Tate's last decade at Harvard: Stark's conjecture, elliptic curves, and class field theory

Henri Darmon

To Dick Gross and John Tate, with gratitude and admiration

Contents

Introduction

John Tate literally "wrote the book" on the three topics evoked in the title ([**Ta84**], [**Ta74**], [**AT67**]), but this essay will only touch on a small part of his contributions: those that were made in the decade from roughly 1977 to 1987 and concern special values of L-functions and their arithmetic interpretations.

Tate's popularization of the Stark conjectures in the late 70's soon led to the p-adic variants proposed by Dick Gross [**Gr81**] and Jean-Pierre Serre [**Se78**], which in turn likely motivated Tate's subsequent collaboration with Barry Mazur and Jeremy Teitelbaum on p-adic analogues of the Birch and Swinnerton-Dyer conjecture [**MTT86**]. The striking richness of the p-adic setting provided fertile ground for the "refined conjectures" of [**MT87**], the theme of the influential Mazur–Tate seminar held during the winter semester of 1986, where tame refinements of the Stark conjectures were also formulated [**Gr88**]. A posteriori, these refinements provided a suggestive conceptual

framework for the theory of Euler Systems initiated by Thaine, Rubin and Kolyvagin in the late 1980's.

Tate's work made it apparent that the conjectures of Stark and of Birch and Swinnerton-Dyer lend themselves to numerous refinements and variants, with far-reaching implications for the explicit analytic construction of class fields, global units, and rational points on elliptic curves.

The directions Tate pursued during his final decade at Harvard were a central part of the intellectual landscape when the author arrived as a graduate student in 1987, a year before Tate left for the University of Texas. What follows is a brief survey of Tate's scientific influence on the author, with occasional digressions into topics that have emerged more recently, such as the surprising connection between Stark units and the design of optimal quantum measurements in Section 3, and the recent breakthroughs of Dasgupta and Kakde on the Brumer–Stark conjectures and Gross's "tower of fields" conjecture. These developments illustrate the continuing vitality of the directions that Tate pioneered. We have not strived for a comprehensive account of Tate's impact on number theory beyond our narrow focus, and ask for the indulgence of the reader for the important developments our subjective treatment has inevitably downplayed or omitted.

Dedication

I asked John Tate to be my thesis advisor right after passing the Harvard preliminary exams. He stared at me with a puzzled look before asking "What is an advisor?" A philosophical question that struck me at the time as disconcertingly modest, coming from one of the most acclaimed mentors on the planet! He then proceeded to offer the best career advice I have ever received: "Why don't you try working with Dick Gross?" This is how I joined the privileged ranks of Tate's mathematical descendants. Sadly, Dick passed away a few months after the Tate Centennial meeting. His deep insights and groundbreaking results have been a constant guide and inspiration throughout my scientific life, and it is an honor to dedicate this essay to him as well as to Tate. May their memories be a blessing.

1. Stark's conjecture

Tate's interest in Stark's conjecture [**St71**] dates back to around 1977, possibly motivated by his prior interest in modular forms of weight one whose L-series provided an important source of concrete examples and experimental evidence for the Stark conjecture at its inception. Tate's influential graduate course at Orsay in 1980 [**Ta84**] shaped the way in which the subject is understood today.

The Stark conjecture can be envisaged as a far-reaching generalisation of the Kronecker limit formula, which gives a closed form expression for the value at $s = 1$ — or, equivalently by the functional equation, the first derivative at $s = 0$ — of the partial zeta function attached to a (non-trivial) ideal class character of an imaginary quadratic field K in terms of *elliptic units* arising from the theory of complex multiplication.

Given any ideal class $\mathfrak{A}$ of K, let

$$L(\mathfrak{A}, s) = \sum_{\mathfrak{a} \in \mathfrak{A}} N(\mathfrak{a})^{-s} \tag{1}$$

be the partial zeta function of $\mathfrak{A}$, the sum in (1) being taken over the integral ideals in the class $\mathfrak{A}$, with $N(\mathfrak{a}) \in \mathbb{Z}^{>0}$ denoting the norm of such an ideal. The first Kronecker limit formula relates the Taylor expansion of this function at $s = 1$ to the values at quadratic imaginary arguments of the Dedekind eta function

$$\eta(z) = q^{1/24} \prod (1 - q^n), \qquad q := e^{2\pi i z}, \tag{2}$$

a holomorphic modular form of weight 1/2 whose 24-th power is the Ramanujan Delta function $\Delta(z)$ of weight 12 and level 1. More precisely, it asserts that

$$L(\mathfrak{A}, s) = \frac{\pi}{s-1} + C - \log(\sqrt{y}|\eta(\tau_{\mathfrak{A}})|^2) + O(s-1), \tag{3}$$

where C is a constant that does not depend on $\mathfrak{A}$, and $\tau_{\mathfrak{A}} = x + iy$ is a complex number attached to $\mathfrak{A}$ by the requirement that $y > 0$ and that $\mathbb{Z} + \mathbb{Z}\tau_{\mathfrak{A}}$ is a projective $\mathcal{O}_K$-module isomorphic to a fractional ideal in the class $\mathfrak{A}$. Thanks to the theory of complex multiplication, the ratios $\Delta(\tau_{\mathfrak{A}})/\Delta(\tau_{\mathfrak{A}'})$ attached to any pair $(\mathfrak{A}, \mathfrak{A}')$ of $\mathcal{O}_K$-ideals are closely related to certain distinguished algebraic units in the Hilbert class field H of K.

Replacing the functions $L(\mathfrak{A}, s)$ by a linear combination

$$L(\chi, s) := \sum_{\mathfrak{A} \in \mathrm{Cl}(K)} \chi^{-1}(\mathfrak{A}) L(\mathfrak{A}, s) \tag{4}$$

weighted by a non-trivial character χ of the class group of K with values in a field $E_\chi \subset \mathbb{C}$, and exploiting the functional equation relating the behaviour of $L(\chi, s)$ at $s = 1$ and $s = 0$, one eventually obtains

$$L'(\chi, 0) = \log |U_\chi|, \tag{5}$$

where

$$U_\chi \in E_\chi \otimes \mathcal{O}_H^\times \tag{6}$$

is an *elliptic unit* constructed from suitable ratios of values of the Dedekind η-function at CM points.

Stark's bold insight is that an identity like (5) should continue to hold even when a suitable counterpart of the theory of complex multiplication is lacking and one disposes of no *independent expression* for the right hand side in terms of CM values of modular functions.

For example, the L-function $L(\chi, s)$ appearing in (5) is also the Hecke L-function attached to the theta series

$$\theta_\chi(z) := \sum_{\mathfrak{a}} \chi(\mathfrak{a}) q^{N(\mathfrak{a})}, \qquad q = e^{2\pi i z}, \tag{7}$$

a holomorphic eigenform of weight one on a Hecke congruence subgroup of $\mathbf{SL}_2(\mathbb{Z})$. Replacing θ_χ by a more general eigenform g of weight one in (5), Stark's conjecture suggests that

$$L'(g, 0) \stackrel{?}{=} \log |U_g|, \qquad \text{with } U_g \in E_g \otimes \mathcal{O}_g^\times, \tag{8}$$

where E_g is the field generated by the Fourier coefficients of g and $\mathcal{O}_g$ is the ring of integers of the field cut out by the odd two-dimensional Artin representation attached to g by the construction of Deligne-Serre [**DS74**].

More generally, let V be any finite-dimensional complex vector space (of dimension $n \geq 1$, say) endowed with an irreducible, non-trivial continuous action of the absolute Galois group $G_{\mathbb{Q}}$ of $\mathbb{Q}$, and let

$$\varrho : G_{\mathbb{Q}} \to \mathrm{Aut}(V) \simeq \mathrm{GL}_n(\mathbb{C}) \tag{9}$$

be the associated Artin representation. Since ϱ is continuous, it has finite image and factors through the Galois group $\mathrm{Gal}(H/\mathbb{Q})$ of a finite extension H of $\mathbb{Q}$, and the image under ϱ of the Frobenius element attached to a choice of prime λ of H above ℓ is a well defined linear endomorphism, denoted σ_ℓ, on the space V^{I_λ} of inertia invariants of V. The characteristic polynomial of σ_ℓ has coefficients in a finite (cyclotomic) extension of $\mathbb{Q}$, denoted $E \subset \mathbb{C}$, and the Artin L-function associated to ϱ is defined by the Euler product

$$L(V,s) \;=\; \prod_{\ell} \det(1 - \sigma_\ell \cdot \ell^{-s})^{-1},$$

which converges for $\Re(s) > 1$ and extends to a meromorphic function on $\mathbb{C}$.

Fix an embedding $H \subset \mathbb{C}$ and let c denote the associated complex conjugation in $\mathrm{Gal}(H/\mathbb{Q})$, for which $H^{c=1} \subset \mathbb{R}$. An extension of Dirichlet's unit theorem shows that

$$r := \dim_E \mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, \mathcal{O}_H^\times \otimes_{\mathbb{Z}} E\big) = \dim V^{c=1}.$$

Stark conjectured [**St71**] that $L(V,s)$ vanishes to order r at $s=0$, and that its leading term satisfies

$$L^{(r)}(V,0) = R(V) \pmod{E^\times}, \tag{10}$$

where $R(V)$ is the *Stark unit regulator* whose definition depends on the choice of bases $\{\phi_1, \ldots, \phi_r\}$ and $\{w_1, \ldots, w_r\}$ of the E-vector spaces

$$\mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, \mathcal{O}_H^\times \otimes_{\mathbb{Z}} E\big) \quad \text{and} \quad V^{c=1}$$

respectively. Note that $\phi_i(w_j)$ can be viewed as an element of $\mathbb{R}^\times$ thanks to the chosen complex embedding of H. The regulator $R(V)$ can then be defined as

$$R(V) := \det\Big(\big(\log \phi_i(w_j)\big)_{1 \leq i,j \leq r}\Big). \tag{11}$$

2. Explicit class field theory

When $r = 1$, the Stark regulator reduces to the logarithm of a *single unit* in H, which can be recovered (up to scaling) from $L(V,1)$ or from $L'(V,0)$. Stark's conjecture therefore suggests analytic constructions of units in the fields cut out by such V, essentially by exponentiating $L'(V,0)$. In this way, equation (8) provides the basis for an explicit albeit conjectural construction of the Artin representations attached to newforms of weight one including the exceptional ones with non-dihedral projective image, leading to an effective procedure for calculating the Galois representations of Deligne and Serre [**DS74**]. This application may have piqued Tate's interest at the outset, in light of his prior interest in weight one cusp forms and their associated Artin representations.

As Stark himself pointed out in [**St76**], the scenario where $r = 1$ is directly germane to explicit class field theory as well. Namely, consider the case where V is induced from a one-dimensional character $\chi : G_K \longrightarrow E^\times$ of a fixed ground field K. Then $r = 1$ precisely when χ is *odd* at all but one distinguished archimedean place of K, denoted ∞. The field K is either totally real when ∞ is itself real, or ∞ is its unique complex

place. In the latter scenario K is referred to as an *almost totally real* (ATR) extension of $\mathbb{Q}$.

Let $\infty = \infty_1$ and $\infty_2, \ldots, \infty_t$ be the archimedean places of K and write

$$v_1 : K \hookrightarrow \mathbb{C}, \qquad v_2, \ldots, v_t : K \hookrightarrow \mathbb{R} \tag{12}$$

for the associated embeddings of K. Class field theory identifies χ with a character of the generalised narrow ideal class group associated to a modulus $\mathfrak{m}$ of K which is odd at the places $v_2, \ldots, v_t$, i.e., satisfies

$$\chi((x)) = \mathrm{sgn}(x) := \mathrm{sign}(v_2(x) \cdots v_t(x)), \qquad \text{for all } x \in 1 + \mathfrak{m}\mathcal{O}_K. \tag{13}$$

Assume for simplicity of the exposition that $\mathfrak{m} = 1$ so that the field which is cut out by χ is contained in the Hilbert class field of K (in the narrow sense). The existence of a character χ satisfying (13) implies that H is complex at all the archimedean places lying above $\infty_2, \ldots, \infty_t$. In particular, H is totally complex (but typically not CM) when K is ATR.

The L-functions $L(\chi, s)$ can be expressed as linear combinations of the odd partial L-functions attached to narrow ideal classes $\mathfrak{A}$ of K, defined as

$$L(\mathfrak{A}, s) = \sum_{I = x\mathfrak{A}} \mathrm{sgn}(x)\mathrm{Norm}(I)^{-s}, \tag{14}$$

where the sum is taken over the integral ideals I in the class of $\mathfrak{A}$ in the wide sense.

Stark's conjecture predicts the existence of a system of units $u_{\mathfrak{A}} \in \mathcal{O}_H^\times \otimes \mathbb{Q}$ indexed by the narrow ideal classes $\mathfrak{A}$ which satisfy

$$L'(\mathfrak{A}, 0) \stackrel{?}{=} \log |\tilde{v}_1(u_{\mathfrak{A}})|, \qquad \text{for all } \mathfrak{A} \in \mathrm{Cl}(K), \tag{15}$$

where $\tilde{v}_1 : H \longrightarrow \mathbb{C}$ is a suitable archimedean embedding of H lying above the distinguished place v_1. These units are further expected to be conjugate to each other under the action of $\mathrm{Gal}(H/K)$. More precisely, letting

$$\mathrm{rec} : \mathrm{Cl}^+(K) \longrightarrow \mathrm{Gal}(H/K) \tag{16}$$

be the reciprocity law of global class field theory, it is expected that

$$\mathrm{rec}(\mathfrak{B})(u_{\mathfrak{A}}) \stackrel{?}{=} u_{\mathfrak{A}\mathfrak{B}^{-1}}, \qquad \text{for all } \mathfrak{A}, \mathfrak{B} \in \mathrm{Cl}^+(K). \tag{17}$$

The complex conjugations (Frobenius elements)

$$c_1, \ldots, c_t \in \mathrm{Gal}(H/K) \tag{18}$$

attached to the archimedean places $\infty_1, \ldots, \infty_t$ of K act on the units $u_{\mathfrak{A}}$ according to the rules

$$c_j(u_{\mathfrak{A}}) = \begin{cases} u_{\mathfrak{A}} & \text{if } j = 1; \\ u_{\mathfrak{A}}^{-1} & \text{if } j = 2, \ldots, t. \end{cases} \tag{19}$$

When K is totally real, it follows that the $u_{\mathfrak{A}}$ belong to the subfield $H^{c_1=1}$ which is real at all the places above ∞_1, and therefore the quantities $\tilde{v}_1(u_{\mathfrak{A}})$ can in principle be recovered (up to a mere sign ambiguity) from (15). This equation therefore provides a compelling approach to explicit class field theory and Hilbert's twelfth problem for totally real fields.

When K is an ATR extension (which encompasses the motivating scenario of (3) and (5) where $t = 1$ and K is a quadratic imaginary field) the Frobenius c_1 attached to

∞_1 is the identity and (19) places no restrictions on the complex numbers $\tilde{v}_1(u_{\mathfrak{A}})$. This leads to the following natural question:

QUESTION 2.1. *When K is an ATR extension, is there an analytic formula for the quantities $\tilde{v}_1(u_{\mathfrak{A}})$ occurring in* (15) *and not just their absolute values?*

A positive answer would constitute a non-trivial refinement of the Stark conjecture, and (as will be further explained in Section 8 below) is somewhat in the spirit of the "refined conjectures" that later emerged from the work of Tate in collaboration with Barry Mazur.

Question 2.1 appears to have been raised explicitly for the first time in Samit Dasgupta's Harvard undergraduate senior thesis [**Das99**], which examines its first non-trivial instance, where K is a complex cubic field. (Cf. the final paragraph in §5.5 of loc.cit.) Tian Ren and Robert Sczech later proposed a conjectural analytic expression for the units $u_{\mathfrak{A}}$ in this case [**RS09**] (up to a small root of unity ambiguity), through a careful refinement of Shintani's cone decompositions.

The article [**CD08**] considers the case where K is a quadratic ATR extension of a totally real field F of degree n over $\mathbb{Q}$, expressing the Stark units $\tilde{v}_1(u_{\mathfrak{A}}) \in \mathbb{C}^\times$ in terms of special values of certain *analytic cocycles* attached to Hilbert modular Eisenstein series of parallel weight two over F. More precisely, a "transgression" of the n-cocycle encoding the periods of such Eisenstein series is used to construct certain $(n-1)$-cocycles

$$\kappa_{eis} \in Z^{n-1}(\Gamma, \mathcal{O}_{\mathcal{H}}^\times), \qquad \Gamma = \mathbf{SL}_2(\mathcal{O}_F), \tag{20}$$

where $\mathcal{O}_{\mathcal{H}}^\times$ is the multiplicative group of analytic functions on the Poincaré upper half-plane $\mathcal{H}$. The Hilbert modular group Γ attached to F is viewed as a subgroup of $\mathbf{SL}_2(\mathbb{R})$ via the real embedding $v_1 : F \longrightarrow \mathbb{R}$ attached to the distinguished complex place ∞ of K, and thereby acts on $\mathcal{H}$ by Möbius transformations.

Assume for simplicity that F has narrow class number one. The choice of a basis of an ideal class $\mathfrak{A}$ of K gives rise to a torus $T \simeq K_1^\times$ in $\mathbf{SL}_{2/F}$ for which

$$T(F) \cap \Gamma = T(\mathcal{O}_F) = (\mathcal{O}_K)_1^\times \simeq \mathbb{Z}^{n-1} \times (\mathbb{Z}/2\mathbb{Z}), \tag{21}$$

and T has a unique fixed point $\tau_{\mathfrak{A}}$ in $\mathcal{H}$. Restriction to $T(\mathcal{O}_F)$ composed with evaluation at $\tau_{\mathfrak{A}}$ gives rise to an *evaluation map*

$$\mathrm{ev}_{\mathfrak{A}} : H^{n-1}(\Gamma, \mathcal{O}_H^\times) \longrightarrow H^{n-1}(T(\mathcal{O}_F), \mathbb{C}^\times) = H^{n-1}(\mathbb{Z}^{n-1}, \mathbb{C}^\times) \simeq \mathbb{C}^\times. \tag{22}$$

The main conjecture which is formulated and tested experimentally in [**CD08**] is that, up to a possible torsion ambiguity,

$$\mathrm{ev}_{\mathfrak{A}}(\kappa_{\mathrm{eis}}) \stackrel{?}{=} \tilde{v}_1(u_{\mathfrak{A}}). \tag{23}$$

A more recent work of Bergeron, Charollois and Garcia [**BCG23**] revisits the Ren-Sczech setting where K is a complex cubic field and offers an explicit conjectural formula for the quantities $\tilde{v}_1(u_{\mathfrak{A}})$ which is closer in spirit to the cocycle-based approach of [**CD08**]. It involves the special values of a remarkable cocycle

$$\kappa_{BCG} \in Z^1(\Gamma, \mathcal{A}^\times), \qquad \Gamma = \mathbf{SL}_3(\mathbb{Z}), \tag{24}$$

where $\mathcal{A}^\times$ is the multiplicative group of complex-valued functions on an archimedean symmetric space attached to $\mathbf{SL}_3(\mathbb{R})$ which are meromorphic on a suitable region. This cocycle is constructed from the "elliptic Gamma function" of Felder and Varchenko

[**FV00**] and is described by explicit infinite products reminiscent of the product expansions of the Dedekind eta function of (2). It can be evaluated at tori in $\mathbf{SL}_3$ corresponding to ideal classes in *arbitrary* cubic ATR extensions of $\mathbb{Q}$, leading to a tantalising analogue of the Kronecker limit formula of (3) for complex cubic fields.

Pierre Morain has extended the construction of Bergeron, Charollois and Garcia by constructing a striking hierarchy of cocycles [**Mo24**]

$$\kappa_n \in Z^{n-2}(\Gamma, \mathcal{A}^\times), \qquad \Gamma = \mathbf{SL}_n(\mathbb{Z}), \tag{25}$$

which can be evaluated at tori attached to (narrow) ideal classes in ATR fields of any degree $n \geq 3$. The approach of Ren and Sczech has also been extended to this setting in the recent PhD thesis of Kairi Giovanna Black [**Bl25**].

The breakthroughs in [**BCG23**] and [**Mo24**] suggest an affirmative answer to Question 2.1, supplying a framework that is more refined than leading terms of L-functions and is able to detect the arguments and not just the absolute values of the quantities $\tilde{v}_1(u_{\mathfrak{A}})$ occurring in (15).

Even when K is totally real, there are good theoretical and practical reasons, some of which will be touched on in the next section, to want to refine Stark's conjecture by removing the sign ambiguity in the formulae for Stark units. Gene Kopp's "Shintani-Faddeev cocycle" in $H^1(\mathbf{SL}_2(\mathbb{Z}), \mathcal{A}^\times)$, where $\mathcal{A}$ is the group of analytic functions on $\mathcal{H}$, leads to an intriguing counterpart of the Kronecker limit formula for real quadratic fields [**Kop25**]. It would be interesting to present a unified perspective encompassing the Shintani-Faddeev cocycle along with those of [**CD08**], [**BCG23**], and [**Mo24**].

3. Equiangular configurations of complex lines

The complex embeddings $\tilde{v}_j$ of H above the real places v_j for $j = 2, \ldots, t$ do not play a role in the Stark conjecture (15) or in Question 2.1. Because of (19), the images $\tilde{v}_j(u_{\mathfrak{A}})$ of the Stark units $u_{\mathfrak{A}}$ are complex numbers of absolute value 1, and hence they satisfy $\log|\tilde{v}_j(u_{\mathfrak{A}})| = 0$. The following is superficially similar to Question 2.1:

Question 3.1. *When K is totally real or ATR, is there an analytic formula for the quantities $\tilde{v}_j(u_{\mathfrak{A}})$ for $2 \leq j \leq t$ arising from the Stark units $u_{\mathfrak{A}}$?*

In contrast to Question 2.1 towards which decisive progress has been made thanks to [**BCG23**] and [**Mo24**], Question 3.1 remains essentially untouched.

Its importance is underscored by a surprising recently discovered geometric application of Stark's conjecture to a question motivated by the theory of quantum designs, which provides a modern illustration of the continuing impact of Tate's ideas, and which we cannot resist describing briefly.

Let V be a complex hermitian space of dimension $d < \infty$. The *complex angle* between a pair (L_1, L_2) of complex lines in V is the one whose cosine is $|\langle e_1, e_2\rangle|$, where e_1 and e_2 are unit vectors spanning L_1 and L_2 respectively. A collection $(L_1, \ldots, L_m)$ of lines in V is said to be *equiangular* if the angle between L_i and L_j is *independent* of the pair (i, j) with $i \neq j$. Letting e_j be a unit vector in L_j, the orthogonal projections π_i onto L_i satisfy

$$\pi_j \pi_i \pi_j = \alpha^2 \pi_j, \quad \text{for all } i \neq j, \qquad \text{where } \alpha := |\langle e_i, e_j\rangle|. \tag{26}$$

These projections are readily seen to be linearly independent in $\mathrm{End}(V)$, and it follows that $m \leq d^2$.

The following terminology dates back to [**RBSC04**], where the connection with the theory of quantum measurements was first observed:

Definition 3.2. *A maximal equiangular configuration* $(L_1, \ldots, L_{d^2})$ *is called a SIC-POVM (*Symmetric Informationally Complete Positive Operator-Valued Measure*), or simply a SIC for short.*

It can be shown that the common angle in a SIC is necessarily given by

$$\alpha = \frac{1}{\sqrt{d+1}}, \tag{27}$$

but the following existence question remains open:

Question 3.3. *Is there a SIC in every dimension* d*?*

It is not even known at present whether SICs exist for infinitely many d. In his 1999 PhD thesis [**Za99, Za11**], Gerhard Zauner conjectured that Question 3.3 has an affirmative answer, and that some notable SICs arise as the orbit of a single line under the action of a finite *Heisenberg group* acting irreducibly on V.

Assume henceforth, for ease of exposition, that d is *an odd prime*. The Heisenberg group G_d of cardinality d^3 is a non-trivial central extension of $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}/d\mathbb{Z}$. It fits into a non-split exact sequence

$$1 \longrightarrow Z \longrightarrow G_d \longrightarrow \mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z} \longrightarrow 1, \qquad Z = \mathbb{Z}/d\mathbb{Z}, \tag{28}$$

where Z is both the center and the commutator subgroup of G_d. One model for G_d realizes it as the group of 3×3 upper-triangular unipotent matrices with entries in the field $\mathbb{Z}/d\mathbb{Z}$.

The (elementary) finite counterpart of the Stone-Von Neumann theorem asserts that an irreducible representation of G_d is completely determined by its central character when the latter is non-trivial. Furthermore, this distinguished representation is faithful, and of dimension d. After fixing a non-trivial central character $\psi : Z \longrightarrow \mathbb{C}^\times$, assume that V realises the irreducible representation of G_d attached to ψ, and let

$$\varrho : G_d \longrightarrow \mathrm{Aut}(V) \tag{29}$$

be the resulting group homomorphism. The center Z of G_d acts trivially on the set of lines in V, and a line L whose stabiliser is equal to Z has an orbit under G_d of size d^2. A SIC of the form

$$S := \{g \cdot L\}_{g \in G_d} \tag{30}$$

is called a *Heisenberg SIC*, the line L is called a *fiducial line* attached to S, and a unit vector $e \in L$ is called a *fiducial vector*.

The search for Heisenberg SICs is vastly facilitated by the fact that these special classes of SICs are entirely determined by a single fiducial vector $e \in V$ rather than by a configuration of d^2 lines. The complex numbers

$$\alpha_g = \langle g \cdot e, e \rangle, \qquad g \in G_d, \tag{31}$$

whose absolute values are all equal to α for all $g \notin Z$, can furthermore be understood as a specific *matrix coefficient* in the representation (V, ϱ). Finally, the fiducial line L is readily recovered from the data of the α_g, as the image of the rank one projector

$$\Pi := \sum_{g \in G_d} \alpha_g \cdot \varrho(g). \tag{32}$$

To find a Heisenberg SIC, it therefore suffices to find a matrix coefficient $\{\alpha_g\}_{g\in G_d}$ for (V,ϱ) satisfying

$$|\alpha_g|^2 = \frac{1}{d+1}, \qquad \text{for all } g \in G_d - Z. \tag{33}$$

A remarkable insight which was first proposed in [**AYZ13**] and was gradually sharpened in a series of experimental works by many physicists predicts that matrix coefficients with this property arise from suitable Stark units in a ray class field of the real quadratic field

$$K = \mathbb{Q}(\sqrt{(d+1)(d-3)}). \tag{34}$$

(Cf. [**AFK26**] for a definitive and up-to-date account, as well as the bibliography therein.) The quadratic order $\mathcal{O}_D = \mathbb{Z}[\frac{D+\sqrt{D}}{2}]$ of discriminant $D := (d+1)(d-3)$ is distinguished by the fact that it has an explicit unit

$$\varepsilon_d = \frac{(d-1)+\sqrt{(d+1)(d-3)}}{2} \tag{35}$$

of relatively small height relative to D.

Let H be the narrow Hilbert class field of K and let H_d be the narrow ray class field of K of conductor (d). Global class field theory identifies the Galois group of H_d over H with

$$\mathrm{Gal}(H_d/H) = (\mathcal{O}_K/d)^\times/\langle\varepsilon_d\rangle. \tag{36}$$

The fact that the image of ε_d in $(\mathcal{O}_K/d)^\times$ has order 3 shows that

$$[H_d : H] = \begin{cases} \frac{d^2-1}{3} & \text{if } d \equiv 1 \pmod 3, \\ \frac{d^2+1}{3} & \text{if } d \equiv 2 \pmod 3. \end{cases} \tag{37}$$

(It also accounts for an empirically observed order 3 symmetry in Heisenberg SIC-POVM's, called the *Zauner symmetry.*) Assume from now on that the prime d is congruent to 2 modulo 3.

Let $u_d \in \mathcal{O}_{H_d}^\times$ be a unit of H_d and let

$$\alpha_g = \psi(i)\cdot \mathrm{rec}(j+k\varepsilon_d)\left(\frac{u_d}{\sqrt{d+1}}\right), \qquad \text{for } g = \begin{pmatrix} 1 & j & i \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}. \tag{38}$$

The following conjecture, which is known in the field as the *twisted convolution conjecture*, asserts something remarkable about Stark units attached to the ray class field H_d. (We state it here in a somewhat vague form, and more precise statements can be found in [**AFK26**].)

Conjecture 3.4. *There is an explicit unit u_d, whose square is a Stark unit in H_d, and for which the function α_g of* (38) *is a matrix coefficient for* (V,ϱ).

This intriguing property of Stark units seems to have entirely escaped the attention of number theorists and was discovered experimentally by physicists engaged in the numerical search for Heisenberg SICs. It is now the basis for the most efficient numerical calculation of Heisenberg SICs, notably the recent `Zauner.jl` package written in the computer language Julia that is documented in [**AFK26**].

The numbers α_g arising from a Stark unit are obtained from an analytic calculation of their images under a real embedding $\tilde{v}_1$ of H_d, as the RM values of Kopp's Shintani-Faddeev modular cocycle. These real numbers are of varying magnitudes and the kernel of the rank one projector

$$\Pi_1 := \sum_{g \in G} \tilde{v}_1(\alpha_g) \cdot \varrho(g) \tag{39}$$

is far from being a fiducial line for a Heisenberg SIC. The Stark conjecture predicts, on the other hand, that the complex numbers $\tilde{v}_2(\alpha_g)$ all lie on the circle of radius $1/\sqrt{d+1}$, for any complex embedding $\tilde{v}_2$ of H_d that lies above ∞_2. Since the rank one property is also satisfied by the projector

$$\Pi_2 := \sum_{g \in G} \tilde{v}_2(\alpha_g) \cdot \varrho(g), \tag{40}$$

the image of the projector Π_2 is a fiducial line for a Heisenberg SIC.

Note that the quantities α_g are calculated in practice by first coaxing the real numbers $\tilde{v}_1(\alpha_g)$ out of the RM values of a modular cocycle encoding the leading terms of suitable abelian L-functions, recognizing these real numbers as algebraic numbers in H_d, and finally applying to them the complex embedding $\tilde{v}_2$. A direct analytic expression for the quantities $\tilde{v}_2(\alpha_g)$, in the spirit of a positive answer to Question 3.1, would lead to a more efficient, robust, and purely analytic procedure for calculating the still mysterious Heisenberg SICs that ostensibly arise from Stark units.

4. The p-adic Stark conjecture

Tate's popularisation of the Stark conjecture spurred the study of its p-adic variants. The time for this was ripe in the early 80's, since p-adic L-functions attached to finite order characters of totally real fields had recently been constructed by Pierrette Cassou-Noguès [**CN79**] by interpolating Shintani's explicit formulae, and by Serre [**Se73**] and Deligne-Ribet [**DR80**] following an approach of Siegel exploiting the occurrence of the relevant classical L-values as constant terms of Hilbert Eisenstein series. It was thus understood how to associate a p-adic L-function $L_p(\chi, s)$ to any totally real field K and finite order character χ of K which is either totally even or totally odd. These p-adic L-functions (following a somewhat non-standard normalisation) interpolate the special values $L(\chi, j)$ at negative integers satisfying

$$j \equiv \begin{cases} 1 \pmod{p-1} & \text{if } \chi \text{ is even,} \\ 0 \pmod{p-1} & \text{if } \chi \text{ is odd.} \end{cases} \tag{41}$$

The parity restriction on χ is required to ensure that the classical L-function $L(\chi, s)$ admits sufficiently many non-zero critical values at negative integers to allow for their p-adic interpolation.

Two entirely different p-adic analogues of the Stark conjectures were proposed almost simultaneously: by Jean-Pierre Serre, with whom Tate maintained a regular correspondence over many decades (cf. [**ST15**, Letter of Sept.12 1978] and [**Se78**]), and by Dick Gross [**Gr81**], who was Tate's PhD student from 1974 to 1978. These conjectures concern the behaviour of abelian L-series at the points $s = 1$ and $s = 0$ respectively, which, in contrast to the archimedean setting, are not related to each other in a simple way under the functional equation.

Serre's p-adic Stark conjecture considers the behaviour at $s = 1$ of p-adic Artin L-series. In the special case of a non-trivial even character χ, which cuts out a totally real abelian extension H of K, the induced representation $V = \mathrm{Ind}_H^{\mathbb{Q}}(\chi)$ occurs with multiplicity $n := [K : \mathbb{Q}]$ in the units of the Galois closure H' of H. Serre conjectures that $L_p(\chi, 1)$ is equal to a non-zero algebraic multiple of a p-adic regulator $R_p(V)$, defined by the same formula as in (11) for $R(V)$, but replacing complex by p-adic logarithms (after fixing a p-adic embedding of H'). Serre's regulator involves the p-adic logarithms of *genuine units*. In part because of this, the p-adic Stark conjecture at $s = 1$ remains just as wide open as its archimedean counterpart. (Cf. also [**So02**] for a more in-depth discussion and a more precise statement in the special case of abelian characters of a totally real field.)

Gross's p-adic Stark conjecture considers the behaviour at $s = 0$ of the p-adic L-function of a *totally odd* character χ, which cuts out a CM abelian extension H of the totally real field K. In this setting, the induced representation $V = \mathrm{Ind}_H^{\mathbb{Q}}(\chi)$ now occurs with *multiplicity zero* in the units of H', reflecting the fact that the classical L-function $L(\chi, s)$ is non-vanishing at $s = 0$.

Choose a prime p at which V is unramified, and let σ_p be an associated Frobenius element in $\mathrm{Aut}(V)$ attached to a prime of H' above p. The p-adic L-function $L_p(\chi, s)$ interpolates the critical values of $L(\chi, s)$ at suitable negative integers, with the Euler factors at the primes above p removed. This Euler factor vanishes at $s = 0$ precisely when there is a prime $\mathfrak{p}$ of K above p for which $\chi(\mathfrak{p}) = 1$. In this case, V occurs with non-zero multiplicity in the p-unit group of H'. Let

$$\begin{aligned} r_p &= \dim_E \hom_{G_{\mathbb{Q}}}(V, (\mathcal{O}_{H'}[1/p])^\times \otimes E) \\ &= \#\{\mathfrak{p}|p \ \text{ s.t. } \chi(\mathfrak{p}) = 1\} = \dim_E V^{\sigma_p = 1} \end{aligned}$$

denote this multiplicity. Gross conjectures that $L_p(\chi, s)$ vanishes to order exactly r_p at $s = 0$ and that its leading term is equal to a non-zero algebraic multiple of a p-adic regulator $R_p^{\mathrm{Gr}}(V)$. This regulator is built from p-units rather than actual units, and depends on a choice of bases $\{\phi_1^{(p)}, \ldots, \phi_{r_p}^{(p)}\}$ and $\{w_1, \ldots, w_{r_p}\}$ of the E-vector spaces

$$\mathrm{Hom}_{G_{\mathbb{Q}}}\big(V, (\mathcal{O}_{H'}[1/p])^\times \otimes E\big) \quad \text{and} \quad V^{\sigma_p = 1} \tag{42}$$

respectively. Note that $\phi_i^{(p)}(w_j)$ can be viewed as an element of $\mathbb{Q}_p^\times$ thanks to the chosen p-adic embedding of H. The Gross regulator $R_p^{\mathrm{Gr}}(V)$ can then be defined just as in (11):

$$R_p^{\mathrm{Gr}}(V) := \det\Big(\big(\log_p \phi_i^{(p)}(w_j)\big)_{1 \le i,j \le r_p}\Big). \tag{43}$$

A rough version of Gross's conjecture asserts that

$$\lim_{s \longrightarrow 0} s^{-r_p} L_p(V, s) = R_p^{\mathrm{Gr}}(V) \pmod{\bar{\mathbb{Q}}^\times}. \tag{44}$$

The Gross regulator is of a fundamentally different nature than the Serre regulator $R_p(V)$, since it involves the p-adic logarithms of p-units rather than of genuine units. Unlike Serre's p-adic Stark conjecture at $s = 1$, Gross's conjecture turns out to be more tractable than its archimedean counterpart, essentially because Gross-Stark p-units can often be "read off" from the p-adic deformation theory of the Artin representation of G_K attached to χ from global class field theory. This key fact is exploited in [**DDP11**], as well as in [**DKV18**], [**DLR15a**], and [**DLR17**].

In the case where $r_p = 1$ which is the most germane to explicit class field theory for K, Gross's conjecture is proved in [**DDP11**] (under certain technical assumptions on χ which were later removed by Kevin Ventullo). The case of general r_p was subsequently disposed of in a series of breakthroughs [**DKV18**], [**DK24**]. All of these works owe much to the deformation theoretic methods used in the proof by Greenberg and Stevens of the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum, which will be touched on in the next section.

5. The Mazur-Tate-Teitelbaum conjecture

The existence of two fundamentally different p-adic analogues of the Stark conjecture underscores the richness of the non-archimedean setting and its propensity to reveal new phenomena not readily apparent in the complex picture. This may have led Tate to reflect on p-adic analogues of the Birch and Swinnerton–Dyer conjecture in the early 1980's, leading to a highly influential collaboration with Barry Mazur and Jeremy Teitelbaum which uncovered a plethora of unexpected features. This sentiment is expressed in the following passage from the introduction of [**MTT86**]:

> It seemed to us, then, to be an appropriate time to embark on the project of formulating a p-adic analogue of the conjecture of Birch and Swinnerton-Dyer, and gathering numerical data in its support. It also seemed, at the outset, that this would be a relatively routine project. The project has proved to be anything but routine, and this article is an attempt to report on our findings so far."

The p-adic L-function attached to an elliptic curve E – or rather, to the associated weight two cusp form f_E, whose existence at the time was only conjectural – had already been constructed in [**MS74**] as a p-adic Mellin transform of a measure on $\mathbb{Z}_p^\times$ arising from the *modular symbol* attached to f_E. This modular symbol is the $\Gamma_0(N)$-invariant additive function

$$m_E\{\ ,\ \} : \mathbb{P}_1(\mathbb{Q}) \times \mathbb{P}_1(\mathbb{Q}) \longrightarrow \mathbb{C} \tag{45}$$

defined by

$$m_E\{x, y\} := \int_x^y 2\pi i f_E(z) dz. \tag{46}$$

It takes values in a lattice $\Lambda_E \subset \mathbb{C}$ commensurable with the Néron lattice of E. Let Ω_E^+ (resp. Ω_E^-) be the positive generator of the projection of Λ_E to the real (resp. imaginary) axis in $\mathbb{C}$, so that

$$\Lambda_E \subset \mathbb{Z} \cdot \Omega_E^+ \oplus \mathbb{Z} \cdot \Omega_E^- \quad \text{with index} \quad \leq 2. \tag{47}$$

The periods Ω_E^+ and Ω_E^- can be used to extract a pair $m_E^\pm\{\ ,\ \}$ of $\mathbb{Z}$-valued modular symbols from $m_E\{\ ,\ \}$ by setting

$$m_E\{x, y\}_E = m_E^+\{x, y\} \cdot \Omega_E^+ + m_E^-\{x, y\} \cdot \Omega_E^- \qquad \text{for all } x, y \in \mathbb{P}_1(\mathbb{Q}). \tag{48}$$

The $\mathbb{Z}$-valued symbol $m_E^+\{\ \ \}$ gives an integral measure μ_{f_E} on $\mathbb{Z}_p^\times$ by setting

$$\int_{a+p^n\mathbb{Z}_p} d\mu_{f_E} := \{a/p^n, \infty\}_E^+, \tag{49}$$

and $L_p(E,s)$ is the Mazur-Mellin transform of this measure:

$$L_p(E,s) = \int_{\mathbb{Z}_p^\times} \langle x\rangle^{s-1} d\mu_{f_E}(x), \qquad \text{with } \langle x\rangle := x \lim_{n\to\infty} x^{-p^n}. \tag{50}$$

With the p-adic L-function in hand, the naive p-adic variant of the Birch Swinnerton–Dyer conjecture would predict that $L_p(E,s)$ has a zero of order $r := \mathrm{rank}(E(\mathbb{Q}))$ and that the leading term $L_p^{(r)}(E,1)$ can be expressed in terms of the usual quantities arising in the Birch and Swinnerton Dyer conjecture – the cardinality of the conjecturally finite Shafarevich-Tate group of E, and a *regulator* whose definition is the same as the classical regulator, after replacing the Néron-Tate canonical height by a p-adic height pairing

$$\langle\ ,\ \rangle_p : E(\mathbb{Q}) \times E(\mathbb{Q}) \longrightarrow \mathbb{Q}_p. \tag{51}$$

This is indeed what the p-adic conjecture predicts in general, with one seemingly minor but crucial exception: when E has *split multiplicative reduction at p* — i.e., when the prime p divides N exactly and the p-th Fourier coefficient of f_E is equal to 1 — the order of vanishing of $L_p(E,s)$ seems to be thrown off, and to *increase by one*. Thus $L_p(E,s)$ acquires what the authors of [**MTT86**] refer to as an *exceptional zero*. On the "arithmetic side", the elliptic curve E is endowed with Tate's celebrated p-adic uniformisation

$$\Phi_{\mathrm{Tate}} : \mathbb{Q}_p^\times / q^{\mathbb{Z}} \longrightarrow E(\mathbb{Q}_p), \tag{52}$$

where $q \in p\mathbb{Z}_p$ is the *Tate period* of E which is one of the key *dramatis personae* in the contributions by Brian Conrad and Dustin Clausen in this volume.

To account for the leading term of $L_p(E,s)$ in the presence of an exceptional zero, Mazur, Tate, and Teitelbaum introduce the *extended Mordell-Weil group*

$$E^\dagger(\mathbb{Q}) := \mathbb{Q}_p^\times \times_{\Phi_{\mathrm{Tate}}} E(\mathbb{Q}) = \{(z,P) \in \mathbb{Q}_p^\times \times E(\mathbb{Q}) \text{ such that } \Phi_{\mathrm{Tate}}(z) = P\}, \tag{53}$$

which fits into an exact sequence

$$1 \longrightarrow q^{\mathbb{Z}} \longrightarrow E^\dagger(\mathbb{Q}) \longrightarrow E(\mathbb{Q}) \longrightarrow 1. \tag{54}$$

It is an abelian group of rank $r+1$ when E has split multiplicative reduction at p, and of rank r otherwise. The main prediction of [**MTT86**] is

Conjecture 5.1 (Mazur-Tate-Teitelbaum). *The p-adic L-function satisfies*

$$\mathrm{ord}_{s=1} L_p(E,s) = \mathrm{rank}(E^\dagger(\mathbb{Q})).$$

To account for the leading term of $L_p(E,s)$ in the presence of an exceptional zero, the p-adic height pairing of (51) is extended to (a finite index subgroup of) $E^\dagger(\mathbb{Q})$ by the rules

$$\langle (z_1,P_1),(z_2,P_2)\rangle_p = \langle P_1,P_2\rangle_p, \qquad \text{whenever } z_1,z_2 \in \mathbb{Z}_p^\times, \tag{55}$$

together with the complementary rules for the extra element $q = (q,0) \in E^\dagger(\mathbb{Q})$ corresponding to the Tate period:

$$\langle q,(P,z)\rangle_p = \log_p(z), \qquad \langle q,q\rangle_p = \log_p(q)/\mathrm{ord}_p(q), \tag{56}$$

where $\log_p : \mathbb{Q}_p^\times/p^{\mathbb{Z}} \longrightarrow \mathbb{Q}_p$ is Iwasawa's branch of the p-adic logarithm satisfying $\log_p(p) = 0$. The *extended regulator* is the determinant

$$R_p^\dagger(E) = \det\left(\langle x_i,x_j\rangle_p\right)_{1\le i,j\le r+1}, \tag{57}$$

where $x_1, \ldots, x_{r+1}$ is a $\mathbb{Z}$-basis of $E^\dagger(\mathbb{Q})$. A more precise version of Conjecture 5.1 asserts that the leading term of $L_p(E,s)$ is a product of this regulator times the other terms that appear in the classical Birch and Swinnerton-Dyer conjecture like the cardinality of the conjecturally finite Shafarevich-Tate group of E.

In the special case where $r = 0$ and p is a prime of split multiplicative reduction for E, the Mazur-Tate-Teitelbaum conjecture asserts that $L_p(E,s)$ has a simple zero at $s = 1$ and that

$$L_p'(E,1) = \frac{\log_p(q)}{\mathrm{ord}_p(q)} \cdot \frac{L(E,1)}{2 \cdot \Omega_E^+}. \tag{58}$$

This strikingly concrete statement asserts that the modular symbol attached to E, obtained from the modular parametrisation of E over $\mathbb{C}$, "knows about" the period q that Tate had discovered more than a decade earlier in his p-adic uniformisation theory of elliptic curves! As the authors comment in the introduction to [**MTT86**],

> It is quite remarkable to see the p-adic digits of this [Tate] p-adic period be "reproduced" in the print-out of a computing machine programmed to compute [...] certain expressions involving modular symbols."

The extensive computer calculations carried out by the authors of [**MTT86**] to test their predictions is one of the high points of what can now be seen as the twilight of the "heroic era" in computational number theory.

Formula (58) came to be known as the "exceptional zero conjecture". It has turned out to be enormously impactful, and has been generalised to a wide variety of settings. The original conjecture was proved in an influential work of Greenberg and Stevens [**GS93**] which exploits several important new ingredients: most significantly, the theory of p-adic deformations of modular forms and their associated Galois representations and p-adic L-functions.

For general r, the Mazur–Tate–Teitelbaum conjecture seems as difficult as the original Birch and Swinnerton-Dyer conjecture, even if one inequality –

$$\mathrm{ord}_{s=1} L_p(E,s) \geq \mathrm{rank}(E^\dagger(\mathbb{Q})), \tag{59}$$

has been proved by Kazuya Kato [**Ka04**] using ideas from the theory of "Euler systems". Kato's proof rests on the Iwasawa-theoretic interpretation of the p-adic L-function $L_p(E,s)$, as encoding the behaviour of E over the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, which is contained in the field

$$\mathbb{Q}_\infty := \bigcup_{n=1}^{\infty} \mathbb{Q}_n, \qquad \mathbb{Q}_n \subset \mathbb{Q}(e^{2\pi i/p^n}), \tag{60}$$

where $\mathbb{Q}_n$ is the maximal p-power degree subfield of the field of p^n-th roots of unity. The Iwasawa main conjecture for $L_p(E,s)$ relates this analytic invariant to the *characteristic power series* of the p-Selmer group of E over $\mathbb{Q}_\infty$, viewed as a module over the Iwasawa algebra $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$. Kato does not prove the full main conjecture but shows that $L_p(E,s)$ is divisible by the characteristic power series of the pro-p Selmer group, which directly implies (59) but is somewhat stronger and more precise.

The mechanism whereby the rank of E forces vanishing of the associated L-function at the central point is thus better understood than in the complex setting, where no elliptic curve E with Hasse-Weil L-function satisfying $\mathrm{ord}_{s=1} L(E,s) > 3$ has so far been

shown to exist. The reverse inequality in (59) also lies much deeper, and does not follow from the corresponding inequality in the Main Conjecture.

6. The Mazur-Tate conjecture

In [**MT87**], Mazur and Tate propose to replace the $\mathbb{Z}_p$-extension of (60) by a *finite abelian extension* H of $\mathbb{Q}$ with Galois group G, the Iwasawa algebra by the integral group ring $\mathbb{Z}[G]$, and the p-adic L-function by a "theta element" in $\mathbb{Z}[G]$, which plays the role of the Stickelberger element in the Brumer-Stark conjecture.

The setting considered primarily in [**MT87**] is the one where $H = H_M := \mathbb{Q}(\zeta_M)^+$ is the real subfield of the M-th cyclotomic field, so that

$$G := G_M = (\mathbb{Z}/M\mathbb{Z})^\times/\langle\pm 1\rangle, \tag{61}$$

and where E is a (modular) elliptic curve over $\mathbb{Q}$. Like the Mazur-Swinnerton-Dyer p-adic L-function of [**MTT86**], the resulting theta element

$$\theta_{E,M} := \sum_{a\in(\mathbb{Z}/M\mathbb{Z})^\times} m_E^+\{a/M,\infty\}\cdot[a] \quad\in\quad \mathbb{Z}[G_M] \tag{62}$$

is constructed from the modular symbol attached to E, and satisfies the interpolation property

$$\chi(\theta_{E,M}) = \frac{\tau(\chi)L(E,\bar\chi,1)}{2\cdot\Omega_E^+} \qquad \text{for all even primitive } \chi : G_M \longrightarrow \mathbb{C}^\times, \tag{63}$$

where $\tau(\chi)$ is a Gauss sum attached to χ. In particular, $\theta_{E,M}$ belongs to the augmentation ideal I of $\mathbb{Z}[G_M]$ whenever $L(E,1)=0$. The "refined Birch and Swinnerton-Dyer conjecture" of [**MT87**] asserts that $\theta_{E,M}$ vanishes to order at least $r := \mathrm{rank}(E(\mathbb{Q}))$ at the trivial character, namely, that

$$\theta_{E,M} \overset{?}{\in} I^r. \tag{64}$$

It also proposes a formula for the leading term of $\theta_{E,M}$, i.e., its natural image in I^r/I^{r+1}, involving the same quantities as in the Birch and Swinnerton–Dyer conjecture, but with the Néron-Tate regulator replaced by a finite "Mazur-Tate regulator". This regulator is defined in terms of the *Mazur-Tate height pairing* that will be discussed in slightly more detail in Section 8 below:

$$\langle\ ,\ \rangle_{E,M} : E_M(\mathbb{Q})\times E(\mathbb{Q}) \longrightarrow G_M = I/I^2, \tag{65}$$

where $E_M(\mathbb{Q})$ is the group of points in the connected component of the Néron model of E over $\mathbb{Z}$ that reduce to the identity modulo M. The Mazur-Tate regulator is the determinant of the associated $r\times r$ pairing matrix, with entries in I/I^2, and therefore naturally lies in I^r/I^{r+1}.

The point of view introduced in [**MT87**] is broadly applicable and reveals a variety of new and intriguing arithmetic phenomena. It was extended almost immediately to the setting of abelian L-series of totally real fields by Dick Gross, leading to "tame refinements" [**Gr88**] of the p-adic Gross-Stark conjecture evoked in Section 4.

It was also Dick Gross who suggested to the author that transposing the Mazur-Tate conjecture to a setting where $\mathbb{Q}$ is replaced by an imaginary quadratic field K would provide a congenial framework for understanding the "Euler system" arguments of Kolyvagin, which burst on the scene shortly after the Mazur-Tate seminar of 1986. In Kolyvagin's approach, certain global objects (the "Kolyvagin cohomology classes")

indexed by the collection of ring class fields of K and constructed from Heegner points are used to control the Selmer group of E over K. When E has odd (analytic) rank over K, the author's PhD thesis [**Dar92**] associates a theta element $\Theta'_{E,K,c}$ to the ring class field H_c of K of any conductor c that is relatively prime to the conductor of E. Writing $G_c = \mathrm{Gal}(H_c/K)$, this theta element is obtained by fixing a Heegner point $P_c \in E(H_c)$ of conductor c, letting

$$\theta'_{E,c} = \sum_{\sigma \in G_c} P_c^\sigma \otimes \sigma, \qquad \bar{\theta}'_{E,c} = \sum_{\sigma \in G_c} P_c^\sigma \otimes \sigma^{-1} \tag{66}$$

be the formal resolvent elements in $E(H_c) \otimes \mathbb{Z}[G_c]$ arising from this Heegner point, and defining

$$\Theta'_{E,K,c} := \theta'_{E,c} \otimes \bar{\theta}'_{E,c} \in E(H_c)^{\otimes 2} \otimes \mathbb{Z}[G_c]. \tag{67}$$

Let

$$h^{\otimes 2} : E(H_c)^{\otimes 2} \longrightarrow \mathbb{R}$$

be the canonical Néron-Tate height, extended to an $\mathbb{R}$-valued linear functional on $E(H_c)^{\otimes 2} \otimes \mathbb{R}$. The Gross-Zagier formula [**GZ86**] for Heegner points over H_c suggests that $\Theta'_{E,K,c}$ satisfies the interpolation property analogous to (63)

$$(h^{\otimes 2} \otimes \chi)(\Theta'_{E,K,c}) = \frac{L'(E/K, \chi, 1)}{\Omega_E^+ \Omega_E^-} \pmod{\bar{\mathbb{Q}}^\times}, \tag{68}$$

where Ω_E^+ and Ω_E^- are as in (47).

For this reason, $\Theta'_{E,K,c}$ can be envisaged as a Mazur-Tate style avatar of the first derivative $L'(E/K, s)$. The main result of the author's thesis [**Dar92**] asserts that

$$\Theta'_{E,K,c} \text{ belongs to } E(H_c)^{\otimes 2} \otimes I^{r-1}, \quad \text{where } r := \mathrm{rank}(E(K)), \tag{69}$$

under suitable technical conditions (and after inverting some problematic small primes in the integral group ring).

The Euler system techniques of Kato were later applied in [**Ot18**] to extend the formal argument in the proof of (69) to the original setting of the Mazur-Tate conjecture, thereby establishing (64).

Building on the foundational work of Gross [**Gr88**], Mazur-Tate style variants in the setting of unit groups of number fields and the Stark conjectures have been further explored in the cyclotomic setting ([**Dar95**], [**MR11**], [**Sa14**], ...) Recent breakthroughs of Dasgupta and Kakde have established the Brumer-Stark conjecture [**DK23**], [**DKSW**] and the Mazur-Tate refinement of the Gross-Stark conjecture of [**Gr88**] for totally odd abelian extensions of totally real fields [**DK24**].

A recent conjecture of Harris and Venkatesh [**HV19**] motivated by Venkatesh's theory of "derived Hecke operators" suggests a tantalising Mazur-Tate analogue of Serre's p-adic Stark conjecture at $s = 1$ for the Artin representation attached to the adjoint of a weight one newform. This might lead to a more systematic framework for the Mazur-Tate conjecture for units in number fields, refining the two types of p-adic Stark conjectures evoked in Section 4: Gross's variant at $s = 0$ and Serre's at $s = 1$. (See [**Zh26**] for some first steps towards bringing this analogy into sharper focus.)

7. A Stark conjecture for elliptic curves

Just as with the Mazur-Tate conjectures, it is natural to explore the Mazur-Tate-Teitelbaum conjecture beyond the cyclotomic setting. A particularly appealing set-up involves replacing the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ in (60) with the *anticyclotomic* $\mathbb{Z}_p$-extension of a quadratic imaginary field K (of discriminant D, say). This $\mathbb{Z}_p$-extension is given by

$$K_\infty := \bigcup_{n=1}^{\infty} K_n, \qquad K_n \subset K\left(j(\mathbb{Z}+p^n\mathcal{O}_K)\right), \tag{70}$$

where K_n is the largest p-power degree extension of K contained in the ring class field of K of conductor p^n. The latter extension is generated over K by the j-invariants of elliptic curves with endomorphism ring equal to the order of conductor p^n in $\mathcal{O}_K$. The availability of a systematic supply of *Heegner points* in $E(K_n)$ under suitable hypotheses suggest that the anticyclotomic setting might involve features not apparent in the original cyclotomic case, and singles it out for special attention [**BD96**].

The relevant p-adic L-function attached to (E, K_∞) is denoted $\mathscr{L}_p(E/K_\infty, s)$, and can also be viewed as belonging to the Iwasawa algebra of $G_\infty := \mathrm{Gal}(K_\infty/K)$,

$$\mathbb{Z}_p[\![G_\infty]\!] := \varprojlim \mathbb{Z}_p[\mathrm{Gal}(K_n/K)] \simeq \mathbb{Z}_p[\![T]\!]. \tag{71}$$

It interpolates the central critical values $L(E/K, \chi, 1)$ as χ ranges over the continuous finite order characters of G_∞. More accurately, it interpolates their *square roots*: this is why it is denoted $\mathscr{L}_p(E/K_\infty, s)$ rather than

$$L_p(E/K_\infty, s) := \mathscr{L}_p(E/K_\infty, s)\mathscr{L}_p(E/K_\infty, s)^*, \tag{72}$$

where $\lambda \mapsto \lambda^*$ denotes the involution on $\mathbb{Z}_p[\![G_\infty]\!]$ sending each group-like element $g \in G_\infty$ to its inverse. The availability of such "square root p-adic L-functions" is a feature of the anticyclotomic setting that gives rise to some of its more compelling arithmetic applications.

When E has good reduction at p, it can be observed that $\mathscr{L}_p(E, K_\infty)$ vanishes identically unless the sign in the functional equation for $L(E/K, s)$ is equal to 1. But when E has multiplicative reduction at p and $K_p := K \otimes \mathbb{Q}_p$ is the quadratic unramified extension of $\mathbb{Q}_p$, an interesting *sign change phenomenon* ensures that the power series $\mathscr{L}_p(E, K_\infty)$ can only be non-zero when the classical L-series $L(E/K, s)$ has sign -1 in its functional equation, and hence a *zero of odd order* at $s = 1$. In that case, $L(E/K, 1) = 0$ and $\mathscr{L}_p(E, K_\infty)$ likewise vanishes at the trivial character: in other words, it belongs to the augmentation ideal I of the completed group ring $\mathbb{Z}_p[[G_\infty]]$. It then becomes natural to consider its *first derivative* with respect to s. In terms of the Iwasawa algebra description of $\mathscr{L}_p(E/K_\infty)$, this derivative corresponds to the canonical image of $\mathscr{L}_p(E, K_\infty)$ in

$$I/I^2 = G_\infty = K_{p,1}^\times, \tag{73}$$

where the first identification is a general algebraic fact and the second follows from global class field theory.

Because E has split multiplicative reduction over K_p, it is equipped with the p-adic analytic *Tate uniformisation* Φ_{Tate} of the previous section.

Suppose that $E(K)$ is of rank one. The extended Mordell-Weil group $E^\dagger(K)$ is spanned (up to torsion) by $q \in \mathbb{Q}_p^\times$ and by a Mordell-Weil generator $P \in E(K)$. The

role of the cyclotomic p-adic height pairing that enters into the Mazur-Tate conjecture is now played by an *anticyclotomic* p-adic height

$$\langle\ ,\ \rangle_K : E^\dagger(K) \times E^\dagger(K) \longrightarrow G_\infty, \tag{74}$$

whose definition owes everything to the general framework for p-adic height pairings initiated by Mazur and Tate.

Remarkably, in this specific instance it turns out that both the point P and the period q are *isotropic* relative to the (extended) anticyclotomic p-adic height:

$$\langle q, q\rangle_K = \langle P, P\rangle_K = 0, \tag{75}$$

while $\langle q, P\rangle_K$ is essentially given by

$$\langle q, P\rangle_K := \log_p(\tilde{P}), \qquad \text{where } \Phi_{\mathrm{Tate}}(\tilde{P}) = P. \tag{76}$$

The Mazur-Tate regulator in this special case acquires a remarkably simple form, namely, it is given by

$$R^\dagger(E/K) = \det\begin{pmatrix} 0 & \log_p(\tilde{P}) \\ \log_p(\tilde{P}) & 0 \end{pmatrix} = -\log_p(\tilde{P})^2. \tag{77}$$

The resulting exceptional zero conjecture leads to the following theorem:

Theorem 7.1 ([**BD98**]). *The leading term*

$$\mathscr{L}_p'(E, K_\infty) \in G_\infty = K_{p,1}^\times \tag{78}$$

maps to a global point in $E(K)$ under the Tate uniformisation.

This result can be viewed as a p-adic analogue of the rank one Stark conjecture, since it recovers a global point on E by applying the Tate "exponential" to the first derivative of a suitable (anticyclotomic) p-adic L-function. Similar phenomena had been uncovered earlier by Karl Rubin [**Ru92**] (in the setting of elliptic curves with complex multiplication) and Perrin-Riou [**PR93**] for modular elliptic curves.

The proof of Theorem 7.1 relies on an explicit p-adic analytic description of *Heegner points* arising from the Cerednik-Drinfeld theory of p-adic uniformisation of Shimura curves, and it can thus also be viewed as a p-adic variant of the Gross-Zagier formula [**GZ86**]. That the global point that accounts for the leading term of $\mathscr{L}_p(E, K_\infty)$ should be obtained via a Heegner point construction is hardly surprising – to this day, Heegner points provide the only systematic way to construct global points on elliptic curves with direct ties to L-function behaviour. But Theorem 7.1 suggests the possibility of transposing the approach of Stark's conjecture to elliptic curves, and of extracting (conjectural) global points on elliptic curves from leading terms of p-adic L-series in settings where Heegner points arising from the theory of complex multiplication are entirely unavailable. It is in this spirit that "elliptic Stark conjectures" involving Mordell-Weil groups over the fields cut out by Artin representations attached to modular forms of weight one were later explored in [**DLR15b**]. The notion of "Stark-Heegner points" over ring class fields of real quadratic fields that will be touched on in Sections 9 and 10 emerges from a similar perspective.

8. Mazur-Tate heights

In the process of formulating the p-adic Birch and Swinnerton-Dyer conjecture of [**MTT86**] and its tame variants [**MT87**], Mazur and Tate were led to a foundational study of Néron-Tate style height pairings on the Mordell-Weil group of an elliptic curve over a number field K [**MT83**]. In that level of generality, they associate a canonical bilinear pairing to any modulus $\mathfrak{m} \subset \mathcal{O}_K$:

$$\langle\ ,\ \rangle_{E,\mathfrak{m}} : E_{\mathfrak{m}}(K) \times E(K) \longrightarrow C_{\mathfrak{m}} := U_{\mathfrak{m}} \backslash \mathbb{A}_K^\times / K^\times, \tag{79}$$

where $E_{\mathfrak{m}}(K)$ is the subgroup of $E(K)$ in the connected component of the Néron model of E consisting of points that reduce to 1 modulo the ideal $\mathfrak{m}$, and $U(\mathfrak{m})$ is the standard compact open subgroup of the idéles of K attached to $\mathfrak{m}$. The classical (additive) Néron-Tate height is recovered by composing $\langle\ ,\rangle_{E,\mathfrak{m}}$ with the logarithm of the idèlic norm to $\mathbb{R}^\times$, and the pairings of (65) with values in finite Galois groups that arise in the statement of the Mazur-Tate conjecture are likewise obtained by composing with the reciprocity map of global class field theory.

In the special case where K is a real quadratic field (assumed to be of class number one, to simplify the statements), the pairing (79) can be parlayed into a pairing with values in a circle group,

$$\langle\ ,\ \rangle_{\mathrm{circ}} : E(K) \times E(K) \longrightarrow \mathbb{R}/\mathbb{Z}\log(\varepsilon_K), \tag{80}$$

where ε_K is the fundamental unit of norm 1 in K, by projecting onto the connected component of the norm 1 elements of the idéle class group.

Assume that all the primes dividing the conductor N of E split in $K/\mathbb{Q}$. This implies that the ring $M_0(N) \subset M_2(\mathbb{Z})$ of matrices that are upper-triangular modulo N contains a subring isomorphic to $\mathcal{O}_K$, which is well-defined up to conjugation. The image of $\varepsilon_K \in (\mathcal{O}_K^\times)_1$ in $\Gamma_0(N) = M_0(N)_1^\times$ gives an element γ_K which has two fixed points τ_K and τ_K' on the boundary of the complex upper half-plane.

Let f be the normalised weight two newform on $\Gamma_0(N)$ that is associated to E by modularity. The period integral

$$I_{E,K} = \int_x^{\gamma_K x} f(z)dz \tag{81}$$

is an explicit non-zero multiple of $L(E/K,1)$. If $I_{E,K} = 0$, [**BD04**, Lemma 2.6] shows that the image of the "derived period integral"

$$I'_{E,K} = \int_x^{\gamma_K x} \log\left(\frac{z-\tau_K}{z-\tau_K'}\right) f(z)dz \quad \in \quad \mathbb{R}/\Omega_E^+\log(\varepsilon_K) \tag{82}$$

does not depend on the choice of base point $x \in \mathbb{P}_1(\mathbb{Q})$ that was made to define it. It can be viewed as a Mazur-Tate style avatar of the "square root" of the *second derivative* of $L(E/K,s)$ at $s=1$. This interpretation motivates the following conjecture

Conjecture 8.1 ([**BD04**], Conj 3.1). *The period integral of* (82) *vanishes when* $\mathrm{rank}(E(K)) > 2$ *and satisfies*

$$I'_{E,K} \stackrel{?}{=} \langle P^+, P^-\rangle_{\mathrm{circ}} \cdot \Omega_E^+ \pmod{\mathbb{Q}^\times \Omega_E^+ \log(\varepsilon_K)} \tag{83}$$

otherwise, where P^+ and P^- are Mordell-Weil generators of the eigenspaces $E(K)^+ = E(\mathbb{Q})$ and $E(K)^-$ for the action of $\mathrm{Gal}(K/\mathbb{Q})$.

Conjecture 8.1 can be envisaged as an archimedean fragment of a refined Birch and Swinnerton-Dyer conjecture involving the Mazur-Tate circle pairing, but it is unclear how to make sense of higher regulators for this pairing to obtain non-trivial invariants when $\mathrm{rank}(E(K)) > 2$.

Like the arguments of Stark units that arise in Questions 2.1 and 3.1, the Mazur-Tate circle pairing takes its values in a circle group and is not readily extracted from the leading terms of L-functions. Conjecture 8.1 is therefore in somewhat the same spirit as these two questions about Stark units, providing a loose analogue of them in the setting of elliptic curves.

9. The exceptional zero conjecture, revisited

The exceptional zero conjecture (58) of Mazur, Tate and Teitelbaum is profoundly evocative, and has been framed in several fruitful ways (cf. [**Or04**] and [**Br10**] for example). As the authors of [**MTT86**] point out, it reveals a surprising interplay between the modular symbol of an elliptic curve E arising from its modular parametrisation over $\mathbb{C}$ and Tate's p-adic uniformisation of E, suggesting that the modular symbol might be parlayed into a p-adic variant of the parametrisation of Shimura-Taniyama-Weil. It turns out that (58) can indeed be recast in this light, as we now proceed to explain.

Assume for simplicity that E has prime conductor p. The modularity of E yields a cusp form f_E of weight two on $\Gamma_0(p)$, and two period homomorphisms

$$\varphi_E^+, \quad \varphi_E^- \quad \in \quad H^1(\Gamma_0(p), \mathbb{Z}), \tag{84}$$

satisfying

$$\int_{z_0}^{\gamma z_0} f_E(z)dz \;=\; \varphi_E^+(\gamma)\cdot\Omega_E^+ + \varphi_E^-(\gamma)\cdot\Omega_E^-, \qquad \text{for all } \gamma\in\Gamma_0(p), \tag{85}$$

where Ω_E^+ and Ω_E^- are related to the Néron period lattice Λ_E of E as in (47). In other words, the class of

$$\Omega_E^+\cdot\varphi_E^+ + \Omega_E^-\cdot\varphi_E^- \;\in\; H^1(\Gamma_0(p), \mathbb{C}) \tag{86}$$

becomes trivial in $H^1(\Gamma_0(p), \mathcal{A}_\infty)$, where $\mathcal{A}_\infty$ is the space of holomorphic functions on the Poincaré upper half-plane equipped with the left translation action $\gamma\cdot F(\tau) := F(\gamma^{-1}\tau)$:

$$\Omega_E^+\cdot\varphi_E^+ + \Omega_E^-\cdot\varphi_E^- = dF_E, \qquad F_E(\tau) := \int_{i\infty}^{\tau} f_E(z)dz. \tag{87}$$

This property characterises the lattice Λ_E up to scaling and commensurability, and F_E is just a complex realisation of the Weil uniformisation of E:

$$F_E : \Gamma_0(p)\backslash\mathcal{H} \longrightarrow \mathbb{C}/\Lambda_E = E(\mathbb{C}). \tag{88}$$

It is natural to ask if the Tate period q of (52) admits a similar characterisation: for instance, if the period homomorphisms

$$q^{\varphi_E^+},\; q^{\varphi_E^-} \;\in\; H^1(\Gamma_0(p), \mathbb{C}_p^\times) \tag{89}$$

can be trivialised after replacing $\mathcal{A}_\infty$ by the multiplicative group $\mathcal{A}^\times$ of *rigid analytic* functions on the Drinfeld p-adic upper-half plane

$$\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p). \tag{90}$$

The fact that $\Gamma_0(p)$ does not act discretely on $\mathcal{H}_p$ precludes the existence of functions in $\mathcal{A}^\times$ satisfying the p-adic analogue of (87). A p-adic avatar of F_E is obtained instead from the cohomology of the *Ihara group*

$$\Gamma \;=\; \mathbf{SL}_2(\mathbb{Z}[1/p]) \;=\; \mathbf{SL}_2(\mathbb{Z}) *_{\Gamma_0(p)} \mathbf{SL}_2(\mathbb{Z}). \tag{91}$$

The Mayer-Vietoris sequence in group cohomology associated to its description as an amalgamated product of two copies of $\mathbf{SL}_2(\mathbb{Z})$ along a common copy of $\Gamma_0(p)$ leads to a canonical inclusion with finite cokernel

$$H^1(\Gamma_0(p), \mathbb{Z}) \hookrightarrow H^2(\Gamma, \mathbb{Z}). \tag{92}$$

The homomorphisms φ_E^+ and φ_E^- can thus be transferred to classes α_E^+ and $\alpha_E^- \in H^2(\Gamma, \mathbb{Z})$, for which the following p-adic analogue of (87) becomes available:

THEOREM 9.1 ([**Dar01**]). *The classes $q^{\alpha_E^\pm} \in H^2(\Gamma, \mathbb{C}_p^\times)$ become trivial in $H^2(\Gamma, \mathcal{A}^\times)$. In particular, there are one-cochains $J_E^\pm : \Gamma \longrightarrow \mathcal{A}^\times$ satisfying*

$$q^{\alpha_E^+} = dJ_E^+, \qquad q^{\alpha_E^-} = dJ_E^-. \tag{93}$$

Although the statement of this theorem resonates with (87), its proof is in a very different spirit. The complex periods of E are related to those of a weight two cusp form by realising E as a quotient of a modular jacobian, so the proof of (87) relies on the full force of Falting's proof of the Tate conjecture for curves. Theorem 9.1 is shown in [**Dar01**] to be a direct formal consequence of the exceptional zero conjecture (58) of [**MTT86**], whose proof by Ralph Greenberg and Glenn Stevens [**GS93**] uses techniques of Galois deformation theory, notably, the two-variable Mazur-Kitagawa p-adic L-function attached to a Hida family interpolating the newform f_E in weight two. Although in a sense Theorem 9.1 was conjectured by Mazur, Tate and Teitelbaum, and proved in [**GS93**] (more than a decade before it was stated!), the geometry underlying Theorem 9.1 remains unclear, lending it an air of mystery which has yet to be dispelled.

Just as the function F_E becomes $\Gamma_0(p)$-invariant modulo periods and therefore gives rise to a class

$$F_E \;\in\; H^0(\Gamma_0(p), \mathcal{A}_\infty/\Lambda_E), \tag{94}$$

the one-cochains J_E^+ and J_E^- of Theorem 9.1 satisfy the one-cocycle relation modulo $q^{\mathbb{Z}}$, and therefore give rise to classes

$$J_E^+,\; J_E^- \;\in\; H^1(\Gamma, \mathcal{A}^\times/q^{\mathbb{Z}}) \tag{95}$$

analogous to the modular parametrisation F_E of (87) and (94). They are called the (even and odd) *elliptic cocycles* attached to E. In passing from F_E to its p-adic counterparts J_E^+ and J_E^-, the shift in cohomological degrees is even more significant than the transition from complex to p-adic analysis.

10. Stark-Heegner points

The values of the complex uniformisation F_E of (88) at quadratic imaginary arguments of the Poincaré upper half-plane give rise to a plentiful collection of global points on E defined over ring class fields of imaginary quadratic fields. The elliptic cocycles $J_E^\pm$ can likewise be evaluated at *real quadratic* elements of $\mathcal{H}_p$, which are characterised

by the fact that their stabiliser in Γ is an abelian group of rank one, with a generator γ_τ (the so-called *automorph* of τ). The value of $J_E = J_E^+$ or J_E^- at τ is defined to be

$$J_E[\tau] := J_E(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times / q^{\mathbb{Z}} = E(\mathbb{C}_p). \tag{96}$$

Conjecture 10.1 ([**Dar01**]). *The value $J_E[\tau]$ at a real quadratic $\tau \in \mathcal{H}_p$ is a global point on E defined over an explicit ring class field of $K := \mathbb{Q}(\tau)$.*

These "real multiplication values" are called *Stark-Heegner points*. They are expected to behave "just like" the Heegner points $F_E(\tau)$ when τ is imaginary quadratic, but lie beyond the scope of the theory of complex multiplication.

While a proof of Conjecture 10.1 would seem to require substantial new ideas, extensive numerical evidence has been gathered in its support, thanks to efficient algorithms [**DP06**] for calculating elliptic cocycles based on the theory of "overconvergent modular symbols" elaborated by Robert Pollack and Glenn Stevens. For the elliptic curve

$$E_{11} = 11A : y^2 + y = x^3 - x^2 - 10x - 20 \tag{97}$$

of conductor 11 and the field $\mathbb{Q}(\sqrt{101})$ of class number one, a numerical evaluation reveals that

$$J_{E_{11}}^+ \left[\frac{1+\sqrt{101}}{2} \right] = (x/t^2, y/t^3) \pmod{11^{200}}, \tag{98}$$

where

$$\begin{aligned} x &= 108162413664469253966708468511684 9 \\ y &= -193914629777492183691609899807062004727621577550 0 \\ &\quad -45034813271762519727132587561686024065704563549 3\sqrt{101} \\ t &= 15711350731963510, \end{aligned}$$

reconstructing a global point of fairly large height in $E_{11}(\mathbb{Q}(\sqrt{101}))$ without any a priori knowledge of it.

The real quadratic field $K = \mathbb{Q}(\sqrt{1297})$ has class number 11 and its Hilbert class field H is therefore a degree 11 extension of K. Let E_{37} be the elliptic curve

$$E_{37} = 37A : y^2 + y = x^3 - x \tag{99}$$

of conductor 37. An analysis of signs in functional equations shows that

$$\mathrm{ord}_{s=1} L(E_{37}/H, s) \geq 11, \tag{100}$$

suggesting that $E_{37}(H)$ has rank ≥ 11. A numerical calculation reveals that

$$J_{E_{37}}^+ \left[\frac{1+\sqrt{1297}}{2} \right] = (x, y) \pmod{37^{50}}, \tag{101}$$

where x satisfies the polynomial

$$\begin{aligned} 961x^{11} - 4035x^{10} - 3868x^9 + 19376x^8 + 13229x^7 - 27966x^6 - 21675x^5 \\ +11403x^4 + 11859x^3 + 1391x^2 - 369x - 37, \end{aligned}$$

whose roots generate H and are the x coordinates of global points on $E_{37}(H)$.

Samit Dasgupta's PhD thesis [**Das04**] [**DD06**] puts the analogy between Stark-Heegner points and Gross-Stark units into sharper focus by replacing the modular

symbols attached to the cusp form f_E in the construction of $J_E^{\pm}$ by the Dedekind-Rademacher homomorphism encoding the periods of weight two Eisenstein series. The values of the resulting "Dedekind-Rademacher cocycle" at elements of K turn out to be Gross-Stark p-units in ring class fields of K, whose local norms to $\mathbb{Q}_p$ encode the derivatives at $s = 0$ of p-adic L-series attached to (totally odd) ring class characters of K. Since this formalism leads to a conjectural recipe for the Gross-Stark units themselves and not just their norms to $\mathbb{Q}_p$, it provides a genuine strengthening of the Gross-Stark conjecture and points to a positive answer to a p-adic analogue of Question 2.1. The construction of [**Das04**] is now unconditional thanks to the proof by Dasgupta and Kakde of Gross's tame refinement of the Gross-Stark conjecture alluded to at the end of Sec. 6. (See also [**DPV24**].)

Dasgupta's thesis is a prototypical instance of the fruitful dictionary between unit groups and elliptic curves, exemplified by the passage from the Mazur-Tate element of (62) to Tate's Stickelberger element in the Brumer-Stark conjecture. The Eisenstein cocycle (20) of [**CD08**] was guided by a similar analogy with the elliptic cocycle of [**DL03**] attached to a Hilbert cuspidal eigenform (attached, say, to an elliptic curve E over the totally real field F), whose values at points in a quadratic ATR extension K of F lead to a conjectural archimedean recipe for points on E over abelian extensions of K.

The rich store of parallel constructions of units in number fields and global points on elliptic curves in settings that lie outside the purview of the theory of complex multiplication hints at a synthesis of the themes Tate explored in his last decade at Harvard: the Stark conjectures and p-adic variants of the Birch and Swinnerton-Dyer conjecture. The influence of Tate's work during this period manifests itself not just through the depth and importance of his theorems, but in how his perspectives have set the stage for later developments and continue to frame our understanding of the subject.

References

[AFK26] M. Appleby, S.T. Flammia, and G.S. Kopp. *A constructive approach to Zauner's conjecture via the Stark conjectures.* Forum Math Pi, to appear.

[AT67] E. Artin, J. Tate, *Class Field Theory*, AMS Chelsea Publishing, 2nd ed., 2009.

[AYZ13] D.M. Appleby, H. Yadsan-Appleby, and G. Zauner. *Galois automorphisms of a symmetric measurement.* Quant. Inf. Comput. **13** 672–720 (2013)

[BCG23] N. Bergeron, P. Charollois, and L. Garcia. *Elliptic units for complex cubic fields*, preprint.

[BD96] M. Bertolini and H. Darmon. *Heegner points on Mumford-Tate curves.* Invent. Math. **126** (1996), no. 3, 413–456.

[BD98] M. Bertolini and H. Darmon. *Heegner points, p-adic L-functions, and the Cerednik-Drinfeld uniformization.* Invent. Math. **131** (1998), no. 3, 453–491.

[BD04] M. Bertolini and H. Darmon, *A Birch and Swinnerton-Dyer conjecture for the Mazur-Tate circle pairing.* Duke Math. J. **122** (2004), no. 1, 181–204.

[Bl25] K. G. Black, *A refinement of the Stark conjecture over ATR fields.* PhD thesis, Duke University, 2025.

[Br10] C. Breuil. *Série spéciale p-adique et cohomologie étale complétée.* Astérisque **331** (2010) 65–115.

[CD08] P. Charollois and H. Darmon. *Arguments des unités de Stark et périodes de séries d'Eisenstein.* Algebra and Number Theory **2** (2008) no.6, 655-688.

[CN79] P. Cassou-Noguès. *Valeurs aux entiers négatifs des fonctions zêtaet fonctions zêta p-adiques*, Inventiones Math. **51** (1979) 29–59.

[Dar92] H. Darmon. *A refined conjecture of Mazur-Tate type for Heegner points.* Invent. Math. **110** (1992), no. 1, 123–146.

[Dar95] H. Darmon. *Thaine's method for circular units and a conjecture of Gross.* Canad. J. Math **47** (1995) 302–317.

[Dar01] H. Darmon, *Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications*, Ann. of Math. (2) **154** (2001), 589–639.

[Das99] S. Dasgupta. *Stark's conjectures.* Undergraduate senior thesis, Harvard University, 1999.

[Das04] S. Dasgupta. *Gross-Stark Units, Stark-Heegner Points, and Class Fields of Real Quadratic Fields*, PhD. Thesis, University of California-Berkeley, May 2004.

[DD06] H. Darmon and S. Dasgupta. *Elliptic Units for Real Quadratic Fields.* Annals of Mathematics, **163** (2006), no. 1, 301–345.

[DDP11] S. Dasgupta, H. Darmon, and R. Pollack. *Hilbert modular forms and the Gross-Stark conjecture.* Ann. of Math. **174** (2011) 439–484.

[DKV18] S. Dasgupta, M. Kakde, and K. Ventullo. *On the Gross-Stark conjecture.* Ann. of Math.(2) **188** (2018) 833–870.

[DK23] S. Dasgupta and M. Kakde. *On the Brumer–Stark Conjecture and Refinements.* Proceedings of the ICM. (2022) 1768–1799.

[DK24] S. Dasgupta and M. Kakde, *Brumer-Stark units and explicit class field theory.* Duke Math. J. **173** (2024), no. 8, 1477–1555.

[DKSW] S. Dasgupta, M. Kakde, J. Silliman, and J. Wang. *The Brumer–Stark Conjecture over $\mathbb{Z}$.* To appear.

[DL03] H. Darmon and A. Logan. *Periods of Hilbert modular forms and rational points on elliptic curves.* Int. Math. Res. Not. (2003) **40**, 2153–2180.

[DLR15a] H. Darmon, A. Lauder, and V. Rotger. *Overconvergent generalised eigenforms of weight one and class fields of real quadratic fields.* Adv. Math. **283** (2015) 130–142.

[DLR15b] H. Darmon, A. Lauder and V. Rotger. *Stark points and p-adic iterated integrals attached to modular forms of weight one.* Forum of Mathematics, Pi (2015), Vol. 3, e8, 95 pages.

[DLR17] H. Darmon, A. Lauder, and V. Rotger. *First order p-adic deformations of weight one newforms.* Contrib. Math. Comput. Sci **10** Springer (2017) 39–80.

[DP06] H. Darmon and R. Pollack, *Efficient calculation of Stark-Heegner points via overconvergent modular symbols* Israel J. Math. **153** (2006), 319–354.

[DPV24] H. Darmon, A. Pozzi, and J. Vonk. *The values of the Dedekind-Rademacher cocycle at real multiplication points.* J. Eur. Math. Soc. (JEMS) **26** (2024), no. 10, 3987–4032.

[DR80] P. Deligne and K. Ribet. *Values of abelian L-functions at negative integers over totally real fields.* Invent. Math. **59** (1980) 227–286.

[DS74] P. Deligne and J.-P. Serre. *Formes modulaires de poids 1.* Ann. Sci. Ecole Norm. Sup. (4) **7** (1974) 507–530.

[FV00] G. Felder and A. Varchenko. *The elliptic Gamma function* $\mathbf{SL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$. Advances in Math **156**(1):44–76 (2000)

[Gr81] B.H. Gross. *p-adic L-series at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **28** (1981), no. 3, 979–994.

[Gr88] B.H.Gross. *On the values of abelian L-functions at $s = 0$.* J. Fac. Sci. Univ. Tokyo Sect. IA Math. **35** (1988), no. 1, 177–197.

[GS93] R. Greenberg and G. Stevens, *p-adic L-functions and p-adic periods of modular forms*, Invent. Math. **111** (1993), 407–447.

[GZ86] B.H. Gross and D.B. Zagier, *Heegner points and derivatives of L-series*, Invent. Math. **84** (1986), 225–320.

[HV19] M. Harris and A. Venkatesh, *Derived Hecke algebra for weight one forms*, Exp. Math. **28** (2019), 342–361.

[Ka04] K. Kato, *p-adic Hodge theory and values of zeta functions of modular forms.* In Cohomologies p-adiques et applications arithmt́iques. III Astérisque **295** (2004), ix, 117–290.

[Kol90] V. Kolyvagin, *Euler systems*, The Grothendieck Festschrift, Vol. II, Progr. Math. **87**, Birkhäuser (1990) 435–483.

[Kop21] G.S. Kopp. *SIC-POVMs and the Stark conjectures.* Int. Math. Res. Not. IMRN **18** (2021) 13812–13838.

[Kop25] G.S. Kopp. *The Shintani–Faddeev modular cocycle: Stark units from q-Pochhammer ratios.* Submitted.

[Ma72] B. Mazur. *Rational points on abelian varieties with values in towers of number fields.* Invent. Math. **18** (1972) 183–266.

[Mo24] P. Morain. *Elliptic units above fields with exactly one complex place.* Preprint, to appear.

[MR11] B. Mazur and K. Rubin. *Refined class number formulas and Kolyvagin systems.* Compos. Math.**147** (2011) 56–74.

[MS74] B. Mazur and P. Swinnerton-Dyer. *Arithmetic of Weil curves.* Invent. Math. **25** (1974) 1–61.

[MT83] B. Mazur and J. Tate. *Canonical height pairings via biextensions.* Progr. Math. **35** BirkhäuserBoston (1983) 195–237.

[MT87] B. Mazur and J. Tate, *Refined conjectures of the "Birch and Swinnerton–Dyer type"*, Duke Math. J. **54** (1987), 711–750.

[MTT86] B. Mazur, J. Tate, and J. Teitelbaum, *On p-adic analogues of the conjectures of Birch and Swinnerton–Dyer*, Invent. Math. **84** (1986), 1–48.

[Or04] L. Orton. *An elementary proof of a weak exceptional zero conjecture.* Canad. J. Math **56** (2004) 373–405.

[Ot18] K. Ota. *Kato's Euler system and the Mazur-Tate refined conjecture of BSD type.* Amer. J. Math. **140** (2018) 495–542.

[PR93] B. Perrin-Riou. *Fonctions L p-adiques d'une courbe elliptique et points rationnels.* Ann. Inst. Fourier (Grenoble) **43** (1993) no. 4, 945–995.

[RBSC04] J.M. Renes, R. Blume-Kohout, A.J. Scott,and C.M. Caves. *Symmetric informationally complete quantum measurements.* J. Math. Phys. **45** (6): 2171–2180. (2004)

[RS09] T. Ren and R. Sczech. *A refinement of Stark's conjecture over complex cubic number fields.* J. Number Theory **129** (2009) no. 4, 831–857.

[Ru92] K. Rubin. *p-adic L-functions and rational points on elliptic curves with complex multiplication.* Invent. Math. **107** (1992), no. 2, 323–350.

[Sa14] T. Sano. *A generalization of Darmon's conjecture or Euler systems for general p-adic representations.* J. Number Theory **144** (2014) 281–324.

[Se73] J.-P. Serre. *Formes modulaires et fonctions zêta p-adiques.* Lecture Notes in Math **350** (1973) 191–268.

[Se78] J.-P. Serre, *Sur le résidu de la fonction zêta p-adique d'un corps de nombres*, C.R. Acad. Sc. Paris série A, **287** (1978) 83–126.

[So02] D. Solomon. *p-adic Abelian Stark conjectures at $s = 1$.* Annales Inst. Fourier **52**, no 2 (2002), p. 379–417.

[ST15] J.-P. Serre and J. Tate. Correspondance Serre-Tate, Vol II., P. Colmez and J.P. Serre eds., in Documents Mathématiques, Société Mathématique de France (2015).

[St71] H.M. Stark, *L-functions at $s = 1$. I. L-functions for quadratic characters*, Adv. Math. **7** (1971), 301–343.

[St76] H.M. Stark. *L-functions at $s = 1$. III. Totally real fields and Hilbert's twelfth problem.* Advances in Math **22** (1976) no. 1, 64–84.

[Ta74] J. Tate, *The arithmetic of elliptic curves.* Invent. Math. **23** (1974), 179–206.

[Ta84] J. Tate, *Les conjectures de Stark sur les fonctions L d'Artin en $s = 0$*, Progress in Math. 47, Birkhäuser, 1984.

[Za99] G. Zauner. *Quantendesigns — Grundzüge einer nichtcommutativen Designtheorie.* PhD thesis, University of Vienna, 1999.

[Za11] G. Zauner. *Quantum designs: foundations of a noncommutative design theory.* Int. J. Quantum Inf. **9** (2011), no. 1, 445–507.

[Zh26] R. Zhang. *The Harris–Venkatesh conjecture for derived Hecke operators II: a unified Stark conjecture.* Preprint, submitted.