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Joël Bellaïche (1974-2022)

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Joël never limited himself to a particular group; he wandered freely between various areas...
He believed in setting knowledge free.

– Shaunak Deo

Joël Bellaïche (1974–2022) was a distinguished French mathematician renowned for his contributions to number theory, particularly in the study of automorphic forms, Galois representations, and p-adic L-functions.

Born in 1974, Bellaïche pursued his higher education at the Université Paris-Sud XI in Orsay, where he completed his Ph.D. in 2002 under the supervision of Laurent Clozel. After earning his doctorate, Bellaïche held academic positions at the University of Padova, the University of Nice and later at Columbia University. In 2010, he joined the faculty at Brandeis University in Waltham, Massachusetts.

Throughout his career, Bellaïche made significant contributions to the understanding of eigenvarieties and their geometric properties. His collaborative work with Gaëtan Chenevier on the geometry of eigenvarieties and their relation to Selmer groups has been particularly influential.

Beyond his research, Bellaïche was a dedicated educator, supervising 9 successful PhD students over a span of about 10 years at Brandeis.

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Joël Bellaïche passed away on May 30, 2022, leaving behind an enduring legacy of mathematical contributions. In the sections below, we recount Joël's mathematical directions through recollections of our interactions with him. We present the perspectives of students (Bergdall, Medvedovsky), collaborators (Chenevier, Dasgupta), advisors (Clozel), and colleagues (Darmon).

Early work of Bellaïche

by Laurent Clozel

I first met Joël when he attended my DEA (Master's) course in 1994-95. (He was the best student.) We became friends attending the AMS conference in Santa Cruz in the summer of '95. He started his thesis that fall. His defense was delayed to 2002 as he was very involved with his job as a 'caïman' (student tutor) at the École normale supérieure. He was very appreciated by his students; he was also busy with the (rather animated) politics at the School.

Bellaïche's thesis (2002). Bellaïche's Master's thesis topic was the famous paper of Ribet on the converse of Herbrand's theorem [Rib76]. At the time it was noticed by Michael Harris and myself that the arithmetic of Shimura varieties for $U(3)$ provided a Galois representation (of the absolute Galois group of a quadratic imaginary field) suggesting an extension of representations predicted by Bloch and Kato, as we explain below. (The Galois representations were obtained by Blasius and Rogawski; their shape confirmed the theory of Arthur, conjectural at the time.)

In his thesis, Bellaïche was able to make this concrete. The proof relied inter alia on two ideas of Ribet: (i) "Raising the level" of modular forms; (ii) Obtaining extensions of Galois representations (mod ℓ) from the reducibility (mod ℓ) of a $\mathbb{Z}_\ell$-representation.

Let E be a quadratic imaginary field, and $G'/\mathbb{Q}$ be a quasi–split unitary group of (absolute) rank 3 relative to the extension $E/\mathbb{Q}$. We write $\pi = \pi_\infty \otimes \pi_f$ for an irreducible representation of $G'(\mathbb{A})$. According to Langlands, there should be automorphic representations of $G'(\mathbb{A})$ associated to homomorphisms $W_\mathbb{Q} \to {}^LG'$, where $W_\mathbb{Q}$ is the Weil group of $\mathbb{Q}$ and ${}^LG'$ the L-group of G'. In particular this yields a homomorphism

$$\psi : W_E \longrightarrow \hat{G}' = \mathrm{GL}(3, \mathbb{C}). \tag{1}$$

Let χ be an algebraic character of $\mathbb{A}_E^\times/E^\times$, of type $z \mapsto z^a(\bar{z})^{1-a}$ on $\mathbb{C}^\times$, where $a \geq 2$. Let

$$\chi_0(z) = \chi(z)|z|^{-1/2} \quad (z \in \mathbb{A}_E^\times),$$

so χ_0 is a unitary character. We assume $\chi_0(zz^c) = 1$, where c denotes complex conjugation. We consider ψ given by

$$w \mapsto \begin{pmatrix} \chi_0(w)|w|^{1/2} & & \\ & \mathbb{1} & \\ & & \chi_0(w)|w|^{-1/2} \end{pmatrix} (w \in W_E). \tag{2}$$

In this situation, the Bloch–Kato conjecture asserts the following. Assume $L(\chi, 0) = L(\chi_0, 1/2) = 0$. Then there exists a nonsplit extension

$$0 \longrightarrow \chi_\ell \longrightarrow U \longrightarrow \mathbb{1} \longrightarrow 0 \tag{3}$$

where $\chi_\ell \colon \mathfrak{g}_E = \mathrm{Gal}(\overline{E}/E) \to L^\times$ ($L/\mathbb{Q}_\ell$ finite) is the Galois character associated to χ, and having good reduction [BC04, §1.1] at all primes. Note that the expression (2) suggests the possibility of such an extension, after deformation of the reducible representation (2) to an irreducible representation.

To ensure the property of good reduction, we want a representation π' of $G'(\mathbb{A})$ having *minimal ramification* at all primes (and associated to ψ); such a π' intervenes only if $\epsilon(\chi_0, 1/2) = 1$. However, it is only a priori if $\epsilon(\chi_0, 1/2) = -1$ that we know that $L(\chi_0, 1/2) = 0$ (necessary according to Bloch–Kato to obtain an extension). We now assume $\epsilon(\chi_0, 1/2) = -1$.

Bellaïche assumes χ is everywhere unramified. He then constructs "mod ℓ" variants of the extension (3). Let L be the field of values of χ, and λ a place of L dividing ℓ. There is then a λ-adic realisation χ_λ of χ, a L_λ-adic representation of $\mathfrak{g}_E$. We denote by $\bar{\chi}_\lambda$ the reduction mod λ, and by ω_ℓ the ℓ-adic cyclotomic character. Let $\mathbb{F}$ be the residue field of L_λ.

Theorem. *There exists a set of places λ of L (of nonzero density) such that there exists a nontrivial extension of representations of $\mathfrak{g}_E$:*

$$0 \longrightarrow (\bar{\chi}_\lambda)^{-1}\bar{\omega}_\ell \longrightarrow U \longrightarrow \mathbb{F} \longrightarrow 0 \tag{4}$$

with good reduction everywhere.

For the place(s) of E dividing ℓ, "good reduction" means "crystalline".[1] I briefly sketch the argument, which requires several results of the 270-page thesis.

(a) Let G be the inner form of G' of type $U(3)$ at infinity. Then the sign condition, wrong for G', is correct for G: there exists an automorphic representation π of $G(\mathbb{A})$ such that $\pi_f = \pi'_f$. (π_∞ is a finite–dimensional representation of $G(\mathbb{R})$.)

(b) Using Chapter VII of [Bel02], Bellaïche shows that, at a suitable prime p (and for a density of ℓ) it is possible to "raise the level" of π_p. Let π_1 be the new representation so obtained.

(c) One then shows that $\pi_{1,f} = \pi'_{1,f}$ for a cuspidal representation π'_1 of $G'(\mathbb{A})$. It is no longer Abelian, i.e. of type (2). I will only consider the case where π'_1 is *stable*, i.e., associated by base change to a cuspidal representation of $\mathrm{GL}(3, \mathbb{A}_E)$. There is then an irreducible L_λ-representation of $\mathfrak{g}_E$ which, reduced mod λ, yields after semi simplification the three (mod λ) λ-adic characters in (2) — translated by $(\bar{\omega}_\ell)^{-1}$; it occurs in the cohomology of a Shimura variety S for G'.

(d) In Chapter VI of [Bel02], Bellaïche proves a generalisation of Ribet's theorem affording extensions of representations mod ℓ (see above). For a statement, I refer to Chenevier's exposition in this homage. In particular [Bel02, 4.1.5] implies the existence of various extensions of $\mathbb{F}$-representations of $\mathfrak{g}_E$, one of

[1] Note that this differs from (3); it is essentially equivalent, see [Bel02, p. 248].

which is (4) above. They are mutually exclusive, so the final job is to exclude all other possibilities — this leads to the exclusion of new values of ℓ. This finally proves the theorem.

The work of Bellaïche–Chenevier (2004). We keep the notation of the previous section, in particular χ, χ_0, G, G'; we assume $\epsilon(1/2, \chi_0) = -1$. We suppose (only) that χ is unramified at p, and p splits in E. We denote by L a *p-adic* field of values for χ_p.[2]

Theorem. *There exists a nontrivial extension*

$$0 \longrightarrow \chi_p \longrightarrow U \longrightarrow \mathbb{1} \longrightarrow 0 \tag{5}$$

of representations of $\mathfrak{g}_E$ *on* L*-vector spaces, having good reduction everywhere*[3].

(We have abused notation by choosing a prime $\mathfrak{p}$ of E dividing p and writing χ_p for $\chi_{\mathfrak{p}}$.) We will only sketch, very roughly, the main steps.

The proof shares many similarities with the one in Joël's thesis. Instead of constructing representations (mod ℓ) using lattices in an ℓ-adic representation, one constructs p-adic representations using reduction mod $\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal associated to a point in a p-adic analytic space. (The quotient field is a p-adic field.) For this the authors insert π (see above) in a 'suitably refined' p-adic family of automorphic forms for the group G, whose existence follows from Chenevier's thesis. The Galois-theoretic arguments “à la Ribet” leading to the construction of a non-trivial extension of **1** by χ_p are formally the same, except that they require work of Kisin to control the ramification above p and some theorems from the theory of pseudo-representations.

With Joël, I have lost not only a student but a friend. I remember with sadness his last years fighting disease, but with longing our meetings: dinners, long walks, and his hospitality (and his wife Clémentine's) as I stayed with them while visiting Columbia. I also remember his deep culture — literature, linguistics, and his interest for ancient history — I understand that he even studied Sanskrit!

Some early work of Bellaïche and memories

by Gaëtan Chenevier

It is with deep nostalgia that I look back at my years of mathematical collaboration with Joël. They really began around the year 2000, when Joël was finishing his thesis, and I was about to start mine. Those were exciting times for young number theorists, with amazing recent results such as the proof of Fermat's last theorem or of the local Langlands conjecture for GL_n. I already knew at the time how lucky I was to have Joël by my side to endlessly discuss the details of these theories that we were learning together — and switch to beers or comic books when we were bored.

Joël's early works, on the construction of certain extensions of Galois representations, were greatly influenced by the famous “Ribet lemma”, as recalled by Laurent above. Joël was especially fond of

[2] Note that Bellaïche's “ℓ” becomes “p”.

[3] I.e., a nontrivial element of the Bloch–Kato group $H^1_f(\mathfrak{g}_E, \chi_p)$ [FPR94].

Serre's point of view on this lemma in terms of the Bruhat–Tits tree, which he liked to popularize (such as in his Honolulu lectures [Bel09]). He generalized it in [Bel03] as follows. Let V be a finite-dimensional $\mathbb{Q}_p$-vector space equipped with an irreducible representation of a compact group G. Such a V admits $\mathbb{Z}_p$-lattices $L \subset V$ stable by G. The associated $\mathbb{F}_p[G]$-modules $L \otimes \mathbb{F}_p$ are non-isomorphic in general, but all have the same semisimplification (Brauer–Nesbitt), which we denote by $\overline{V}$. Let $r_1, \ldots, r_s$ be the distinct irreducible summands of $\overline{V}$. Joël considers the oriented graph Γ with vertices the r_i, and with an arrow $r_i \to r_j$ if there exists a non-trivial extension of r_i by r_j obtained as the quotient of two G-stable lattices in V. His theorem asserts that the oriented graph Γ is connected (and any finite oriented graph can be obtained in this way). Ribet's original case is $\dim V = s = 2$ (easy graph!). When $\overline{V}$ is multiplicity-free, the set of G-stable lattices of V is included in an apartment of the building of $\mathrm{PGL}(V)$: Joël describes in [BG06a] its possible geometries and proves an analogue *modulo* p^n of the previous statement.[4]

In the case of interest in Joël's thesis, we have $s = 3$, G is the absolute Galois group of an imaginary quadratic field, and $\overline{V} = 1 \oplus \overline{\omega}_p \oplus \overline{\chi}_p$ with ω_p the p-adic cyclotomic character and χ_p the p-adic character he is interested in. The key to constructing an extension of 1 by $\overline{\chi}_p$ is then to eliminate the possibility of an extension of 1 by $\overline{\omega}_p$. This elimination is the most delicate part of Joël's argument, and assuming the ramification has been controlled carefully at every prime when choosing V, it eventually follows from the fact that an imaginary quadratic field has no unit of infinite order!

Joël and I were fortunate that our theses combined so well, in the sense that we could attack the very same problem of his thesis using my just-born unitary eigenvarieties rather than his congruences based on (more complicated!) level raising arguments (see [Bel02], [BG06b]). A decisive advantage of this variant, already discussed by Laurent, is that it allowed to construct à la Ribet an extension of 1 by χ_p, rather than only its reduction mod p. To go further, an especially interesting question from the point of view of the BSD conjecture was whether it was possible to construct several linearly independent extensions of 1 by χ_p, by studying the p-adic eigenvariety $\mathscr{E}$ of U(3) at the specific point x we were considering[5]. We soon discovered that at this point, the local representation at a decomposition group at p was deforming irreducibly in a very strong sense, quite the opposite of the "ordinary" situation in the cases of Ribet or Mazur–Wiles, and forcing the global reducibility locus to be reduced to the closed point $\{x\}$. As a baby case, we first studied in [BC06] the critical Eisenstein points of the eigencurve. As a key ingredient to understand those facts, we started to develop a theory of *trianguline deformations* (and *refined families*), building on works of Kisin and Colmez, and this was another exciting new topic at the time. These would eventually become the two main themes of our book [BC09], on which we worked roughly between 2004 and 2007. Joël was in New York at the time, and I was in Paris, and my mailbox tells me that we exchanged more than 640 emails in this very intense period!

Let me say a bit more about some results in our book. It is easy to see that it is impossible to construct independent extensions by Ribet's method over a DVR in the residually multiplicity free case. Another slightly annoying (but fortunate?) fact is that eigenvarieties do not quite carry families of Galois

[4] Of course, all results remain valid if we replace $\mathbb{Q}_p$ by any complete discretely valued field.
[5] with Galois representation $1 \oplus \omega_p \oplus \chi_p$ and the choice of "anti-ordinary" p-refinement.

representations, but only a family of *pseudo-representations* in the sense of Wiles and Taylor, obtained by interpolation from classical points. So, it is in the context of pseudo-representations $T : G \to A$, with A an arbitrary Henselian local ring, and under a residual multiplicity 1 hypothesis, that we first revisited Ribet's lemma in [BC09]. Our statement, inspired by [BG06a] and the work of Mazur–Wiles, involves the *reducibility loci* associated with T, and its proof, the new notion of *Generalized Matrix Algebra* (in short *GMA*). We then derived some criteria for producing independent extensions, involving both the structure of the associated GMA and the geometry of certain reducibility loci of T. Using that the global reducibility locus at x is the closed point $\{x\}$, our final and main result was that if $\mathscr{E}$ is not smooth at x, then there are two independent extensions of 1 by χ_p (with good reduction).

It was at this point that Joël began to work on p-adic L-functions, and in particular, tried to link the latter to the geometry of eigenvarieties. He then found some beautiful continuations of the above ideas to the construction of p-adic L-functions in the critical cases [Bel12a], and even later wrote a full book on eigenvarieties [Bel21]. Joël also pursued the study of pseudo-representations. In particular, he obtained a useful description in [Bel12b] of the space of pseudo-deformations of a multiplicity-free representation, and he studied many concrete examples in the context of mod-p modular forms [Bel12c; BK15; Bel19]. Anna, Henri, John and Samit will say more about these topics.

Joël was an amazing collaborator, because of his extraordinary creativity, energy and enthusiasm. He usually had the impression we would eventually prove X (a big conjecture), to which I usually replied that in my opinion we would most probably prove Y (a trivial statement), and we finally proved many interesting theorems! He was also a very good friend, and he is now deeply missed.

Modular symbols, eigenvarieties, and p-adic L-functions

by John Bergdall

I came to Brandeis in Fall 2008 and graduated with a Ph.D. in 2013, under Joël. Joël's other students and I were lucky to have the chance to be both the focus of his energy and the target of his generosity. It was not so much that we had long meetings with question after question being answered. In fact, meetings were challenging to schedule. Instead, he had what I've come to understand is a genuine excitement for good mathematics and for sharing his love with anyone within arm's reach.

Just during my five-year Ph.D., Joël gave topics courses or led seminars on elliptic curves, Galois representations, eigenvarieties, Deligne's paper on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, Kisin's paper on interpolating crystalline periods and Colmez's theory of trianguline representations, and more. For the reader who works at a large school, this might not seem like a lot. But at the time the Brandeis faculty were only ten people. It took a tremendous amount of energy for Joël to do so much in so little time for his research group. And, not to degrade my other former professors, but Joël was easily the most generous Brandeis faculty in terms of encouraging of his students to decamp from Waltham and infiltrate the halls of Harvard, MIT, Boston University, etc. He wanted us to soak up all the knowledge we could, so that we stood a better chance in the highly-competitive research markets in which he believed we belonged.

Around 2008 or so, Joël began to pursue research on p-adic L-functions. The purpose was to invite L-functions to more clearly enter his and Chenevier's strategy for attacking the Bloch–Kato conjecture. He sought to not just reinforce the existing foundations but also to link special geometric behavior on eigenvarieties with strange properties of L-series. Joël's published work in this direction uniformly focuses on the eigencurve, though he expands on the motivation for higher rank groups in Section 1.6 of [Bel12a] and Mathoverflow post #80181. His work in the era featured important collaborations with Samit Dasgupta, Mladen Dimitrov, and Robert Pollack, but I will focus on his primary paper [Bel12a], a modern classic on p-adic L-functions in the context of the eigencurve.

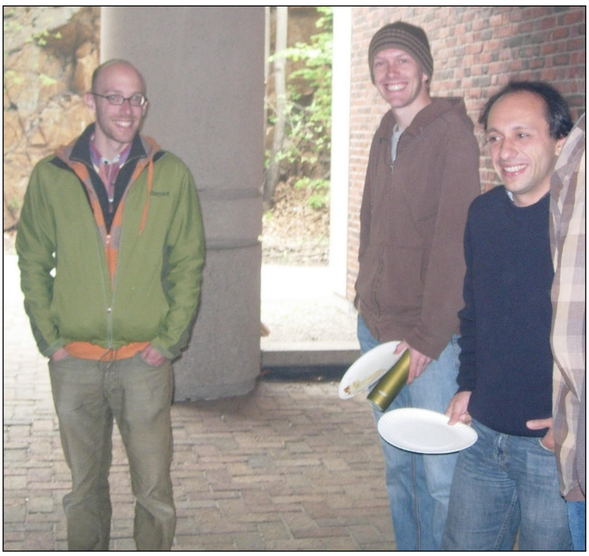
In it, he achieves two main goals. First, he explains the details of a construction of the eigencurve in terms of *modular symbols*. Modular symbols are cohomological incarnations of modular forms that are often used to study algebraicity of L-functions. The concept of the eigencurve constructed this way was around already for 15 years, in unpublished writings and seminars of Glenn Stevens. Joël took on the task of nailing down and clarifying details, since they would be vital for his second goal (explained in the next paragraph). He also gave a more leisurely explanation of all this material, aimed at learners, in *The eigenbook* [Bel21]. That text came out of just one of the courses he had taught at Brandeis (in Fall 2010). Especially the first eight chapters are perfect examples of how Joël's writing mixed classical (Bourbakian, as Medvedovsky writes) instincts with more generous and highly illustrative examples. He often includes full explanations of details where others would include less. I always took that as a sign of his belief that the mathematics can explain itself, if only the author would present things with utmost clarity and guide the reader directly to the most important points. His writing outside prestigious journal articles also shows a more playful side. In *The eigenbook*, he cites Issac Newton for the definition of binomial coefficients, and in his famous Clay Research School notes on the Bloch–Kato conjectures, he complains he cannot write more since he must go to the beach :-).

His second aim was analyzing the eigencurve at critical eigenforms. A critical eigenform (conjecturally) arises in a fairly simple fashion. You start with an eigenform with complex multiplication such that the prime p splits in the associated quadratic field. This eigenform will have one ordinary ("unit root") and one non-ordinary stabilization to level with $\Gamma_0(p)$-structure. The critical eigenforms are the non-ordinary stabilizations. They are interesting because the local p-adic Galois representations are generically irreducible along the eigencurve near a critical eigenform, but at the critical point itself those local representations become totally decomposable. This representation-theoretic phenomenon is the basic setting of Ribet's lemma and so a basic example in which strategic elements of the Bellaïche–Chenevier program can be tested. To be clear, Joël understood there were no new or remarkable theorems that would be proven on the eigencurve and which would directly lead to breakthroughs for higher rank groups. Instead, what he was aimed to do more than anything was try to convince everything to line up just the way he wanted them to.

In any case, what analysis did Joël carry out in critical cases? First, he constructs their p-adic L-functions in a completely canonical way, by constructing canonical modular symbols for critical eigenforms and then integrating those cohomology classes to generate an L-series. The *canonical* nature of the work is exactly what you cannot deduce as a consequence of Stevens's work — it is really a major achievement of Joël. He does it by considering not just one space of modular symbols, but entire families of them, parametrized by p-adic weights. What he proves (which he says he actually learned from Chenevier) was the eigencurve at a critical point is non-singular, even though the weight parameter is not a smooth parameter for the curve. This absolute smoothness then forces the number of eigensymbols at each point to be constant, and in particular the number of eigensymbols is the same at the critical points as the other classical points (where the answer is just one up to scalar, by Stevens's work).

The second piece to the analysis is studying this new canonical p-adic L-function. He proves it is divisible by a rather easy-to-write-down factor, with well-understood zeros. More importantly, he observes that the level of this divisibility is more or less a faithful reflection of the length of the ramification of the weight parameter. That was the outcome he really sought, to connect p-adic L-functions to the geometric structure of the eigencurve in an impossible to mistake way. By the way, he built the link a second time in the final chapter of *The eigenbook*. There, he expands on the 2006 Berkeley Ph.D. thesis of Walter Kim [Kim06], in which the p-adic properties of adjoint L-series are even more directly (and perhaps more conceptually) linked to ramification of weight parameters on eigenvarieties.

In the decade since, Joël's work has been extended in numerous ways. I've worked with collaborators (Balasubramanyam and Longo [BBL25] and Hansen [BH24]) on versions of the story of Hilbert modular forms. For higher rank groups, the p-adic L-function story (in critical contexts!) lags behind the geometric analysis. But in terms of geometry, Breuil, Hellman, and Schraen made a deep study of critical phenomena on eigenvarieties for higher rank groups; what they showed was that singularities abound [BHS17; BHS19]. Even more, in follow-up work of Hellmann, Schraen, and Hernandez [HHS24], the singularities are shown to have a profound and unexpected impact in *invalidating* the multiplicity one principle that came out of



Courtesy Daniel Ruberman

Robert Pollack, John Bergdall and Joël Bellaïche (left to right) at Brandeis University, spring 2010.

Joël's research on the eigencurve. It remains to see how researchers will connect that new phenomenon with the p-adic behavior of L-functions.

What I have described is just one slice of Joël's work testing the theory of p-adic L-functions in the context of the eigencurve. You can see more of it in Dasgupta's contribution below. From it, one can learn a lot of mathematics, but you will also find happiness. His attitude, in private and sometimes in public, was always that the eigencurve was some kind of playground in which aspects of the Langlands program would play alongside each other. He felt that if he watched for long enough, with enough focus, he would see his way to understanding and creating adaptions for higher rank groups, which would in turn lead to progress on conjectures about Selmer groups and L-functions.

p-adic L-functions of critical Eisenstein series

by Samit Dasgupta

I met Joël at Harvard during my time as a postdoc there, but I really got to know him at the Clay Summer School in Hawaii in 2009. I had finished my Ph.D. thesis under Ribet five years earlier, and my most recent paper employed a general version of Ribet's Lemma. It was therefore a great surprise that Joël's beautiful lectures on Ribet's method were able to completely change how I view the theory. Just as memorable as Joël's lectures were our meals together at Duke's on Waikiki beach. Inspired by our stimulating mathematical discussions and new friendship, I invited Joël to speak at the Bay Area Number Theory Day at UCSC in Fall 2010.

As described above in Bergdall's contribution, over the previous years Joël had defined and studied the p-adic L-functions of critical forms. A natural question arises to calculate this function in the case of a critical Eisenstein series. The classical L-function of an Eisenstein series is easily seen to equal the product of two Dirichlet L-functions, as follows from the well-known formula for the Fourier expansion of Eisenstein series. For the case of *ordinary* Eisenstein series, the p-adic L-function can be seen to equal the product of two Kubota–Leopoldt p-adic L-functions by interpolating the classical formula. However, for *critical* Eisenstein series, Pollack and Stevens had computationally discovered a new interesting phenomenon: they conjectured that the p-adic L-function of a critical Eisenstein series is equal to a product of two Kubota–Leopoldt p-adic L-functions together with a product of $k-2$ terms of the form $\log_p(s+n)$ for forms of weight k. From Bellaïche's perspective, these parasitic log factors arise from the application of the θ operator, which maps the (p-adic) ordinary Eisenstein series of weight $2-k$ to the critical Eisenstein series of weight k.

In order to convert this philosophy into a proof, one had to construct the modular symbol of measures (as studied by Greenberg–Stevens) giving rise to the two-variable p-adic L-function associated to an Eisenstein series. At face value, this is problematic because Eisenstein series are necessarily irregular at some cusps, so one cannot literally define modular symbols supported at such cusps.

Joël mentioned this difficulty to Henri Darmon, who suggested working with me, since it was one of the topics of my thesis. The trick is to stabilize the Eisenstein series to be regular at a subset of the cusps containing 0 and ∞, and to consider *partial modular symbols* that are supported on this subset. When he visited UCSC in 2010, Joël described the problem to me, and I explained the constructions of my thesis. The main task that remained in order to calculate the p-adic L-functions of critical Eisenstein series was to relate Joël's original definition with full modular symbols to the new construction with partial modular symbols. Our collaboration culminated in the publication [BD15], which included a proof of the Pollack–Stevens conjecture on the evaluation of the p-adic L-functions of critical Eisenstein series. I am forever grateful to Darmon for facilitating this collaboration — working with Joël during walks on the Santa Cruz boardwalk or in coffee shops in Boston is one of the highlights of my mathematical life.

Modular forms modulo p and their density

by Anna Medvedovsky

Joël Bellaïche was my Ph.D. advisor at Brandeis through the early 2010s, a golden age for Brandeis number theory graduate students — at one point our group was eight strong, a real community, all needy and clamoring for his attention. Whenever this last was on you and your problems (math, it was always math) — when Joël's attention was on you, it was glorious: you a sunflower (or a lizard), basking in the sun. The resulting phototropism engendered some frenzy among us huddled masses. Have you heard? Joël's in the science library today!... He was on MathOverflow just an hour ago — send that email now!

Joël had arrived in the US steeped in the Bourbaki tradition of organizing mathematical information — calm and clear-eyed, incrementally specializing from the general to particular. As a teacher he was an extraordinary expositor, both in courses that he taught and in notes he released. At the same time — and in slight tension with the Bourbaki perspective — he again and again clearly articulated the idea, poignant and bold, that we all (we mathematicians, presumably) ought to protect our love of mathematics. Mathematical curiosity is a gift that deserves nurture for its own sake; we should not waste attention and energy on difficult problems merely because they are difficult. In short, we should work on problems that we find important, inspiring, or simply interesting. Sometimes that might mean walking away without dotting every i and crossing every t.

In the fall of 2011, Joël attended a talk by J-P. Serre at Harvard on Serre's then-recent results with Nicolas on the "big" mod-2 level-1 Hecke algebra $A(1, \mathbb{F}_2)$. Let Δ be the unique normalized level-1 cuspform of weight 12, first studied by Ramanujan. Serre and Nicolas had managed to show, in part through brute force computation with the Hecke recurrence satisfied by the sequence $\{T_\ell(\Delta^n) \bmod 2\}_n$ for small odd primes ℓ, that $A(1, \mathbb{F}_2) = \mathbb{F}_2[[T_3, T_5]]$ [NS12b; NS12a]. At the time, the fact $A(1, \mathbb{F}_2)$ has dimension 2 was a surprise — though here it is worth noting that Joël always pointed out that how surprised you are depends on what you know. In the years following, the surrounding sea rose; now the Nicolas–Serre result is just one of several addressing lower bounds on dimensions of big mod-p Hecke algebras [BK15; Deo17; Med15; DM24]; these use a variety of methods — but each owes a debt to Bellaïche.

Indeed, Serre's talk unleashed a torrent of activity on mod-p modular forms lasting until 2019 or so, Joël's most variedly collaborative period. His answer of a question of Serre about the representation carried by $A(1, \mathbb{F}_2)$ [Bel12c] led to ongoing collaborations with Serre and Nicolas on mod-2 modular forms, which remain partially unpublished to this day (though see [BN16] and the forthcoming [Bel]). At least part of the initial motivation for Joël was the lure of potential applications to partition parity, a simple-to-state question with no simple resolution. The fact that the partition generating function $\prod_n (1-q^n)^{-1}$ essentially coincides mod 2 with "$\Delta^{-\frac{1}{3}}$" (this cartoon can easily be made precise) explains the connection to mod-2 modular forms. In fact, it seems likely that this observation was close to Nicolas's original motivation for studying powers of Δ modulo 2, leading to the collaboration with Serre.

A possible access point to understanding the parity of coefficients of powers of Δ is Joël's notion of *density* of a mod-p modular form: that is, the proportion of primes whose Fourier coefficients are nonzero [Bel19]. As was already clear to Serre in the 1970s, the set of such primes is *frobenian*, in the sense that membership is determined by Frobenius conjugacy classes in some finite Galois extension of $\mathbb{Q}$, which therefore controls the Fourier coefficients of the form, even if the form is not an eigenform. In [Bel19], a technical and exhausting paper closing this era, Joël furthered Pink–Lie theory to study the density of mod-p modular forms for $p \geq 3$, though more work remains. The generic density result for mod-2 powers of Δ (all but a thin set of which Joël expected to have density $\frac{1}{8}$, an equidistribution statement in this setting) also remains unproven. In the same time period Joël also considered the question of density for *all* coefficients, not just prime ones, for classical forms together with Soundararajan [BN16], and for half-integral forms with Soundararajan and Green [BGS18].

In his last article, "Correspondences on curves" [Bel23], published posthumously, Joël yet again returned to his master expositor form, laying out a conceptual framework for understanding the behavior of iterations of Hecke correspondences on modular curves in characteristic p, possibly with an eye towards a geometric approach to reconceptualizing the post-Nicolas–Serre methods for understanding mod-p Hecke algebras. ("A characteristic-p result deserves a characteristic-p proof.")

For what it's worth, I believe that it's a mistake to view this period of Joël's mathematical-creative work as a distraction from what some might see as his "core concern" (something like the geometry of eigenvarieties as it informs the p-adic automorphic forms they parametrize). Understanding mod-p modular forms *was* a core concern, borne out of a genuine curiosity and hope, a love of mathematics. He has left behind a deeper understanding of a beautiful and accessible topic, his ideas not yet fully understood or incorporated into the mainstream — and the story still tantalizingly incomplete.

Weight one points on eigenvarieties

by Henri Darmon

I encountered Joël's mathematics before meeting Joël himself. The setting was a 2002 Paris summer school on the Birch–Swinnerton-Dyer conjecture, where breaking news of Joël's ongoing work with Gaëtan Chenevier was a frequent topic of conversation among the participants. Joël's thesis *Congruences endoscopiques et représentations galoisiennes* had just been defended a few months before, adapting the

"Ribet–Mazur–Wiles" approach to new settings where congruences with Eisenstein series are replaced by congruences with endoscopic lifts to $U(2,1)$. This had potential implications for the anti-cyclotomic Iwasawa theory of elliptic curves, which previously had been approached largely through the study of Heegner points following a program initiated by Barry Mazur. Many of us were keen to bring to these investigations the congruence methods dramatically popularised by Andrew Wiles less than a decade earlier in his proof of the modularity of elliptic curves. The program of Joël and Gaëtan opened a fresh chapter in the subject and made a vivid impression on me at the time.

I met Joël at a thematic semester on p-adic families of modular forms at Harvard in 2006, where he gave some striking lectures on his soon to be submitted work with Gaëtan, which would appear three years later in an influential Astérisque volume [BC09]. Joël and I quickly bonded over our common Jewish North African roots, his free-thinking perspectives on politics and philosophy, and, of course, our shared mathematical interests. Joël was just starting to be drawn into the study of p-adic L-functions and although he was a relative newcomer to the topic he was full of original insights and I learned a lot from talking with him.

After he moved to Brandeis, Joël made fairly regular visits to Montreal. On one occasion he introduced me to an excellent Ashkenazi bakery that I hadn't noticed in my several decades living in the city. This was typical of Joël, who also had a knack for spotting unfamiliar features in mathematical landscapes that I wrongly thought to have thoroughly explored.

Joël's frequent exhortations to pay close attention to singular points of eigenvarieties are an example of this. They emerged from his idea that the "extra tangent directions" at singular points allow the construction of multiple non-trivial extension classes, following the philosophy that the local geometry of the eigenvariety "sees all possible Galois deformations". The paper [BD16] by Joël with Mladen Dimitrov specialises this study to classical points on the Coleman–Mazur eigencurve attached to weight one cusp forms where smoothness is not forced by Hida's classicality theorem. If g is such a cusp form whose associated odd two-dimensional Artin representation ϱ_g has distinct frobenius eigenvalues at p, then Joël and Mladen show, building on earlier work of Cho and Vatsal, that this point is nonetheless smooth, and even étale over weight space *unless* ϱ_g is induced from a character of a real quadratic field in which p is split.

The circumstance of non-étale points on eigenvarieties in the latter scenario leads to the existence of weight one overconvergent generalised eigenforms for Hecke operators which are of considerable arithmetic interest, and can be envisaged as a p-adic analogue of the "weak harmonic Maass forms" growing out of Ramanujan's theory of mock theta functions. The generalised eigenvalues attached to the weight one forms of Bellaïche–Dimitrov can be expressed as p-adic logarithms of algebraic numbers in class fields of real quadratic fields. This offers a key to "explicit class field theory" for such fields, and was one of the crucial hints that led to the discovery, by Jan Vonk and myself a few years later, of singular moduli for real quadratic fields [DV21].

Joël was planning to spend the winter term of 2022 on sabbatical in Montreal, and had generously offered to give a course on p-adic L-functions during a stay that was to be funded largely by the Simons

Foundation. Unfortunately, this visit was postponed to the Fall because of the resurgence of the omicron variant of the Covid virus, and then tragically canceled after Joël's unexpected passing later that summer. In the months prior to his planned visit, Joël and I exchanged a brief mathematical correspondence where he explained some of the directions that he was hoping to pursue. This included a number of intriguing ideas whose full understanding I postponed to his visit. If only I had paid closer attention at the time! Joël's original perspectives on mathematics and on life, as well as his unique friendship, will be greatly missed.

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