \sum_{\alpha \in \Gamma} \alpha \mathcal{D} = \sum_{i=1}^r \sum_{\alpha \in \gamma^{\mathbb{Z}} \backslash \Gamma} \sum_{k=-\infty}^{+\infty} \gamma^k \alpha \mathcal{D}_{L_i},$$

we have

$$\hat{\mathcal{D}} \cap \mathcal{A}_\gamma = \sum_{i=1}^r \sum_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} \gamma^{k_{i,\alpha}} \alpha \mathcal{D}_{L_i}.$$

On the other hand, using (14) and that  $\xi^+$  and  $\xi^-$  are fixed by  $\gamma$  we can write

$$j_{\mathcal{D}}(\gamma) = \prod_{i=1}^r \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \gamma^{k_{i,\alpha}} \alpha \mathcal{D}_{L_i}].$$

Combining the last two equalities, and specifying the order of multiplication on the last expression for  $j_{\mathcal{D}}(\gamma)$  given by the increasing union of (12), we obtain the desired result. □

We can obtain similar expressions for  $j_{\mathcal{D}}(\gamma)$  when  $\mathcal{D}$  is a divisor of strong degree 0 supported on preimages of edges by the reduction map.

**Proposition 5.4** *Let  $\mathcal{D}$  be a divisor on  $\mathcal{H}_p$  supported on preimages of edges of  $\mathcal{T}$  by the reduction map. Let  $\hat{\mathcal{D}} = \sum_{\alpha \in \Gamma} \alpha \mathcal{D} \in \text{Div}^\dagger(\mathcal{H}_p)$ . We have:*

(1) *If  $\mathcal{D} = \mathcal{D}_e$  is of strong degree 0 in an even sense, then*

$$j_{\mathcal{D}_e}(\gamma) = \lim_{n \rightarrow +\infty} [(\xi^+) - (\xi^-); \hat{\mathcal{D}} \cap \mathcal{W}_{\gamma,e}^{\leq n}].$$

(2) *If  $\mathcal{D} = \mathcal{D}_o$  is of strong degree 0 in an odd sense, then*

$$j_{\mathcal{D}_o}(\gamma) = \lim_{n \rightarrow +\infty} [(\xi^+) - (\xi^-); \hat{\mathcal{D}} \cap \mathcal{W}_{\gamma,o}^{\leq n}].$$

**Proof** We only give the proof for the first case, the second being similar. Write  $\mathcal{D} = \sum_{i=1}^r \mathcal{D}_{L_i}$ , where  $\mathcal{D}_{L_i}$  is a degree 0 divisor supported on  $\mathcal{W}_{L_i} - \mathcal{A}_{L_i}$  and the set  $\{L_1, \dots, L_r\}$  consists of even vertices of  $\mathcal{T}_0$ . Hence, we have

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}] = \prod_{i=1}^r \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \alpha \mathcal{D}_{L_i}].$$

Now, for every class  $[\alpha] \in \gamma^{\mathbb{Z}} \setminus \Gamma$  and  $L_i$  even vertex as above, there exists precisely one representative  $\gamma^{k_{i,\alpha}} \alpha$  such that  $\gamma^{k_{i,\alpha}} \alpha L_i \in \mathcal{T}_{0,\gamma}$ . It follows from there that the divisor  $\gamma^{k_{i,\alpha}} \alpha \mathcal{D}_{L_i}$  is supported on  $\gamma^{k_{i,\alpha}} \alpha \mathcal{W}_{L_i} - \gamma^{k_{i,\alpha}} \alpha \mathcal{A}_{L_i} = \mathcal{W}_{\gamma^{k_{i,\alpha}} \alpha L_i} - \mathcal{A}_{\gamma^{k_{i,\alpha}} \alpha L_i} \subset \mathcal{W}_{\gamma,e}$ . This implies that if

$$\hat{\mathcal{D}} = \sum_{i=1}^r \sum_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} \sum_{k=-\infty}^{+\infty} \gamma^k \alpha \mathcal{D}_{L_i},$$

we have

$$\hat{\mathcal{D}} \cap \mathcal{W}_{\gamma,e} = \sum_{i=1}^r \sum_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} \gamma^{k_i, \alpha} \alpha \mathcal{D}_{L_i}.$$

On the other hand,

$$j_{\mathcal{D}}(\gamma) = \prod_{\alpha \in \gamma^{\mathbb{Z}} \setminus \Gamma} [(\xi^+) - (\xi^-); \gamma^{k_i, \alpha} \alpha \mathcal{D}_{L_i}]. \tag{15}$$

Note that  $\mathcal{W}_{\gamma,e}^{\leq n}$  can be written as a union of sets of the form  $\mathcal{W}_{L_i} - \mathcal{A}_{L_i}$ , where the union is over even vertices in  $\mathcal{T}_{0,\gamma}^{\leq n}$ . Hence,  $\hat{\mathcal{D}} \cap \mathcal{W}_{\gamma,e}^{\leq n}$  is a degree 0 divisor (because  $\mathcal{D}$  is of strong degree 0). Moreover, the increasing union over  $n$  of the sets  $\mathcal{W}_{\gamma,e}^{\leq n}$  covers  $\mathcal{W}_{\gamma,e}$ , as we deduced in (13). This implies that we can use these sets to specify an order of multiplication on (15) to obtain the desired expression.  $\square$

### 6 Abel–Jacobi images of Heegner divisors

We use the results of Sect. 5 to compute Abel–Jacobi images of the Heegner divisors introduced in Sect. 2. More precisely, let  $\Phi$  be a convenient Schwartz–Bruhat function and fix  $D \in \mathcal{D}_S$ . Choose  $\mathcal{D}_{\Phi}(D)_e, \mathcal{D}_{\Phi}(D)_o$  divisors on  $\mathcal{H}_p$  of strong degree 0 in an even and odd sense respectively that lift  $\Delta_{\Phi}(D)$  and let  $\hat{\mathcal{D}}_{\Phi}(D) = \pi^* \Delta_{\Phi}(D)$ . If  $\Delta_{\Phi}(D)$  is supported on preimages of vertices under the reduction map, we suppose that  $\mathcal{D}_{\Phi}(D)_e = \mathcal{D}_{\Phi}(D)_o$  and we will drop the subindices  $e$  and  $o$ . At last, let  $\gamma \in \Gamma$  be an element hyperbolic at  $p$ . We will compute  $j_{\mathcal{D}_{\Phi}(D)_e} \cdot j_{\mathcal{D}_{\Phi}(D)_o}(\gamma)$ .

#### 6.1 Values of theta functions associated to Heegner divisors

Since  $\Phi$  is invariant under multiplication by  $p$ , we have  $\Delta_{\Phi}(D) = \Delta_{\Phi}(Dp^{2n})$  for every  $n \geq 0$ . Therefore, we will assume here and for the rest of the section that  $D$  is an element of  $\mathcal{D}_S$  with  $\text{ord}_p(D) \in \{0, 1\}$ . In view of the notion of depth of a lattice with respect to  $\gamma$ , which was introduced in Definition 5.2, the following definition will be relevant.

**Definition 6.1** Let  $v \in V$  be a vector such that  $Q(v) = D$ . The *depth* of  $v$  with respect to  $\gamma$  is

$$\text{depth}(v) := \min_{L \ni v} \{\text{depth}(L)\},$$

where the minimum is taken over all unimodular  $\mathbb{Z}_p$ -lattices in  $V_{\mathbb{Q}_p}$  such that  $v \in L$ .

Note that by Lemma 2.4 there exist at most two unimodular  $\mathbb{Z}_p$ -lattices containing  $v$ . If there is a unique unimodular lattice containing  $v$ , denote it by  $L_v \in \mathcal{T}_0$ . If there are two unimodular lattices containing  $v$ , denote by  $e_v \in \mathcal{T}_1$  the edge connecting them.

We now present the computation of  $j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma)$ , which is slightly different according to the  $p$ -adic valuation of  $D$ . Let  $n \geq 1$ . If  $\text{ord}_p(D) = 0$ , consider

$$\hat{\mathcal{D}}_\Phi(D) \cap \mathcal{A}_\gamma^{\leq n} = \sum_{\substack{Q(v)=D, \\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \Phi(v)\Delta(v),$$

where the sum is over the vectors  $v \in V$ . Since this divisor is of degree zero, as  $\Phi$  is convenient, the function on  $C_V$  given by

$$\xi \mapsto \prod_{\substack{Q(v)=D, \\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \langle \tilde{\xi}, v \rangle^{\Phi(v)},$$

where  $\tilde{\xi}$  is any vector in the isotropic line generated by  $\xi$  in  $V_{\mathbb{C}_p}$ , is well-defined and has divisor equal to  $\hat{\mathcal{D}}_\Phi(D) \cap \mathcal{A}_\gamma^{\leq n}$ . Therefore, if  $\tilde{\xi}^+$  and  $\tilde{\xi}^-$  are vectors in  $V_{\mathbb{C}_p}$  generating the  $\mathbb{C}_p$ -lines  $\xi^+$  and  $\xi^-$  introduced in Sect. 5, Proposition 5.3 implies

$$j_{\mathcal{D}_\Phi(D)}(\gamma) = \lim_{n \rightarrow \infty} \prod_{\substack{Q(v)=D, \\ L_v \in \mathcal{T}_{0,\gamma}^{\leq n}}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)} = \lim_{n \rightarrow \infty} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \setminus V \\ Q(v)=D, \\ \text{depth}(v) \leq n}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)}. \tag{16}$$

Here, the second equality follows from the fact that the terms appearing in the expression of the middle do not change if we replace  $v$  by  $\gamma v$ . If  $\text{ord}_p(D) = 1$ , we can proceed similarly. In that case, the function on  $C_V(\mathbb{C}_p)$  given by

$$\xi \mapsto \prod_{\substack{Q(v)=D \\ e_v \in \mathcal{T}_{1,\gamma,e}^{\leq n}}} \langle \xi, v \rangle^{\Phi(v)} \prod_{\substack{Q(v)=D \\ e_v \in \mathcal{T}_{1,\gamma,o}^{\leq n}}} \langle \xi, v \rangle^{\Phi(v)}$$

has divisor equal to

$$\hat{\mathcal{D}}_\Phi(D) \cap \mathcal{W}_{\gamma,e}^{\leq n} + \hat{\mathcal{D}}_\Phi(D) \cap \mathcal{W}_{\gamma,o}^{\leq n}.$$

It then follows from Proposition 5.4 that

$$\begin{aligned} j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma) &= \lim_{n \rightarrow +\infty} \prod_{\substack{Q(v)=D \\ e_v \in \mathcal{T}_{1,\gamma,e}^{\leq n}}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)} \prod_{\substack{Q(v)=D \\ e_v \in \mathcal{T}_{1,\gamma,o}^{\leq n}}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)} \\ &= \lim_{n \rightarrow +\infty} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \setminus V \\ Q(v)=D \\ \text{depth}(v) \leq n-1}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)} \prod_{\substack{v \in \gamma^{\mathbb{Z}} \setminus V \\ Q(v)=D \\ \text{depth}(v) \leq n}} \left( \frac{\langle \tilde{\xi}^+, v \rangle}{\langle \tilde{\xi}^-, v \rangle} \right)^{\Phi(v)}, \end{aligned} \tag{17}$$

where the second equality is obtained by rearranging the terms of the products in the following way. Recall that  $\mathcal{T}_{1,\gamma,e}^{\leq n}$  is the union of edges that have an endpoint in  $\mathcal{T}_{0,\gamma}$  which is even and at distance less than or equal to  $n$  from the geodesic  $g_\gamma$ . Similarly,  $\mathcal{T}_{1,\gamma,o}^{\leq n}$  is given by replacing even by odd in the definition above. It follows from this description that, up to  $\gamma^{\mathbb{Z}}$ -equivariance, the double product in the first equality considers vectors  $v \in V$  such that  $e_v$  is at distance less than or equal than  $n - 1$  twice (as  $e_v$ , or a  $\gamma^{\mathbb{Z}}$ -translate of it, belongs to both  $\mathcal{T}_{1,\gamma,e}^{\leq n}$  and  $\mathcal{T}_{1,\gamma,o}^{\leq n}$ ). On the other hand, the double product of the first equality considers  $v \in V$  such that  $e_v$  is at distance equal to  $n$  exactly once (as  $e_v$  belongs to either  $\mathcal{T}_{1,\gamma,e}^{\leq n}$  and  $\mathcal{T}_{1,\gamma,o}^{\leq n}$ ). The equality can be deduced from there and from the fact that the expression  $\langle \tilde{\xi}^+, v \rangle^{\Phi(v)} / \langle \tilde{\xi}^-, v \rangle^{\Phi(v)}$  does not change if  $v$  is replaced by  $\gamma v$ .

### 6.2 Vectors of length $D$ in $\gamma^{\mathbb{Z}} \backslash V$

Recall that  $D \in \mathcal{D}_S$  is such that  $\text{ord}_p(D) \in \{0, 1\}$ . We give a concrete choice of representatives of the quotient

$$\left\{ v \in \gamma^{\mathbb{Z}} \backslash V \mid Q(v) = D, \text{depth}(v) \leq n \right\}. \tag{18}$$

This will lead to a relation between  $j_{\mathcal{D}_\Phi(D)}(\gamma)$  and Fourier coefficients of theta series in the next section. We begin by recalling some of the notation introduced in Sect. 5.1. Recall that the conjugation action of  $\gamma$  on  $V_{\mathbb{Q}_p}$  diagonalizes, with eigenvalues  $\{\varpi, \varpi^{-1}, 1\}$ , and that  $\text{ord}_p(\varpi) = 2t \in 2\mathbb{Z}$ . Let  $w^+, w^-, e$  be the corresponding eigenvectors, scaled so that  $L_0 = \langle w^+, w^-, e \rangle$  is a unimodular  $\mathbb{Z}_p$ -lattice of  $V_{\mathbb{Q}_p}$ . For  $j \in \mathbb{Z}$ , let

$$w_j^+ := p^j w^+, \quad w_j^- := p^{-j} w^-$$

and consider the  $\mathbb{Z}_p$ -lattice  $L_j = \langle w_j^+, w_j^-, e \rangle$ . Then, as discussed in Sect. 5,  $L_0, L_1, \dots, L_{2t-1}$  form a set of representatives modulo  $\gamma^{\mathbb{Z}}$  of the vertices in the geodesic of  $\mathcal{T}$  stabilized by  $\gamma$ .

Define

$$L_j^+[n] := \left\{ v \in L_j \cap V \mid Q(v) = Dp^{2n} \text{ and } \langle v, w_j^+ \rangle \in \mathbb{Z}_p^\times \right\}$$

and define  $L_j^-[n]$  in a similar way as above but replacing the symbol  $+$  by the symbol  $-$  everywhere. The motivation for the definition of  $L_j^+[n]$  and  $L_j^-[n]$  is the following. Let  $\mathcal{T}_j^+[n]$  be the subset of  $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$  of elements  $x \in \mathcal{T}$  that are at distance equal to  $n$  from  $L_j$  and satisfy that:

- (1) If  $x = L$  is a vertex and  $\text{Parent}(L) = L_k$ , then  $k \geq j$ .
- (2) If  $x$  is an edge, for any of its endpoints  $L$  we have that if  $\text{Parent}(L) = L_k$ , then  $k \geq j$ .

Define  $\mathcal{T}_j^- [n]$  in a similar way but replacing the symbol  $\geq$  by the symbol  $\leq$  everywhere. It then follows from the description of the action of  $\gamma^{\mathbb{Z}}$  in  $\mathcal{T}$  that the disjoint union

$$\bigcup_{j=0}^{2t-1} \left\{ v \in V \mid Q(v) = D, \text{red}(\Delta(v)) \subset \mathcal{T}_j^+ [n] \right\}$$

gives a set of representatives of (18). Similarly, the same holds if we replace the symbol  $+$  by the symbol  $-$ .

**Lemma 6.2** *The map*

$$L_j^+ [n] \xrightarrow{\sim} \left\{ v \in V \mid Q(v) = D, \text{red}(\Delta(v)) \subset \mathcal{T}_j^+ [n] \right\}, u \mapsto u/p^n$$

*is bijective. The same result holds if we replace the symbol  $+$  by the symbol  $-$  everywhere.*

**Proof** We start proving that the map is well-defined. Let  $u \in L_j^+ [n]$  and let  $v = u/p^n$ . Since  $u$  is primitive,  $L_v$  (resp.  $e_v$ ) is at distance  $n$  from  $L_j$  if  $\text{ord}_p(D) = 0$  (resp.  $\text{ord}_p(D) = 1$ ). Moreover, the condition  $\langle u, w_j^+ \rangle \in \mathbb{Z}_p^\times$  implies that if for any unimodular lattice  $L$  containing  $v$  we denote  $\text{Parent}(L) = L_k$ , we have  $k \geq j$ . Hence,  $\text{red}(\Delta(v)) \subset \mathcal{T}_j^+ [n]$ .

The injectivity of the map is clear, so we are left to prove surjectivity. For that, let  $v \in V$  be such that  $Q(v) = D$  and  $\text{red}(\Delta(v)) \subset \mathcal{T}_j^+ [n]$ . Since there is a unimodular  $\mathbb{Z}_p$ -lattice in  $V_{\mathbb{Q}_p}$  containing  $v$  at distance  $n$  from  $L_j$ , we have that  $p^n v \in L_j$ . Note that  $\langle p^n v, w_j^+ \rangle \neq 0$ . Indeed, for the sake of contradiction suppose that  $\langle p^n v, w_j^+ \rangle = 0$ . This implies that

$$p^n v = aw_j^+ + be,$$

for  $a, b \in \mathbb{Z}_p$ . Then,

$$\gamma \cdot (p^n v) = a\varpi w_j^+ + be.$$

Subtracting these two equations, we get that  $\gamma v - v \in V$  is either 0 or it is an eigenvector for the  $\mathbb{Q}$ -linear action of  $\gamma$  on  $V$  of eigenvalue  $\varpi$ . Since  $\gamma v - v \in V$  and  $\varpi \notin \mathbb{Q}$ , the only possibility is that  $\gamma v - v = 0$ . This implies that  $v \in \langle e \rangle$ , giving a contradiction with the fact that  $\sqrt{-D} \notin \mathbb{Q}_p$ . We can therefore choose  $i \leq j$  such that  $p^n v \in L_i^+ [n]$ . Now, the fact that the map is well-defined applied to the index  $i$ , together with the observation that the sets  $\mathcal{T}_i^+ [n]$  and  $\mathcal{T}_j^+ [n]$  are disjoint if  $i \neq j$  proves that  $i = j$  and we are done.  $\square$

We can combine the information of Lemma 6.2 for  $j = 0, \dots, 2t - 1$  to obtain the following result.

**Proposition 6.3** *Let  $n \geq 0$ , we have a bijection*

$$L_0^+[n] \cup \dots \cup L_{2t-1}^+[n] \xrightarrow{\sim} \left\{ v \in \gamma^{\mathbb{Z}} \setminus V \mid Q(v) = D, \text{depth}(v) \leq n \right\}$$

given by  $v \mapsto [p^{-n}v]$ , where  $[p^{-n}v]$  denotes the class of  $p^{-n}v \in V$  modulo  $\gamma^{\mathbb{Z}}$ . Moreover, the same result is true if we replace the symbol  $+$  by the symbol  $-$ .

**Proof** By Lemma 6.2 we have that the map

$$L_0^+[n] \cup \dots \cup L_{2t-1}^+[n] \xrightarrow{\sim} \bigcup_{j=0}^{2t-1} \left\{ v \in V \mid Q(v) = D, \text{red}(\Delta(v)) \subset \mathcal{T}_j^+[n] \right\},$$

$$u \longmapsto u/p^n$$

is bijective. We conclude the proof by recalling that the right hand side gives a set of representatives of

$$\left\{ v \in \gamma^{\mathbb{Z}} \setminus V \mid Q(v) = D, \text{depth}(v) \leq n \right\}.$$

□

As a consequence, we obtain the following expression for  $j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma)$ .

**Theorem 6.4** *Consider the same notation as above.*

(1) *If  $\text{ord}_p(D) = 0$ , we have*

$$j_{\mathcal{D}_\Phi(D)}(\gamma) = \lim_{n \rightarrow +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n]} \langle w_0^+, v \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n]} \langle w_0^-, v \rangle^{\Phi(v)}}.$$

(2) *If  $\text{ord}_p(D) = 1$ , we have*

$$j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma) = \lim_{n \rightarrow +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n] \cup L_j^+[n+1]} \langle w_0^+, v \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n] \cup L_j^-[n+1]} \langle w_0^-, v \rangle^{\Phi(v)}}.$$

**Proof** Suppose that  $\text{ord}_p(D) = 0$ . By (16) and Proposition 6.3, we have

$$j_{\mathcal{D}_\Phi(D)}(\gamma) = \lim_{n \rightarrow +\infty} \prod_{j=0}^{2t-1} \frac{\prod_{v \in L_j^+[n]} \langle w_0^+, vp^{-n} \rangle^{\Phi(v)}}{\prod_{v \in L_j^-[n]} \langle w_0^-, vp^{-n} \rangle^{\Phi(v)}}.$$

Here we used that  $w_0^+$  (resp.  $w_0^-$ ) generates the line  $\xi^+$  (resp.  $\xi^-$ ) and that  $\Phi(pv) = \Phi(v)$  for every  $v \in V$ . Since the divisor  $\mathcal{D}_\Phi(D)$  is of degree 0, the product of the factors  $p^{-n\Phi(v)}$  is equal to 1, leading to the desired expression. The case when  $\text{ord}_p(D) = 1$  is proven in an analogous way, but using (17), instead of (16). □

### 6.3 Computation of $p$ -adic valuations

We end the section by using the previous calculations to compute the  $p$ -adic valuation of  $j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma)$ .

**Proposition 6.5** *Let  $\Phi$  be a convenient Schwartz–Bruhat function. Then*

$$\text{ord}_p(j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma)) = 0$$

for all  $\gamma \in \Gamma$ . In particular, we have

$$G_\Phi^+(q) \in \text{Hom}(\Gamma, \mathbb{Z}_p^\times)[[q]].$$

**Proof** Using Theorem 6.4, together with the fact that  $\text{ord}_p(\langle w_0^+, v \rangle) = -j$  if  $v \in L_j^+[n]$ , and  $\text{ord}_p(\langle w_0^-, v \rangle) = j$  if  $v \in L_j^-[n]$ , we deduce that it is enough to show that, for every  $j \in \{0, \dots, 2t - 1\}$  and for every  $n \geq 0$ , we have

$$\sum_{v \in L_j^+[n]} \Phi(v) + \sum_{v \in L_j^+[n+1]} \Phi(v) = 0$$

and that the same statement replacing the symbol  $+$  with  $-$  everywhere holds (which is proven analogously). Note that the quantity on the left hand side can be interpreted as follows. Denote by  $\mathcal{A}_j^+[n]$  the preimage of  $T_j^+[n]$  under the reduction map. Then,

$$\begin{aligned} & \hat{\mathcal{D}}_\Phi(D) \cap (\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1]) \\ &= \sum_{\substack{v \in V \\ Q(v)=D \\ \text{red}(\Delta(v)) \subset T_j^+[n]}} \Phi(v)\Delta(v) + \sum_{\substack{v \in V \\ Q(v)=D \\ \text{red}(\Delta(v)) \subset T_j^+[n+1]}} \Phi(v)\Delta(v), \end{aligned}$$

and

$$\begin{aligned} & \text{deg} \left( \hat{\mathcal{D}}_\Phi(D) \cap (\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1]) \right) \\ &= \sum_{\substack{v \in V \\ Q(v)=D \\ \text{red}(\Delta(v)) \subset T_j^+[n]}} 2\Phi(v) + \sum_{\substack{v \in V \\ Q(v)=D \\ \text{red}(\Delta(v)) \subset T_j^+[n+1]}} 2\Phi(v) \\ &= \sum_{v \in L_j^+[n]} 2\Phi(v) + \sum_{v \in L_j^+[n+1]} 2\Phi(v), \end{aligned}$$

where in the last equality we used Lemma 6.2 together with the fact that  $\Phi$  is invariant under multiplication by  $p$ . Recall that for a given lattice  $L$  we defined the wide open



$\mathcal{W}_L \subset \mathcal{H}_p$  in Sect. 4.2. Since we have the disjoint union

$$\mathcal{A}_j^+[n] \cup \mathcal{A}_j^+[n+1] = \bigcup_{L \in \mathcal{T}_j^+[n+1] \cap \mathcal{T}_0} \mathcal{W}_L \cup \bigcup_{L \in \mathcal{T}_j^+[n] \cap \mathcal{T}_0} \mathcal{A}_L$$

and  $\Delta_\Phi(D)$  is of strong degree 0, the result follows. □

### 7 First order $p$ -adic deformations of ternary theta series

Fix a convenient Schwartz–Bruhat function  $\Phi$ . By Proposition 6.5 above we know that  $G_\Phi^+(q)$  belongs to  $\text{Hom}(\Gamma_{\text{ab}}, \mathbb{Z}_p^\times)[[q]]$ . In order to prove modularity of  $G_\Phi^+(q)$  it is therefore enough to prove that

$$\log_\gamma(G_\Phi^+(q)) := \sum_{D \in \mathcal{D}_S} \log_p(j_{\mathcal{D}_\Phi(D)_e} \cdot j_{\mathcal{D}_\Phi(D)_o}(\gamma))q^D \in \mathbb{Q}_p[[q]]$$

is a modular form for every  $\gamma \in \Gamma$  hyperbolic at  $p$ , which we fix from now on. Here  $\log_p$  denotes the branch of the  $p$ -adic logarithm such that  $\log_p(p) = 0$ . In this section, we use  $\gamma$  and  $\Phi$  to construct a  $p$ -adic family of theta series  $\Theta_k$ , of weight  $k + 3/2$  and level  $\Gamma_0(4N)$ , satisfying the following two properties. First,  $\Theta_0 = 0$ . Second, if we denote by  $\Theta'_0$  the derivative with respect to the  $p$ -adic variable  $k$  evaluated at  $k = 0$ , and  $e_{\text{ord}}$  the so-called  $p$ -ordinary projector, then  $e_{\text{ord}}(\Theta'_0) \in S_{3/2}(\Gamma_0(4N), \mathbb{Q}_p)$ . Furthermore, the generating series  $\log_\gamma(G_\Phi^+(q))$  is the projection to the  $U_{p^2} = 1$  eigenspace of  $2e_{\text{ord}}\Theta'_0$ . In particular, it is a cusp form of weight  $3/2$  and level  $\Gamma_0(4N)$ , which proves Theorem 2.7.

#### 7.1 Ordinary subspaces

Let  $k$  be a non-negative integer. For  $\ell \nmid 2N$  denote by  $T_{\ell^2}$  the associated Hecke operator acting on  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z})$  and if  $\ell \mid N$  denote by  $U_{\ell^2}$  the associated Hecke operator acting on  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z})$ . We similarly consider the Hecke operators  $T_\ell$  if  $\ell \nmid N$ , and  $U_\ell$  if  $\ell \mid N$  acting on  $S_{2k+2}(\Gamma_0(2N), \mathbb{Z})$ . The following key theorem relates these spaces of modular forms.

**Theorem 7.1** *For  $k \geq 0$ , we have an isomorphism*

$$\mathcal{S} : S_{k+3/2}(\Gamma_0(4N), \mathbb{Q}) \xrightarrow{\sim} S_{2k+2}(\Gamma_0(2N), \mathbb{Q})$$

which is Hecke-equivariant, i.e.

$$\begin{aligned} \mathcal{S} \circ T_{\ell^2} &= T_\ell \circ \mathcal{S} \quad \forall \ell \nmid 2N, \\ \mathcal{S} \circ U_{\ell^2} &= U_\ell \circ \mathcal{S} \quad \forall \ell \mid N. \end{aligned}$$

**Proof** The result follows from the work of Niwa [32, §1 and Remark 4], the fact that the  $\mathbb{C}$ -span of  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$  is equal to the space of cusp forms of weight  $k + 3/2$  and level  $\Gamma_0(4N)$  by the theorem of Serre and Stark in [33], together with the fact that Hecke operators preserve  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$ . Although the calculations in [32] are done for  $k \geq 1$ , they are also valid for  $k = 0$  as pointed out by Kohnen on page 59 of [24].  $\square$

Consider the space  $\mathbb{Z}_p[[q]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  equipped with the norm

$$\left| \sum_{n \geq 0} a_n q^n \right| = \max_n \{|a_n|\}.$$

Since the eigenvalues of  $U_p$  acting on  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Q})$  are algebraic integers, the operator

$$e_{\text{ord}}: S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p) \longrightarrow S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p), f \longmapsto \lim_{m \rightarrow +\infty} U_p^{m!}(f)$$

is well-defined. Denote by  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  the image of this map, and similarly define  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$ . We also consider the analogous definition for integral weight cusp forms and use similar notation.

**Proposition 7.2** *The rank of the finitely generated modules  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  is constant as long as  $k$  varies over non-negative integers such that  $k \equiv 0 \pmod{(p-1)/2}$ .*

**Proof** It is enough to prove that  $\dim_{\mathbb{Q}_p} S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$  is constant as long as  $k \in \mathbb{Z}_{\geq 0}$  and  $k \equiv 0 \pmod{(p-1)/2}$ . It follows from Theorem 7.1 that we have an isomorphism

$$S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p) \xrightarrow{\sim} S_{2k+2}^{\text{ord}}(\Gamma_0(2N), \mathbb{Q}_p).$$

But the dimensions of the right hand side are constant as long as  $k \equiv 0 \pmod{(p-1)}$ . (See the proof of Theorem 3 in Section 7.2 of [18].) The result follows.  $\square$

### 7.2 $\Lambda$ -adic forms of half-integral weight

We study the space of  $\Lambda$ -adic modular forms of half-integral weight and prove a classicality result in this setting. We follow [18] and [19].

Let  $\Lambda = \mathbb{Z}_p[[T]]$  denote the Iwasawa algebra over  $\mathbb{Z}_p$  and put  $u = 1+p \in 1+p\mathbb{Z}_p$ . A  $\Lambda$ -adic cusp form of half-integral weight is a formal power series

$$F = \sum_{n \geq 1} A_n q^n \in \Lambda[[q]]$$

such that there exists  $k_0$  (dependent on  $F$ ) satisfying that for all  $k \geq k_0$  and  $k \equiv 0 \pmod{p-1}$ , the so-called weight  $k$  specialization

$$F_k := F(u^k - 1) := \sum_{n \geq 1} A_n(u^k - 1)q^n \in \mathbb{Z}_p[[q]],$$

belongs to  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$ . We denote the space of such forms by  $\mathbb{P}$ . We define ordinary  $\Lambda$ -adic cusp forms of half-integral weight in the same way as above but replacing  $S_{k+3/2}(\Gamma_0(4N), \mathbb{Z}_p)$  by  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$ , and we denote this space by  $\mathbb{P}^{\text{ord}}$ .

A key input to study the space  $\mathbb{P}^{\text{ord}}$  is the fact that  $r^{\text{ord}} = \text{rank}_{\mathbb{Z}_p} S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  is constant as long as  $k \geq 0$  and  $k \equiv 0 \pmod{p-1}$ , proven in Proposition 7.2.

**Theorem 7.3**  $\mathbb{P}^{\text{ord}}$  is free of finite rank over  $\Lambda$ . In particular,  $\text{rank}_{\Lambda}(\mathbb{P}^{\text{ord}}) \leq r^{\text{ord}}$ .

**Proof** A proof of this statement can be found in Proposition 4 of [19]. There, Hida considers different level structures than the ones considered here, but the same reasoning works in this case. □

For every  $k \geq 0$ , we can define a map

$$\varphi_k : \mathbb{P}^{\text{ord}} / P_k \mathbb{P}^{\text{ord}} \longrightarrow \mathbb{Z}_p[[q]], \quad F \longmapsto F_k,$$

where  $P_k = T - (u^k - 1) \in \mathbb{Z}_p[[T]]$ , which is injective. The image of this map is a submodule of  $\mathbb{Z}_p[[q]]$ . We can also view  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  as a submodule of  $\mathbb{Z}_p[[q]]$ . The relation between these two submodules is the so-called control theorem, which is again a consequence of Proposition 7.2.

**Theorem 7.4** Let  $k \geq 0$  such that  $k \equiv 0 \pmod{p-1}$ . Then, the map  $\varphi_k$  induces an isomorphism

$$\varphi_k : \mathbb{P}^{\text{ord}} / P_k \mathbb{P}^{\text{ord}} \xrightarrow{\sim} S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p).$$

**Proof** The analogous statement for ordinary cuspidal  $\Lambda$ -adic forms of integral weight is known. A proof can be found in Theorem 3, Section 7.3 of [18]. The same proof given there works for the case of half-integral weight forms once we have Proposition 7.2.

Indeed, it can be proven that every element  $f \in S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  is in the image  $\varphi_k$  as in the case of integral weight forms. For example, this is done in Proposition 5 of [19]. Since  $\mathbb{P}^{\text{ord}}$  is free of finite rank, this already implies the result for  $k$  large enough. To obtain the result for all  $k$  note that

$$S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p) \subset \text{Im}(\varphi_k) \subset \mathbb{Z}_p[[q]]. \tag{19}$$

This implies

$$r^{\text{ord}} \leq \text{rank}_{\mathbb{Z}_p}(\text{Im}(\varphi_k)).$$

Since  $\text{rank}_{\mathbb{Z}_p}(\text{Im}(\varphi_k)) \leq \text{rank}_{\Lambda}(\mathbb{P}^{\text{ord}})$ , the previous inequality and Proposition 7.3 imply  $r^{\text{ord}} = \text{rank}_{\mathbb{Z}_p}(\text{Im}(\varphi_k))$ . Hence, it follows from (19) that  $\text{Im}(\varphi_k) = S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$  and we are done.  $\square$

Fix a  $\Lambda$ -basis  $\{B_1, \dots, B_r\}$  of  $\mathbb{P}^{\text{ord}}$  and write

$$B_i = \sum_{n \geq 1} A_{i,n} q^n \in \Lambda[[q]].$$

By Theorem 7.4, the set  $\{B_1(0), \dots, B_r(0)\}$  forms a  $\mathbb{Z}_p$ -basis of  $S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Z}_p)$ . Thus, there exist  $n_1, \dots, n_r$  such that

$$\det((A_{i,n_j}(0))_{1 \leq i, j \leq r}) \neq 0.$$

Since  $\det((A_{i,n_j}(T))_{1 \leq i, j \leq r}) \in \Lambda$ , it follows by continuity that there exists  $k_0$  such that if  $k \geq k_0$  and  $k \equiv 0 \pmod{p-1}$ , then

$$\det((A_{i,n_j}(u^k - 1))_{1 \leq i, j \leq r}) \neq 0. \tag{20}$$

Now define

$$\underline{b}_i = \sum_{n \geq 1} a_{i,n} q^n,$$

where  $a_{i,n} : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$  is the analytic function determined by  $a_{i,n}(k) = A_{i,n}(u^k - 1)$  for every  $k \geq 0$  such that  $k \equiv 0 \pmod{p-1}$ .

We will now prove that certain first order derivatives of  $\Lambda$ -adic modular forms of half-integral weight are modular forms themselves. Let  $F$  be a  $\Lambda$ -adic modular form of half-integral weight such that  $F_0 = 0$ . Let

$$F' := \frac{d}{dk} F_k|_{k=0} = \lim_{k \rightarrow 0} \frac{F_k}{k} \in \mathbb{Z}_p[[q]]$$

be the first derivative of  $F$  with respect to  $k$  evaluated at  $k = 0$ . It is a weight  $3/2$  analogue of a  $p$ -adic modular form in the sense of Serre. Here the limit is taken in  $\mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$  with respect to the norm introduced above. Recall that  $U_{p^2}$  has the following expression at the level of  $q$ -expansions:

$$\sum_{n \geq 0} a_n q^n \mapsto \sum_{n \geq 1} a_{np^2} q^n.$$

Since  $|U_p^2 f| \leq |f|$  for any  $f \in \mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$  and  $U_{p^2}$  is linear, it follows that we can define the  $p$ -adic modular form of weight  $3/2$

$$e_{\text{ord}}(F') := \lim_{k \rightarrow 0} e_{\text{ord}}\left(\frac{F_k}{k}\right).$$

Moreover, it is a calculation to verify that the limit

$$\lim_{m \rightarrow +\infty} U_p^{m!}(F')$$

exists in  $\mathbb{Z}_p[[q]] \otimes \mathbb{Q}_p$  and is equal to  $e_{\text{ord}}(F')$ .

**Corollary 7.5** *For every  $F \in \mathbb{P}$  with  $F_0 = 0$  the  $p$ -adic modular form  $e_{\text{ord}}(F')$  is classical. More precisely, it belongs to  $S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$ .*

**Proof** By definition,

$$e_{\text{ord}}(F') = \lim_{k \rightarrow 0} e_{\text{ord}}\left(\frac{F_k}{k}\right).$$

Now, Theorem 7.4 implies that, for every  $k > 0$  and  $k \equiv 0 \pmod{p-1}$ , we can write

$$e_{\text{ord}}\left(\frac{F_k}{k}\right) = \sum_{i=1}^r x_i(k) \underline{b}_i(k),$$

where  $x_i(k) \in \mathbb{Q}_p$  for every  $i$ . Let  $n_1, \dots, n_r$  be as above, and note that  $(x_i(k))_i$  is the solution of the linear system of equations

$$(a_{i,n_j}(k))_{j,i} (x_i(k))_i = \left( a_{n_j} \left( \frac{F_k}{k} \right) \right)_j.$$

Moreover, since the determinant of the matrix defining this system is an analytic function, which is non-zero if  $k \geq k_0$  and  $k \equiv 0 \pmod{p-1}$  by (20) and the discussion above it, we deduce that for every  $i$  the limit  $\lim_{k \rightarrow 0} x_i(k)$  exists in  $\mathbb{Q}_p$ . Denote it by  $x_i(0)$ . Then,

$$e_{\text{ord}}(F') = \lim_{k \rightarrow +\infty} e_{\text{ord}}\left(\frac{F}{k}\right) = \sum_{i=1}^r x_i(0) \underline{b}_i(0)$$

and it follows from Theorem 7.4 in the particular case that  $k = 0$  that the right hand side belongs to  $S_{k+3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$ , which concludes the proof.  $\square$

### 7.3 $p$ -adic families of theta series

Recall that the element  $\gamma \in \Gamma$  determines a collection of  $\mathbb{Z}_p$ -lattices  $L_j$  of depth zero, and let  $w_j^+, w_j^-$  and  $e$  be as in Sect. 5.1, so that  $L_j = \langle w_j^+, w_j^-, e \rangle$ . Note that  $w_j^+, w_j^-$  can be viewed both as elements of  $V_{\mathbb{Q}(\gamma)}$  and  $V_{\mathbb{Q}_p}$ , using the embedding  $\mathbb{Q}(\gamma) \hookrightarrow \mathbb{Q}_p$  satisfying that  $\text{ord}_p(\varpi) = 2t > 0$ . These data, together with  $\Phi$ , can be used to define the following Schwartz–Bruhat functions

$$\Phi_j^+ = \Phi \otimes \mathbf{1}_{\{v \in L_j \mid \langle v, w_j^+ \rangle \in \mathbb{Z}_p^\times\}}, \text{ and } \Phi_j^- = \Phi \otimes \mathbf{1}_{\{v \in L_j \mid \langle v, w_j^- \rangle \in \mathbb{Z}_p^\times\}}$$

on  $V_{\mathbb{A}^\infty}$  for every  $j \in \{0, \dots, 2t - 1\}$ . We have that  $\Phi$  is invariant under  $K_0(4N/p)^{(p)}$  by assumption. Moreover,

$$\begin{aligned} 1_{\{v \in L_j \mid \langle v, w_j^+ \rangle \in \mathbb{Z}_p^\times\}} &= 1_{L_j} - 1_{L_j \cap L_{j-1}}, \quad 1_{\{v \in L_j \mid \langle v, w_j^- \rangle \in \mathbb{Z}_p^\times\}} \\ &= 1_{L_j} - 1_{L_j \cap L_{j+1}} \end{aligned} \tag{21}$$

and  $L_j$  is unimodular, while  $L_j \cap L_{j-1}$  has level  $p$  for every  $j$ . It follows that  $\Phi_j^\pm$  is invariant under  $K_0(4N)$  for every  $j$ . Since  $w_j^+$  and  $w_j^-$  are isotropic, the functions  $v \mapsto \langle w_j^\pm, v \rangle^k$  are harmonic polynomials on  $V_{\mathbb{Q}(\gamma)}$  for all integers  $k \geq 0$ . Hence, the  $q$ -series

$$\Theta_k := \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^+(v) \langle w_j^+, v \rangle^k q^{Q(v)} - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^-(v) \langle w_j^-, v \rangle^k q^{Q(v)} \tag{22}$$

is a linear combination of classical theta-series with coefficients in the quadratic imaginary field  $\mathbb{Q}(\gamma)$  of weight  $k + 3/2$  and level  $\Gamma_0(4N)$  by [4, Theorem 4.1]. Via the embedding  $\mathbb{Q}(\gamma) \hookrightarrow \mathbb{Q}_p$ , we can also view the Fourier coefficients of  $\Theta_k$  as elements in  $\mathbb{Z}_p$ . Moreover, since the non-zero terms in the infinite sum defining  $\Theta_k$  solely involve elements of  $V$  for which  $\langle w_j^+, v \rangle$  (resp.  $\langle w_j^-, v \rangle$ ) are  $p$ -adic units, it follows that the Fourier coefficients of  $\Theta_k$  vary analytically as functions of the variable  $k \in (\mathbb{Z}/(p - 1)\mathbb{Z}) \times \mathbb{Z}_p$ . We can therefore define  $\Theta_k$  for  $k \in (\mathbb{Z}/(p - 1)\mathbb{Z}) \times \mathbb{Z}_p$ . It gives a prototypical instance of a  $\Lambda$ -adic modular form of half-integral weight, in the sense that there exists a  $F \in \mathbb{P}$  such that  $F_k = \Theta_k$  for every  $k \equiv 0 \pmod{p - 1}$ .

**Lemma 7.6** *The weight 3/2 specialization  $\Theta_0$  is identically zero.*

**Proof** By (21), we have

$$\begin{aligned} \Theta_0 &= \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi(v) (1_{L_j} - 1_{L_j \cap L_{j-1}})(v) q^{Q(v)} \\ &\quad - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi(v) (1_{L_j} - 1_{L_j \cap L_{j+1}})(v) q^{Q(v)} \\ &= - \sum_{v \in V} \Phi(v) 1_{L_0 \cap L_{-1}}(v) q^{Q(v)} + \sum_{v \in V} \Phi(v) 1_{L_{2t-1} \cap L_{2t}}(v) q^{Q(v)} = 0, \end{aligned}$$

where in the last equality we used that  $\gamma(L_0 \cap L_{-1}) = L_{2t} \cap L_{2t-1}$  and that the functions  $\Phi$  and  $v \mapsto Q(v)$  are invariant under the action of  $\gamma^{\mathbb{Z}}$ .  $\square$

Lemma 7.6 together with Corollary 7.5 immediately imply the following:

**Corollary 7.7** *The  $p$ -adic modular form  $e_{\text{ord}}(\Theta'_0)$  is classical. More precisely, it belongs to  $S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$ .*

We now relate  $e_{\text{ord}}(\Theta'_0)$  with the generating series  $\log_\gamma(G_{\mathbb{F}}^+)(q)$ .

**Lemma 7.8** For every  $D \in \mathcal{D}_S$  and every  $n \geq 0$  the following equality holds:

$$a_{Dp^{2n}}(\log_\gamma(G_\Phi^+(q))) = a_{Dp^{2n}}(e_{\text{ord}}((1 + U_{p^2})\Theta'_0)).$$

Note that, since  $a_{Dp^{2n}}(\log_\gamma(G_\Phi^+(q))) = a_D(\log_\gamma(G_\Phi^+(q)))$  for all  $n \geq 0$ , the right hand side does not depend on  $n$ .

**Proof** It is enough to prove the formula when  $\text{ord}_p(D) \in \{0, 1\}$ . Using (22), we can compute

$$a_{Dp^{2m}}(\Theta'_0) = \sum_{j=0}^{2t-1} \sum_{v \in L_j^+[m]} \Phi(v) \log_p(\langle w_j^+, v \rangle) - \sum_{j=0}^{2t-1} \sum_{v \in L_j^-[m]} \Phi(v) \log_p(\langle w_j^-, v \rangle).$$

Hence, it follows from Theorem 6.4 that

$$a_D(\log_\gamma(G_\Phi^+(q))) = \lim_{m \rightarrow +\infty} a_{Dp^{2m}}(\Theta'_0) + a_{Dp^{2(m+1)}}(\Theta'_0). \tag{23}$$

and that the limit on the right hand side exists. Indeed, the case when  $\text{ord}_p(D) = 1$  follows directly from the second part of Theorem 6.4. The case  $\text{ord}_p(D) = 0$  can be deduced by noting that the first part of Theorem 6.4 implies  $\log_p(j_{\mathcal{D}_\Phi(D)}(\gamma)) = \lim_{m \rightarrow +\infty} a_{Dp^{2m}}(\Theta'_0)$  (and therefore also  $\log_p(j_{\mathcal{D}_\Phi(D)}(\gamma)) = \lim_{m \rightarrow +\infty} a_{Dp^{2(m+1)}}(\Theta'_0)$ ) and taking the sum of these two equalities.

On the other hand, from the expression of the ordinary projection given above, we have that for every  $n \geq 0$

$$a_{Dp^{2n}}(e_{\text{ord}}(\Theta'_0 + U_{p^2}(\Theta'_0))) = \lim_{m \rightarrow +\infty} a_{Dp^{2(n+m+1)}}(\Theta'_0) + a_{Dp^{2(n+m+1)}}(\Theta'_0).$$

Since the right hand side of the previous equation is a subsequence of the right hand side of (23), we deduce that

$$a_D(\log_\gamma(G_\Phi^+(q))) = a_{Dp^{2n}}(e_{\text{ord}}(\Theta'_0 + U_{p^2}(\Theta'_0))),$$

which proves the desired equality. □

The action of  $U_{p^2}$  on  $S_{3/2}^{\text{ord}}(\Gamma_0(4N), \overline{\mathbb{Q}}_p)$  diagonalizes. This can be justified, for example, using Theorem 7.1 and the fact that the analogous statement for weight 2 forms of level  $\Gamma_0(2N)$  is well-known. In particular, we can consider

$$\text{pr}_1 : S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p) \longrightarrow S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$$

to be the projection to the  $U_{p^2} = 1$  eigenspace. Its image consists of  $p$ -newforms. Again, this follows from Theorem 7.1 and the corresponding statement for weight 2 forms, which is a consequence of the Weil conjectures for abelian varieties. We prove the main identity of this work, which implies Theorem 2.7.

**Theorem 7.9** *The identity*

$$\log_\gamma(G_\Phi^+)(q) = 2\text{pr}_1(e_{\text{ord}}(\Theta'_0))$$

of formal power series holds. In particular,  $\log_\gamma(G_\Phi^+)(q)$  is an element of  $S_{3/2}^{\text{ord}}(\Gamma_0(4N), \mathbb{Q}_p)$ .

**Proof** Since  $U_{p^2}$  and  $e_{\text{ord}}$  commute, it is enough to prove that if

$$f = e_{\text{ord}}(\Theta'_0 + U_{p^2}\Theta'_0) = \sum_{n \geq 1} a_n(f)q^n,$$

we have  $\log_\gamma(G_\Phi^+)(q) = \text{pr}_1(f)$ . Note that by Theorem 7.8, we have that if  $D \in \mathcal{D}_S$ ,

$$a_{Dp^{2n}}(\log_\gamma(G_\Phi^+)(q)) = a_{Dp^{2n}}(f).$$

for every  $n \geq 0$ . Therefore, the equality of the theorem follows from proving that, if  $D \in \mathbb{Z}_{\geq 0}$  is such that  $\text{ord}_p(D) \in \{0, 1\}$  and  $N \geq 0$ , then:

- (1) If  $\left(\frac{-D}{p}\right) = 1$ ,  $a_{Dp^{2n}}(\text{pr}_1(f)) = 0$ .
- (2) If  $\left(\frac{-D}{p}\right) \in \{0, -1\}$ ,  $a_{Dp^{2n}}(\text{pr}_1(f)) = a_{Dp^{2n}}(f)$ .

We start by proving the first point. Since  $\text{pr}_1(f)$  is a  $p$ -newform, Theorem 1 of [30] implies that the Atkin–Lehner involution at  $p$  acts by multiplication with  $-1$  on  $\text{pr}_1(f)$ . Then, (1) follows from the description of the eigenspaces of the Atkin–Lehner involution given in [30, Remark 2]. (See also [24, Proposition 4].) We proceed to prove the second point. Write  $f$  as a sum of eigenvectors for  $U_{p^2}$ , namely

$$f = \sum_{i=1}^r f_i,$$

where  $f_i \in S_{3/2}^{\text{ord}}(\Gamma_0(4N), L)$  and there exists  $\alpha_i$  such that  $U_{p^2}f_i = \alpha_i f_i$  for every  $i$ . Here  $L$  is a finite extension of  $\mathbb{Q}_p$  containing all the elements  $\alpha_i$ . We can suppose without loss of generality that  $\alpha_i \neq \alpha_j$  if  $i \neq j$  and that  $\alpha_1 = 1$ . In particular,  $f_1 = \text{pr}_1(f)$  (which is possibly zero). Let  $D$  be such that it satisfies the conditions of (2). For every  $n \geq 0$ , we can consider the  $Dp^{2n}$ -th Fourier coefficient of each side of the previous equality to obtain

$$a_D(f) = a_D(f_1) + \sum_{i=2}^r \alpha_i^n a_D(f_i),$$

where we used that  $a_D(f) = a_{Dp^{2n}}(f)$  for every  $n \geq 0$ , which holds by Theorem 7.8. Considering this equality for  $n = 0, \dots, r-1$  and using that the Vandermonde matrix associated to  $\{1, \alpha_2, \dots, \alpha_r\}$  is non-singular we deduce that we must have  $a_D(f) =$



$a_D(f_1)$ , implying the desired equality. Once we have  $\log_\gamma(G_\Phi^+)(q) = 2\text{pr}_1(e_{\text{ord}}(\Theta'_0))$ , the fact that  $\log_\gamma(G_\Phi^+)(q) \in S_{3/2}(\Gamma_0(4N), \mathbb{Q}_p)$  follows from Corollary 7.7.  $\square$

### 8 Numerical example

We conclude by presenting a concrete example where we numerically compute the  $p$ -adic family  $\Theta_k$  and the reduction modulo  $p$  of  $e_{\text{ord}}(\Theta'_0/p)$ . In future work, we aim to present a complete numerical computation of  $e_{\text{ord}}(\Theta'_0/p)$ . However, in this section, we focus on its reduction modulo  $p$ , as these computations are technically simpler and already illustrate interesting phenomena.

Let  $S = \{7, 13, \infty\}$ , let  $p = 7$  and consider  $B$  be the quaternion algebra over  $\mathbb{Q}$  ramified exactly at  $\{13, \infty\}$ . It can be viewed as the algebra over  $\mathbb{Q}$  generated by  $i, j, k$  where

$$i^2 = -2, \quad j^2 = -13, \quad ij = -ji = k.$$

Let  $\tilde{R}$  be the maximal  $\mathbb{Z}[1/p]$ -order of  $B$  given by  $\langle 1/2 + j/2 + k/2, i/4 + j/2 + k/4, j, k \rangle$ , let  $\alpha = 1 + i \in B^\times$ , which has reduced norm  $\ell = 3 \notin S$ , and consider the Eichler  $\mathbb{Z}[1/p]$ -order  $R = \tilde{R} \cap \alpha \tilde{R} \alpha^{-1}$  of level 3. Denote by  $\Gamma$  the group of norm one units in  $R$  modulo  $\{\pm 1\}$ . The quotient  $\Gamma \backslash \mathcal{H}_p$  is isomorphic to the  $\mathbb{C}_p$ -points of the Shimura curve  $X$ .

#### 8.1 Construction of the $p$ -adic family $\Theta_k$

Recall the definition of the  $p$ -adic family  $\Theta_k$  given in (22). This family depends on a choice of a Schwartz–Bruhat function  $\Phi$ , an element  $\gamma \in \Gamma$  hyperbolic at  $p$  and the eigenvectors of the action of  $\gamma$  on  $V_{\mathbb{Q}_p}$ . We proceed to fix these data. Let  $\tilde{R}_0$  be the subgroup of elements of  $\tilde{R}$  of reduced norm zero. As before, write  $1_{\tilde{R}_0}$  for the characteristic function of  $\tilde{R}_0 \otimes \hat{\mathbb{Z}}^{(p)}$ . Consider the  $\hat{R}^\times \times K_0(4 \cdot 13)^{(p)}$ -invariant Schwartz–Bruhat function

$$\Phi = 1_{\tilde{R}_0} - 1_{\tilde{R}_0} \cdot \alpha^{-1}.$$

Since we have the factorization of ideals  $(7) = (7, x + 3)(7, x + 4)$  in the ring of integers of  $\mathbb{Q}[x]/(x^2 + 5)$ , the element  $(x + 3)/(-x + 3) = 3x/7 + 2/7$  is a  $p$ -unit in  $\mathbb{Q}[x]/(x^2 + 5)$ . Its image in  $B$  with respect to the embedding

$$\mathbb{Q}[x]/(x^2 + 5) \hookrightarrow B, \quad x \longmapsto \frac{i}{4} + \frac{j}{2} + \frac{k}{4},$$

is equal to

$$\gamma = \frac{2}{7} + \frac{3i}{28} + \frac{3j}{14} + \frac{3k}{28},$$

and it can be verified that  $\gamma \in \Gamma$ . Let  $\mathfrak{p}$  be the prime ideal spanned by 7 and  $x + 4$  and fix the embedding

$$\mathbb{Q}[x]/(x^2 + 5) \hookrightarrow \mathbb{Q}_p \quad (24)$$

such that  $\text{ord}_p((\mathfrak{p})) = 1$ . Using that  $\gamma$  is hyperbolic at  $p$ , we deduce that its action on  $V \otimes \mathbb{Q}[x]/(x^2 + 5)$  (and therefore on  $V_{\mathbb{Q}_p}$ ) diagonalizes. The eigenvectors of  $\gamma$  are

$$\begin{aligned} w^+ &= i + \left(\frac{4x}{39} - \frac{2}{39}\right)j + \left(-\frac{4x}{39} - \frac{1}{39}\right)k \\ e &= i + 2j + k \\ w^- &= i + \left(-\frac{4x}{39} - \frac{2}{39}\right)j + \left(\frac{4x}{39} - \frac{1}{39}\right)k \end{aligned}$$

with eigenvalues  $\varpi = -12x/49 - 41/49$ , 1 and  $\varpi^{-1}$  respectively. Since  $v_{\mathfrak{p}}(\varpi) = 2$ , we have that  $t = 1$ . Note that  $\langle w^+, w^- \rangle \in \mathbb{Z}_p^\times$ , which implies that  $\{w^+, w^-\}$  generate a hyperbolic plane. Finally, consider the unimodular  $\mathbb{Z}_p$ -lattices

$$\begin{aligned} L_0 &= \langle w^+, e, w^- \rangle = \langle i, j, k \rangle \\ L_1 &= \langle pw^+, e, w^-/p \rangle = \left\langle i + 2j + k, 14i - \frac{28j}{39} - \frac{14k}{39}, \frac{i}{7} + j + \frac{8k}{7} \right\rangle. \end{aligned}$$

We can therefore consider the  $p$ -adic family  $\Theta_k$  given in (22) attached to the data  $\Phi$ ,  $\gamma$  and  $\{w^+, e, w^-\}$ .

## 8.2 Calculation of $\Theta_0$ and $e_{\text{ord}}(\Theta'_0)$

Consider the same notation as above. For every  $M \leq 421 \cdot p^2$  we can run over the following sets:

$$\begin{aligned} &\left\{ v \in V \mid \langle v, v \rangle = M, 1_{\tilde{R}_0}(v) \cdot 1_{L_0}(v) = 1, \langle v, w^+ \rangle \text{ or } \langle v, w^- \rangle \in \mathbb{Z}_p^\times \right\}, \\ &\left\{ v \in V \mid \langle v, v \rangle = M, 1_{\tilde{R}_0}(v) \cdot 1_{L_1}(v) = 1, \langle v, pw^+ \rangle \text{ or } \langle v, p^{-1}w^- \rangle \in \mathbb{Z}_p^\times \right\}, \\ &\left\{ v \in V \mid \langle v, v \rangle = M, 1_{\alpha \cdot \tilde{R}_0}(v) \cdot 1_{L_0}(v) = 1, \langle v, w^+ \rangle \text{ or } \langle v, w^- \rangle \in \mathbb{Z}_p^\times \right\}, \\ &\left\{ v \in V \mid \langle v, v \rangle = M, 1_{\alpha \cdot \tilde{R}_0}(v) \cdot 1_{L_1}(v) = 1, \langle v, pw^+ \rangle \text{ or } \langle v, p^{-1}w^- \rangle \in \mathbb{Z}_p^\times \right\}. \end{aligned}$$

From there, it is possible to compute the first  $421 \cdot p^2$  Fourier coefficients of  $\Theta_k$ , for  $k \in \mathbb{Z}$ , as well as of  $\Theta'_0$ . The necessity to calculate exactly this number of Fourier coefficients comes from the Sturm bound, which is used in the proof of Proposition 8.1

**Table 1** First Fourier coefficients of the theta series  $\Theta_{\tilde{R}_0, L_j}^\pm$

Theta series	q-expansion																
	2	5	6	7	8	11	13	15	18	19	20	21	24	26	28	31	32
$\Theta_{\tilde{R}_0, L_0}^+$	2	2	4	4	6	8	2	8	6	6	8	8	8	6	6	10	14
$\Theta_{\tilde{R}_0, L_0}^-$	2	0	4	2	6	8	0	8	6	4	8	8	12	6	6	8	14
$\Theta_{\tilde{R}_0, L_1}^+$	2	0	4	2	6	8	0	8	6	4	8	8	12	6	6	8	14
$\Theta_{\tilde{R}_0, L_1}^-$	2	2	4	4	6	8	2	8	6	6	8	8	8	6	6	10	14

below. In particular, define

$$\Theta_{\tilde{R}_0, j}^+ := \sum_{\substack{v \in V \\ (v, p^{+j}w^+) \in \mathbb{Z}_p^\times}} 1_{\tilde{R}_0}(v) \cdot 1_{L_j}(v)q^{Q(v)}$$

and define  $\Theta_{\tilde{R}_0, j}^-$  with the same expression but replacing the symbol  $+$  by the symbol  $-$  everywhere. Define also  $\Theta_{\alpha \cdot \tilde{R}_0, j}^+$  and  $\Theta_{\alpha \cdot \tilde{R}_0, j}^-$  analogously. Then,

$$\Theta_0 = \left( \Theta_{\tilde{R}_0, L_0}^+ + \Theta_{\tilde{R}_0, L_1}^+ - \Theta_{\tilde{R}_0, L_0}^- - \Theta_{\tilde{R}_0, L_1}^- \right) - \left( \Theta_{\alpha \cdot \tilde{R}_0, L_0}^+ + \Theta_{\alpha \cdot \tilde{R}_0, L_1}^+ - \Theta_{\alpha \cdot \tilde{R}_0, L_0}^- - \Theta_{\alpha \cdot \tilde{R}_0, L_1}^- \right)$$

and we verify that the first  $421 \cdot p^2$  Fourier coefficients are 0. For example, the first 4 terms that appear in the previous expression are given below (Table 1).

The coefficients of  $q^n$  for  $n < 32$  that do not appear in the table are 0, as theta series attached to lattices in  $V$  have non-zero Fourier coefficients only if  $-D$  is not a square modulo 13. We will follow a similar convention from now on. The forms on the previous table belong to  $M_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Q})$ , a space of dimension 32, and these coefficients fully determine them.

From (22), we see that the derivative of  $\Theta_k$  with respect to  $k$  evaluated at  $k = 0$  is equal to

$$\Theta'_0 = \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^+(v) \log_p(p^j w^+, v)q^{Q(v)} - \sum_{j=0}^{2t-1} \sum_{v \in V} \Phi_j^-(v) \log_p(p^{-j} w^-, v)q^{Q(v)}. \tag{25}$$

Note that the dot products  $\langle v, w^\pm \rangle$  belong to  $\mathbb{Q}[x]/(x^2 + 5)$  and have  $p$ -adic valuation 0. Using the embedding (24), we can view them as elements in  $\mathbb{Z}_p^\times$ . Therefore, the  $p$ -adic logarithm of these numbers lies in  $p\mathbb{Z}_p$ . We can then consider  $\Theta'_0/p$  as an element in  $\mathbb{Z}_p[[q]]$  and study its reduction modulo  $p$ .

Similarly as above, we can calculate the first  $421 \cdot p^2$  Fourier coefficients of  $\Theta'_0/p$  modulo  $p$ . The first ones are

$$\begin{aligned} \frac{\Theta'_0}{p} &= 2q^2 + 3q^5 + 2q^6 + 4q^7 + 5q^8 + 4q^{11} + 3q^{13} \\ &\quad + 3q^{15} + 2q^{18} + 3q^{20} + 6q^{21} + \dots \end{aligned}$$

Since it is possible to calculate the first  $421 \cdot p^2$  Fourier coefficients of  $\Theta'_0/p \pmod p$ , we obtain the first 421 Fourier coefficients of  $U_{p^2}(\Theta'_0/p)$  modulo  $p$ . The first ones are

$$\begin{aligned} U_{p^2}(\Theta'_0/p) &= 3q^2 + 3q^5 + 5q^6 + 2q^7 + 3q^{11} \\ &\quad + 6q^{13} + 3q^{15} + 5q^{18} + q^{19} + 2q^{20} + 2q^{21} \\ &\quad + 2q^{24} + 3q^{26} + q^{28} + 6q^{31} + q^{32} + q^{33} + 4q^{34} + q^{37} \\ &\quad + 3q^{39} + q^{44} + \dots \end{aligned}$$

The following proposition, which is verified experimentally using the calculations mentioned above and Magma, is key for the next calculations.

**Proposition 8.1** *There exists a cusp form in  $S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Z})$  whose reduction modulo  $p$  is equal to  $U_{p^2}(\Theta'_0/p) \pmod p$ .*

**Proof** Since  $\Theta_0 = 0$ , we deduce from the expressions of  $\Theta'_0$  in (25) and of  $\Theta_k$  in (22) that

$$\frac{\Theta'_0}{p} \equiv \frac{\Theta_{p-1}}{p(p-1)} \pmod p = 7.$$

In particular,  $U_{p^2}(\Theta'_0/p)$  is the reduction mod  $p$  of an element  $g_1 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ . We can then verify experimentally using Magma that the first 421 Fourier coefficients of  $g_1$  are congruent modulo  $p$  to the first 421 Fourier coefficients of a modular form  $g_2 \in S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Z})$ .

We claim that this implies  $g_1 \equiv g_2 \pmod p$ . Indeed, let  $\tilde{g}_2 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$  be such that  $g_2 \equiv \tilde{g}_2 \pmod p$ . Then, the modular form  $g_1 - \tilde{g}_2 \in S_{3/2+6}(\Gamma_0(4 \cdot 91), \mathbb{Z})$  has the first 421 Fourier coefficients equal to 0 modulo  $p$ . This implies that the first  $4 \cdot 421$  Fourier coefficients of  $(g_1 - \tilde{g}_2)^4 \in S_{30}(\Gamma_0(4 \cdot 91), \mathbb{Z})$  are congruent to 0 modulo  $p$ . Since

$$421 \cdot 4 > \frac{30 \cdot [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(4 \cdot 91)]}{12} = 1680 = 420 \cdot 4,$$

it follows from the Sturm bound [36, Theorem 1] that  $g_1 - \tilde{g}_2 \equiv 0 \pmod p$ , implying the desired result. □

Using a basis of  $S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Z})$  given by Magma, and using Proposition 8.1, we can then compute  $U_{p^2}(\Theta'_0/p)$  and verify the following:

- (1)  $\frac{1}{2}(U_{p^2} + U_{p^2}^2)(\Theta'_0/p) \pmod p$  is an eigenvector for  $U_{p^2}$  of eigenvalue 1.
- (2)  $\frac{1}{2}(U_{p^2} - U_{p^2}^2)(\Theta'_0/p) \pmod p$  is an eigenvector for  $U_{p^2}$  of eigenvalue  $-1$ .

It follows from there that, modulo  $p$ ,

$$\begin{aligned} e_{\text{ord}}\left(\frac{\Theta'_0}{p}\right) &= \lim_{n \rightarrow +\infty} U_{p^2}^n \frac{\Theta'_0}{p} = \lim_{n \rightarrow +\infty} U_{p^2}^{n-1} U_{p^2} \left(\frac{\Theta'_0}{p}\right) \\ &= \lim_{n \rightarrow +\infty} U_{p^2}^{n-1} \left(\frac{1}{2}(U_{p^2} + U_{p^2}^2)(\Theta'_0/p) + \frac{1}{2}(U_{p^2} - U_{p^2}^2)(\Theta'_0/p)\right) \\ &= \frac{1}{2}(U_{p^2} + U_{p^2}^2)(\Theta'_0/p) - \frac{1}{2}(U_{p^2} - U_{p^2}^2)(\Theta'_0/p) = U_{p^2}^2(\Theta'_0/p). \end{aligned}$$

Based on this decomposition of  $e_{\text{ord}}(\Theta'_0/p)$ , we will write

$$\text{pr}_1(e_{\text{ord}}(\Theta'_0/p)) = (U_{p^2}(\Theta'_0/p) + U_{p^2}^2(\Theta'_0/p))/2,$$

$$\text{pr}_{-1}(e_{\text{ord}}(\Theta'_0/p)) = -(U_{p^2}(\Theta'_0/p) - U_{p^2}^2(\Theta'_0/p))/2.$$

The results of the calculation are summarized in the following table (Table 2).

**Remark 8.2** In the decomposition  $e_{\text{ord}}(\Theta'_0/p) = \text{pr}_1(e_{\text{ord}}(\Theta'_0/p)) + \text{pr}_{-1}(e_{\text{ord}}(\Theta'_0/p))$  both summands are non-zero. The first summand is related to the Gross–Kohnen–Zagier generating series, as proved in Theorem 7.9. It would be interesting to find an arithmetic interpretation of the second summand, namely  $\text{pr}_{-1}(e_{\text{ord}}(\Theta'_0/p))$ .

### 8.3 Shimura lift and Hecke equivariance

The space  $S_2^{\text{new}}(\Gamma_0(7 \cdot 13))$  has dimension 7, and there is a unique (up to scalars) cuspidal form such that  $U_7$  acts by 1 and has odd analytic rank (so in particular, the Hecke operator  $U_{13}$  acts also by 1). Its Fourier expansion is given by

$$\begin{aligned} f &= q - 2q^3 - 2q^4 - 3q^5 + q^7 + q^9 + 4q^{12} + q^{13} \\ &\quad + 6q^{15} + 4q^{16} - 6q^{17} - 7q^{19} + \dots \\ &\equiv q + 5q^3 + 5q^4 + 4q^5 + q^7 + q^9 + 4q^{12} + q^{13} \\ &\quad + 6q^{15} + 4q^{16} + q^{17} \dots \pmod p = 7. \end{aligned}$$

Recall the Shimura lift

$$\mathcal{S}_D := \mathcal{S}_{D,0,91} : S_{3/2}(\Gamma_0(4 \cdot 91), \mathbb{Q}) \longrightarrow S_2(\Gamma_0(2 \cdot 91), \mathbb{Q})$$

defined in Sect. 7.1, where  $D$  is a square-free integer such that  $-D > 0$ . We computed the Shimura lift of  $e_{\text{ord}}(\Theta'_0/p)$  for different values of  $D$  and obtained the following

**Table 2**  $D$ th Fourier coefficients of linear combinations of  $U_{p^2}^{n_1}(\Theta'_0/p)$  for  $D$  such that  $\left(\frac{-D}{13}\right) \neq 1$

Modular form mod $p = 7$	$q$ -expansion														
	2	5	6	7	8	11	13	15	18	19	20	21	24	26	28
$\Theta'_0/p$	2	3	2	4	5	4	3	3	2	0	3	6	0	3	4
$U_{p^2}(\Theta'_0/p)$	3	3	5	2	0	3	6	3	5	1	2	2	2	3	1
$U_{p^2}^2(\Theta'_0/p)$	3	4	2	4	0	3	1	3	5	6	5	6	5	4	4
$(U_{p^2}(\Theta'_0/p) + U_{p^2}^2(\Theta'_0/p))/2$	3	0	0	3	0	3	0	3	5	0	0	4	0	0	6
$(U_{p^2}(\Theta'_0/p) - U_{p^2}^2(\Theta'_0/p))/2$	0	3	5	6	0	0	1	0	0	1	2	5	2	3	2
$e_{\text{ord}}\Theta'_0/p$	3	4	2	4	0	3	1	3	5	6	5	6	5	4	4

For every  $D$ , we consider the color code blue:  $\left(\frac{-D}{p}\right) = -1$ , grey:  $\left(\frac{-D}{p}\right) = 1$ , red:  $\left(\frac{-D}{p}\right) = 0$

identities modulo  $p$

$$\begin{aligned} \mathcal{S}_{-2}(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta'_0/p))) &\equiv 6f \pmod{p}, \\ \mathcal{S}_{-11}(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta'_0/p))) &\equiv 3f + U_2f \pmod{p}, \\ \mathcal{S}_{-15}(\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta'_0/p))) &\equiv 3f + 6(U_2f) \pmod{p}. \end{aligned} \tag{26}$$

In particular, we see that  $\mathrm{pr}_1 e_{\mathrm{ord}}(\Theta'_0/p)$  is a Hecke eigenvector (mod  $p$ ) with Hecke eigenvalues congruent to those of  $f$ .

The Schwartz–Bruhat function  $\Phi$  is convenient. Indeed, since  $\Phi$  is the difference of characteristic functions of the trace zero elements of two maximal orders, we deduce that  $\Delta_\Phi(D)$  is of degree 0 for every  $D$ . Moreover,  $\mathrm{deg}_{\mathcal{T}_0}(\Phi)$  lands in the subspace of  $\mathrm{Func}(\Gamma \backslash \mathcal{T}_0, \mathbb{Z})$  corresponding to weight two cusp forms of level  $\Gamma_0(13 \cdot 3)$  that are old at 3. Since  $S_2(\Gamma_0(13), \mathbb{Q}) = 0$ , we deduce  $\mathrm{deg}_{\mathcal{T}_0}(\Phi) = 0$  implying the desired claim by the proof of Lemma 4.6. Applying Theorem 7.9 to the Schwartz–Bruhat function  $\Phi$  one obtains the equality

$$\log_\gamma(G_\Phi^+(q)) = 2\mathrm{pr}_1(e_{\mathrm{ord}}(\Theta'_0)).$$

In particular,  $(1/p) \log_p(G_\Phi^+(\gamma)) \pmod{p}$  is a Hecke cuspidal eigenform of weight  $3/2$  with the same Hecke eigenvalues as the cusp form  $f$  of weight 2 and level  $\Gamma_0(91)$ .

On the other hand, consider  $G_\Phi(q) \in J(\mathbb{Q}_{p^2})_{\mathbb{Q}}[[q]]$  and observe:

- The classes  $[\Delta_\Phi(D)]$  are invariant under the action of  $R^\times$  for every  $D \in \mathcal{D}_S$ .
- The projection of the class  $[\Delta_\Phi(D)]$  to a Hecke eigenspace is non-zero only if the eigenspace corresponds to an eigenform of rank 1 by the Gross–Zagier formula.
- Via Jacquet–Langlands the Hecke action of  $\mathbb{T}^N$  on  $J(\mathbb{Q}_{p^2})$  factors through the action on  $S_2^{91-\mathrm{new}}(\Gamma_0(91 \cdot 3), \mathbb{Q}_{p^2})$ .
- Since the divisors  $\Delta_\Phi(D)$  on  $X$  are obtained via pullback from divisors of a Shimura curve  $\tilde{X}$  that is  $p$ -adically uniformized by  $\tilde{\Gamma} \backslash \tilde{\mathcal{H}}_p$ , with  $\tilde{\Gamma}$  the norm 1 units of  $\tilde{R}$ , it follows that the classes  $[\Delta_\Phi(D)]$  belong to the subspace corresponding to forms that are old at 3.

Hence, the functionals  $\varphi: J(\mathbb{Q}_{p^2}) \rightarrow \mathbb{Q}_{p^2}$  such that  $\varphi(G_\Phi(q))$  is non-zero are generated by projections to eigenspaces where  $\mathbb{T}^N$  acts with the same eigenvalues as it acts on eigenforms on  $S_2^{\mathrm{new}}(\Gamma_0(91))$  which have rank 1 and  $U_7 = 1$ . As we discussed above, there is a unique (up to scaling) such eigenform in  $S_2^{\mathrm{new}}(\Gamma_0(91))$ , which is  $f$ . Uniqueness implies that  $G_\Phi(q) \in J(\mathbb{Q}_{p^2})_{\mathbb{Q}}[[q]]$  is a non-zero multiple of  $\log_\gamma(G_\Phi^+(q))$ , which has the same Hecke eigenvalues of  $f$  modulo  $p$ . Hence, the calculation we presented gives an example (modulo  $p$ ) of the Hecke equivariance property of the geometric theta lift provided by the Gross–Kohnen–Zagier generating series.

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## Declarations

**Conflict of interest** On behalf of all authors, there is no conflict of interest.

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