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p-adic higher Green's functions

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Abstract

Heegner cycles are higher weight analogues of Heegner points. Their arithmetic intersection numbers also appear as Fourier coefficients of modular forms and often belong to abelian extensions of imaginary quadratic fields. Rotger and Seveso [RS12] propose a precise recipe for the p-adic Abel–Jacobi images of cycle classes whose existence is predicted by conjectures of Bloch and Beilinson and which would be a real quadratic analogue to Heegner cycles. These are the so called Stark–Heegner cycles.

The aim of this thesis is to generalize Darmon–Vonk's theory of rigid meromorphic cocycles [DV21] to higher weight, producing a higher Green's pairing of real quadratic divisors on the p-adic upper half-plane, which seems to be the real quadratic analogue of the pairing of Heegner cycles. Computation of these values for "Principal cycles" gives evidence that they lie in abelian extensions of real quadratic fields. The algebraicity of certain values of the higher Green's function is indirect evidence of the existence of algebraic Stark–Heegner cycles.

Résumé

Les cycles de Heegner sont des analogues des points de Heegner pour des poids plus élevés. Leurs nombres d'intersections arithmétiques sont aussi des coefficients de Fourier de formes modulaires et appartiennent souvent à des extensions abéliennes de corps quadratiques imaginaires. Rotger et Seveso [RS12] proposent une recette précise pour les images p-adiques d'Abel—Jacobi de classes de cycles dont l'existence est prédite par des conjectures de Bloch et Beilinson et qui seraient l'analogue quadratique réel de cycle de Heegner. Ceux-ci sont dénommés les cycles de Stark–Heegner.

Le but de cette thèse est de généraliser la théorie de Darmon et Vonk sur les cocycles méromorphiques rigides à des poids plus élevés, produisant un couplage en poids élevés de Green pour les diviseurs quadratiques réels sur le demi-plan p-adique. Ce couplage semble être l'analogue quadratique réel du couplage des cycles de Heegner. Le calcul de ces valeurs pour les "cycles principaux" semble indiquer qu'ils appartiennent à des extensions abéliennes de corps quadratiques réels. L'algébricité de certaines valeurs de la fonction de Green pour poids élevés est un signe qu'il existe des cycles de Stark—Heegner algébriques.

Contribution

This work generalizes Darmon–Vonk's theory of rigid meromorphic cocycles to higher weight. We relate rigid meromorphic functions on a p-adic upper half-plane to boundary distributions. This approach leads to a full classification of higher weight rigid meromorphic cocycles beyond the examples in Negrini's thesis. We introduce a formalism to describe the evaluation and obstruction to lifting of these higher weight cocycles. We implemented this evaluation in the computer algebra system SageMath, and provide the results to our numerical calculations which inform the conjectures we provide on the values of rigid meromorphic cocycles.

Except for the appendix, the content of this thesis was written solely by the author.

Chapter 2 consists mainly of background knowledge which is available in the literature, except for Sections 2.6, 2.7 and 2.8 which are the result of the author's original work.

Chapters 3, 4, 5 and 6 are the result of the author's original work.

The appendix: Anti-symmetry of the Green's function is an excerpt from an ongoing collaboration with Isabella Negrini and Vinayak Vatsal to which all authors contributed equally. We obtained permission from Isabella Negrini and Vinayak Vatsal to include this result in this thesis.

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Contents

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Introduction

Arithmetic Geometry. The point $(1, 1)$ is a rational solution to the equation

$$E : y^2 = x^3 - x + 1.$$

In fact, the equation E has infinitely many rational solutions which are all recursively generated from this single solution $(1, 1)$. The problem of understanding and solving elliptic curves is one of the longest standing open problems in Mathematics. The *Birch and Swinnerton-Dyer (BSD) conjecture*, one of the Clay Millennium Prize problems, implies that this arithmetic behavior of E is encoded in the analytic behavior of its L-function

$$L(E, s) = 1 - \frac{3}{3^s} - \frac{2}{5^s} - \frac{4}{7^s} + \frac{6}{9^s} + \frac{2}{11^s} - \frac{5}{13^s} + \cdots .$$

The main challenge in the Birch and Swinnerton-Dyer conjecture rests in procedurally finding rational solutions to elliptic curves. The only known procedure generates

Heegner points. The construction and properties of these points formed the key input for the only solved cases of the conjecture in the late 1980s.

The connection between arithmetic objects and the analytic behavior of their L-function is a central theme in arithmetic geometry. The Birch and Swinnerton-Dyer conjecture is the first interesting case in a deep generalization known as the *Beilinson–Bloch–Kato* conjecture. It generalizes from solutions of *elliptic curves*, such as E, to solutions of any collection of polynomial equations in any dimension.

The higher dimensional nature of the conjecture makes it likely far out of reach of current technology. The first serious obstacle is our understanding of higher co-dimensional algebraic cycles and their relation to the cohomology of algebraic varieties. This is best encapsulated in Grothendieck's standard conjectures on algebraic cycles. The second key obstacle is in finding methods of constructing algebraic cycles on varieties.

The Beilinson–Bloch–Kato conjecture has therefore been studied in few special cases. These are typically varieties related to Shimura varieties. These varieties have the advantage that they are often equipped with families of special subvarieties, therefore there are natural candidates for algebraic cycles that account for the conjecture. Furthermore, the L-functions for these varieties are related to automorphic L-functions, which are often convenient to work with as they are known to have analytic continuation and a functional equation. The use of automorphic forms is also favorable since analytic tools such as theta correspondences are often helpful in relating height pairings of algebraic cycles to special values of derivatives of L-functions.

Complex Multiplication and Higher Green's Functions. The starting point of this investigation is the seminal papers of Gross–Zagier[GZ84] and Gross–Kohnen–Zagier[GKZ87], which investigate the Néron–Tate height pairing of Heegner points on Jacobians of modular curves $X_0(N)$. The archimedean contribution to this height pairing is realized as a limit of CM-values of automorphic functions:

$$G_s(\tau, \sigma), \quad s > 1.$$

For positive integer values of s, the functions $G_s(z_1, z_2)$ are known as the higher Green's functions. These are real analytic functions on $\mathcal{H} \times \mathcal{H}$ which are invariant under the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ and have a logarithmic singularity along the diagonal. In [GZ84] and [GKZ87] it is shown that the CM-values of higher Green's functions appear as Fourier coefficients of certain higher weight modular forms. Motivated by this observation, Gross, Kohnen and Zagier made two predictions:

1. The CM-values $G_s(\tau, \sigma)$ have an interpretation as the archimedean contribution to a global height pairing of some higher dimensional algebraic cycles.

2. *The Gross–Zagier algebraicity conjecture.* Certain "principal" linear combinations of CM-values $G_s(\tau, \sigma)$ are logarithms of algebraic numbers.

Heegner Cycles. The first prediction was addressed in [Zha97]. Let $\pi : \mathcal{E} \to X_0(N)$ denote the universal generalized elliptic curve. The Kuga–Sato variety, $\mathcal{Y}_k$, is the canonical resolution of the $(k-2)$-fold fibre product of $\mathcal{E}$ over $X_0(N)$. Associated to each CM-point, τ, there is a Heegner cycle

$$y_{k,\tau} \in \widehat{\mathrm{Ch}}^{k/2-1}(\mathcal{Y}_k),$$

which lies in the fiber over the τ. S. Zhang showed that the archimedean contribution to the height pairing for two distinct Heegner cycles $y_{k,\tau}$ and $y_{k,\sigma}$ satisfies

$$\langle y_{k,\tau}, y_{k,\sigma} \rangle_\infty = \frac{1}{2} G_{k/2}(\tau, \sigma).$$

Now to motivate the algebraicity conjecture of Gross and Zagier, consider a cycle $\mathcal{D}_1$ on an arithmetic variety $\mathcal{X}$. This cycle is said to be principal if it vanishes in the Chow group of $\mathcal{X}$. This is accounted for by the existence of a formal linear combination of rational functions

$$\sum_i c_i(f_i, W_i)$$

whose divisor is $\mathcal{D}_1$. Given another cycle, $\mathcal{D}_2$, the archimedean contribution of the

Arakelov height pairing $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle_\infty$ becomes a sum of logarithms

$$\sum_i c_i \log |f_i(\mathcal{D}_2 \cdot W_i)|$$

i.e. it is a sum of logarithms of algebraic numbers belonging to the compositum of the fields of definition of $\mathcal{D}_1$ and $\mathcal{D}_2$.

This principle led Gross and Zagier to their conjecture that "principal" linear combinations of CM-values, $G_s(\tau, \sigma)$, are logarithms of algebraic numbers lying in the compositum of class fields of imaginary quadratic fields. In fact, they showed that when averaged over the Galois orbit of a pair of CM-points, principal linear combinations of the values

$$\sum_{\theta \in \mathrm{Gal}(H_\tau \cdot H_\sigma)} G_s(\tau^\theta, \sigma^\theta)$$

yield sums of logarithms of rational numbers. Here H_τ and H_σ denote the class fields of their respective CM point.

In [Zha97], Zhang provided a proof of the algebraicity conjecture in the case where τ and σ belong to the same imaginary quadratic field, which is conditional on the non-degeneracy of the height pairing. Mellit's thesis [Mel09] also uses an algebraic approach to give a proof of the algebraicity conjecture for $N = 1$, $k = 4$ and $\tau = i$.

However, a full unconditional proof of the algebraicity conjecture through this geometric interpretation seems to be currently out of reach. The main obstacles to this line of attack are deep conjectures on algebraic and arithmetic cycles such as Gillet–Soulé's arithmetic analogues of Grothendieck's standard conjectures on algebraic cycles [GS94]. Concretely, these conjectures would imply that the action of Hecke operators on cycles factors through the Chow group, and that the arithmetic height pairing is non-degenerate.

Nevertheless, the algebraicity conjecture could be approached through the analytic machinery of regularized theta lifts. This direction was followed in the work of Viazovska [Via15], Bruinier–Ehlen–Yang [BEY21] and a full proof of the conjecture was completed in [Li22, BLY22].

Real Multiplication and Stark-Heegner Cycles. Let p be a prime number, E a real quadratic field in which p is inert, and H_c/E the narrow ring class field of conductor c. For any eigenform $f \in S_k\left(\Gamma_0(p)\right)$ and any character χ of the group $\mathrm{Gal}(H_c/E)$ the sign of the functional equation for $L(f \otimes E, \chi, s)$ is -1. Therefore, the order of vanishing of $L(f \otimes H_c, s)$ at $k/2$ is at least $|\mathrm{Gal}(H_c/E)|$. Thus the conjecture of Bloch and Beilinson predicts the existence of a collection of cycles in $\mathrm{Ch}^{k/2}(\mathcal{Y}_k)$ defined over H_c, which would account for the vanishing of $L(f \otimes E, s)$ and would play the role of Heegner cycles in the real quadratic setting.

In weight 2, the emerging theory of Real Multiplication (RM) studies Stark–Heegner points as conjectural substitutes for Heegner points. In [Dar01], Darmon defines Stark–Heegner points through a p-adic integration theory which packages period integrals of modular forms into a measure and produces p-adic points on elliptic curves that are conjectured to be rational over abelian extensions of real quadratic fields. Similarly to their CM counterparts, Stark–Heegner points are related to Fourier coefficients of modular forms [DT08a] and they appear to be non-torsion precisely when the derivative of a Hasse–Weil L-function does not vanish [BD09].

Stark–Heegner points seem to fall outside current methods of constructing algebraic cycles. Their construction in general does not seem to come from an embedding of a special subvariety in a Shimura variety, although their construction relies essentially on modular tools which help to relate them to special values of p-adic L-functions, and to relate their relative positions to half-integral modular forms.

In the absence of a moduli theoretic interpretation for Stark–Heegner points, it remains unclear how to define the real quadratic analogues of Heegner cycles, which might be called Stark-Heegner cycles. Rotger and Seveso [RS12, Sev12] generalize Darmon's p-adic integration theory to higher weight and in doing so define local cohomology classes that they conjecture to be in the image of a global Selmer group associated to the Kuga–Sato varieties $\mathcal{Y}_k$. This conjecture was verified [Sev12] for linear combinations of Stark-Heegner cycles by genus characters. An explicit geometric construction of Stark–Heegner cycles remains a mystery.

Rigid Meromorphic Cocycles. In the analogy between Complex Multiplication and Real Multiplication, the theory of rigid meromorphic cocycles leads to the counterpart to Green's functions on modular curves. Let

$$\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p)$$

be the p-adic upper half-plane. The action of $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ on $\mathcal{H}_p$ by Möbius transformation is not discrete. It follows that if $\tau \in \mathcal{H}_p$ is a real-quadratic point, then there does not exist a Γ-invariant rigid meromorphic function on $\mathcal{H}_p$ whose divisor is supported on the Γ-orbit of τ. Instead, Darmon–Vonk [DV21] define a cocycle

$$J_\tau \in Z^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$$

whose divisor is supported on the Γ-orbit of τ. Here, $\mathcal{M}^\times$ denotes the Γ-module of rigid meromorphic functions on $\mathcal{H}_p$, i.e. for every $\gamma \in \Gamma$, the zeros and poles of $J_\tau(\gamma)(z)$ belong to $\Gamma \cdot \tau$. When there is a lift of J_τ to

$$\mathfrak{J}_\tau \in H^1(\Gamma, \mathcal{M}^\times),$$

then we can evaluate $\mathfrak{J}_\tau$ at another RM-point, σ, whose fundamental stabilizer is $\gamma_\sigma \in \Gamma$. This evaluation is given by

$$J_p(\tau, \sigma) := \mathfrak{J}_\tau[\sigma] := \mathfrak{J}_\tau(\gamma_\sigma)(\sigma).$$

Darmon and Vonk have recently announced a proof of the algebraicity of the norms of $J_p(\tau, \sigma)$ to $\mathbb{Q}_p$. Their proof also yields an explicit prime factorization formula for the norm of $J_p(\tau, \sigma)$, which is reminiscent of Gross–Zagier's factorization of singular moduli. The RM-values of rigid meromorphic cocycles behave analogously to the archimedean contribution of intersections of Heegner points; although the algebraicity of Stark–Heegner points is not known, one might speculate that the RM-values above encode some local contribution to a p-adic height pairing of Stark–Heegner points.

For example, it is observed in [DV21] that if $p = 3$ and $\phi = \frac{1+\sqrt{5}}{2}$, then up to 100

digits of 3-adic accuracy

$$J_3(\phi, 2\sqrt{5}) \stackrel{?}{=} \frac{174 + 832\sqrt{-1}}{2 \cdot 5^2 \cdot 17}, \tag{1.1}$$

and

$$J_3(\sqrt{5}, 2\sqrt{5}) \stackrel{?}{=} \frac{-4623 + 136\sqrt{-1}}{5^3 \cdot 37}. \tag{1.2}$$

Note that the narrow class field associated to the order $\mathcal{O}_{2\sqrt{5}}$ is $\mathbb{Q}(\sqrt{5}, i)$.

Main Construction: p-adic Higher Green's Functions. In this thesis, we generalize the theory of rigid meromorphic cocycles to higher weight, i.e. to cocycles of $H^1(\Gamma, \mathcal{M}(k))$, where $\mathcal{M}(k)$ is the module of rigid meromorphic functions with a weight k action

$$f \mid_k \gamma = (cz+d)^{-k} f(\gamma z), \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We define the higher Green's functions as periods of such higher weight meromorphic cocycles.

We classify the higher weight rigid meromorphic cocycles. For any formal linear combination, $\mathcal{D}$, of RM-points we define a canonical cocycle class (Lemma 4.1.6)

$$J_{k,\mathcal{D}} \in H^1_{par}(\Gamma, \mathcal{M}(k)),$$

whose divisor is supported on the Γ-orbits of the points of $\mathcal{D}$. Conversely, we will show that every such parabolic cohomology class is a linear combination of the meromorphic cocycle classes attached to real-quadratic points and an analytic cocycle.

The p-adic higher Green's function $G_k(\tau, \sigma)$ is defined as a period of the cocycle $J_{k,\tau}$; if there exists a cocycle $\mathfrak{J}_{k,\tau}$ which takes values in a space of Coleman primitives on $\mathcal{H}_p$ and satisfies

$$\frac{d^{k-1}}{dz^{k-1}} \mathfrak{J}_{k,\tau}(\gamma)(z) = J_{k,\tau}(\gamma)(z) \quad \text{for all } \gamma \in \Gamma,$$

then the higher Green's function is equal to

$$G_k(\tau,\sigma)=\delta_{2-k}^{(k-2)/2}\left(\mathfrak{J}_{k,\tau}(\gamma_\sigma)\right)(\sigma),$$

where $\delta_i=\frac{d}{dz}+\frac{i}{z-\bar{z}}$ is the usual raising operator applied formally to the function $\mathfrak{J}_{k,\tau}(\gamma_\sigma)$, and conjugation is the non-trivial automorphism of $\mathbb{Q}(\sigma)$. When $k=2$, the Green's function recovers Darmon–Vonk's real-quadratic singular moduli:

$$G_2(\tau,\sigma)=\log_p\left(J_p(\tau,\sigma)\right).$$

Main Conjecture and Supporting Evidence. We implemented an algorithm to compute some values of G_k on the computer algebra system SageMath. The resulting values of G_k at "principal" linear combinations of RM-orbits appear to be p-adic logarithms of algebraic numbers lying in the compositum of class fields of the real quadratic fields associated to the RM-orbits.

For example, let $p=3$ and $k=4$. In this case, $S_4(\Gamma_0(3))$ is trivial, which implies that every divisor is principal. Let $\phi=\frac{1+\sqrt{5}}{2}$ be the golden ratio. Up to 140 digits of 3-adic accuracy we compute

$$\begin{aligned}&40G_4(9[\phi]-2[\sqrt{5}],[2\sqrt{5}])\\&\stackrel{?}{=}5\log_p\left(\frac{2-i}{2+i}\right)+17\log_p\left(\frac{6+i}{6-i}\right)+63\log_p\left(\frac{4+i}{4-i}\right)+26\log_p\left(\frac{5+i}{5-i}\right).\end{aligned}$$

Note that i generates the narrow class field of the order $\mathcal{O}_{2\sqrt{5}}$ over $\mathbb{Q}(\sqrt{5})$. The prime factors in the factorization above are all inert or ramified in $\mathbb{Q}(\sqrt{5})$. Apart from the prime 13, all the other primes appear either in the factorization of $J_\phi[2\sqrt{5}]$ or $J_{\sqrt{5}}[2\sqrt{5}]$ given in (1.1) and (1.2).

For technical reasons, the computations are simpler to carry for divisors of "strong degree zero". We expect that a similar algebraicity statement holds for $G_4([\phi],[2\sqrt{5}])$ and $G_4([\sqrt{5}],[2\sqrt{5}])$.

Conjecture 1.0.1. *Let*

$$\mathcal{D} = \sum_i m_i \tau_i$$

be a finite formal linear combination of RM-points, which is principal i.e. $\mathrm{Obs}_0(J_{k,\mathcal{D}})$ *(Definition 4.1.1) is trivial. Then for any other RM-points* σ*, the value of the Green's function* $G_k(\mathcal{D}, \sigma)$ *is a linear combination of logarithms of algebraic numbers belonging to the compositum of narrow class fields associated with the quadratic orders* $\mathcal{O}_{\tau_i}$ *and* $\mathcal{O}_\sigma$*.*

The norms of the algebraic numbers that appear in Conjecture 1.0.1 also seem to admit factorization formulas similar to those of singular moduli. The primes appearing in the factorization of the norm of $G_k(\tau, \sigma)$ are non-split in both orders, and their multiplicities seem to be determined by weighted intersection numbers of geodesics on Shimura curves associated to definite quaternion algebras. Conjecture 5.3.2 gives a more precise formulation of this observation.

The algebraicity of the values $G_k(\tau, \sigma)$ lends evidence to the conjectured existence of rational Stark–Heegner cycles and further adds to the growing analogy between the theories of real multiplication and complex multiplication.

Meromorphic Cocycles of Higher Weight. In the multiplicative setting of [DV21], the meromorphic cocycles are defined explicitly as infinite products which are proven to converge by hand. In higher weight, the analogous infinite sums need not converge. The key idea to remedy this is as follows. If f is a rigid meromorphic function on $\mathcal{H}_p$ whose poles all lie in a finite extension $E/\mathbb{Q}_p$, then the restriction $f|_{\mathcal{H}_E}$ is a rigid analytic function on $\mathcal{H}_E := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(E)$. Meromorphic functions on $\mathcal{H}_p$ can therefore be constructed as integral transforms of certain boundary measures on $\mathbb{P}^1(E)$.

For example, the rigid meromorphic period function j_τ is defined by

$$j_\tau := \prod_{w \in \Gamma\cdot\tau} t_w(z)^{\delta_\infty(w)}, \quad \text{where } t_w(z) = \begin{cases} z - w & |w| \leq 1, \\ z/w - 1 & |w| > 1. \end{cases}$$

for a certain function $\delta_\infty(w) \in \{-1, 0, 1\}$. The logarithmic derivative,

$$\mathrm{dlog}(j_\tau) = \sum_{w \in \Gamma\cdot\tau} \frac{\delta_\infty(w)}{z - w},$$

belongs to the module, $\mathcal{M}(2)$, of weight 2 rigid meromorphic functions and has poles only at points in the orbit of τ. The boundary measure associated to this function is then simply the indicator function at each w in the orbit $\Gamma \cdot \tau$, weighted by $\delta_\infty(w)$; note that while $\Gamma \cdot \tau$ is not discrete in $\mathbb{P}^1(E)$, the subset of elements such that $\delta_\infty(w) \neq 0$ is discrete.

In [Neg23], Negrini shows that any parabolic cocycle in $Z^1_{par}(\Gamma, \mathcal{M}(k))$ has its poles supported only on a finite union of Γ-orbits of RM-points, and she produced cocycles of rigid meromorphic functions whose poles are supported on the union of Γ-orbits of an RM-point τ and its conjugate $\overline{\tau}$. The boundary measures described above will produce parabolic cocycles whose divisors are supported on the orbit of τ only, and therefore allows us to complete the classification of parabolic cocycles.

Another difference in the higher weight setting is the lack of an analogue to the logarithmic derivative map

$$\mathrm{dlog} : \mathcal{M}^\times \longrightarrow \mathcal{M}(2).$$

Real quadratic singular moduli are associated to the multiplicative cocycles. In higher weight, we consider instead the modules $\mathcal{M}_{\mathcal{L}}(2-k)$ defined to be the set of Coleman primitives of functions $f \in \mathcal{M}(k)$ under iterated differentiation $(k-1)$-times and with a choice of p-adic logarithm such that $\log_{\mathscr{L}}(p) = \mathscr{L}$. Note that the iterated differentiation map

$$d^{k-1} : \mathcal{M}_{\mathscr{L}}(2-k) \longrightarrow \mathcal{M}(k)$$

is a Γ-module surjection. The spaces $\mathcal{M}_{\mathscr{L}}(2-k)$ are the meromorphic counterpart to those considered by Breuil [Bre10] [Bre04]. In weight 2, the image of $\mathcal{M}^\times$ under the logarithm, $\log_{\mathscr{L}}$, is dense in $\mathcal{M}_{\mathscr{L}}(0)$.

Finally, to define $G_k(\tau, \sigma)$ we use Breuil duality, to pair the meromorphic cocycle

$$J_\tau^{\mathscr{L}} \in H^1(\Gamma, \mathcal{M}_{\mathscr{L}}(2-k))$$

on $\mathcal{H}_p$ with a locally analytic functions on the boundary of the form

$$\frac{(t-\sigma)^{(k-2)/2}(t-\overline{\sigma})^{(k-2)/2}}{(\sigma-\overline{\sigma})^{(k-2)/2}}\log_{\mathscr{L}}(t-\sigma).$$

We remark that throughout the thesis, we will restrict to p and odd prime. This is solely for the fact that for odd p, all quadratic extensions of $\mathbb{Q}_p$ are contained in a degree 4 extension, up to isomorphism. For $p=2$ all quadratic extensions are contained in a degree 8 extension and one would have to slightly adapt some proofs presented here. More precisely, the proof of Theorem 3.5.8 as presented would have to use instead the combinatorics of the Bruhat–Tits tree of the degree 8 extension.

Thesis Outline.

Chapter 2 Introduces rigid analysis on the p-adic upper half-plane. Much of this material is standard, see [DT08b] for an introduction. In our presentation of higher weight rigid meromorphic functions, we emphasize de Shalit's [dS09a] approach of viewing them as sections of a local system on $\mathcal{H}_p$. We continue with this approach when we introduce Breuil's space of log-rigid analytic functions; this point of view is discussed in an unpublished article of de Shalit [dS09b]. We also introduce the Morita and Breuil dualities, and present some explicit computations which are well-known that will be useful further on.

The novel contribution of this chapter is the section on rigid meromorphic functions. We explain the passage from rigid meromorphic functions on $\mathcal{H}_p$ to rigid analytic functions on a "smaller" upper half-plane. We give a criterion for rigid analytic functions on the "smaller" space to extend to a rigid meromorphic function on $\mathcal{H}_p$. We extend Vishik and Amice-Velu's theorem of p-adic integration to produce meromorphic functions, and we present a detailed proof of this result.

Chapter 3 Starts by compiling the standard definitions of group homology and cohomology, modular symbols and relations between these various objects. We review the theory of rigid analytic cocycles, which is discussed in [DT08b, Dar01, Das05] in the weight 2 setting and generalized in [Ort04, RS12, Sev12] to higher weight, and in other sources. We introduce binary quadratic forms attached to real-quadratic points, and define modular symbols which package divisors associated to real-quadratic points as functions on edges of a Bruhat-Tits tree.

We then recall Darmon–Vonk's multiplicative rigid meromorphic cocycles and describe their relation to the aforementioned modular symbols. We then generalize this construction to higher weight by integrating against the modular symbols taking values in harmonic cochains on edges. This leads to a full classification of parabolic rigid meromorphic cocycles in all weights, which extends Negrini's [Neg23] partial classification. Finally, we introduce a different approach to the construction of rigid meromorphic cocycles which is more refined but no longer necessarily takes values in parabolic cocycles. This refined approach is a generalization of Darmon–Vonk's later construction in [DV25].

Chapter 4 We discuss different "integrals" of rigid cocycles which leads us to the definition of the obstruction maps to finding a primitive as well as the evaluation pairing for cocycles. We use the evaluation pairing to define the p-adic higher Green's function. Many of the ideas from this chapter are motivated by [dS09a] and [RS12, Sev12].

Chapter 5 We describe an algorithm to efficiently compute certain values of the higher Green's function. We describe the results of some numerical experiments we carried out that seem to produce algebraic numbers in the compositum of abelian extensions of quadratic fields. From these computations we conjecture the algebraicity of principal values of the Green's function, we give a conjectural formula for the prime factorization of these algebraic numbers.

Notation

k	an even positive integer	
n	$k-2$	
p	a prime number	
Γ	$\mathrm{SL}_2(\mathbb{Z}[1/p])$	
F	a finite field extension of $\mathbb{Q}_p$	
$\mathcal{O}_F$	the ring of integers of F	
$\mathscr{P}_n(A)$	polynomials of degree at most n with coefficients in A, usually $\mathscr{P}_n(\mathbb{C}_p)$	
K	The bi-quadratic extension of $\mathbb{Q}_p$	
$\mathcal{H}_F$	p-adic upper half-plane for a p-adic field F	15
$\mathcal{T}_F$	The Bruhat-Tits tree for $\mathcal{H}_F$	16
red_F	the reduction map $\mathcal{H}_{\mathcal{F}} \to \mathcal{T}_F$	18
$\mathcal{O}(k, \mathcal{H}_F)$	Rigid analytic functions on $\mathcal{H}_F$ with a weight k action	18
$\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$	log-rigid analytic functions on $\mathcal{H}_F$ with a weight $2-k$ action	18
β	map relating higher-weight functions to local systems	21
$\Sigma_{\mathscr{L}}(k, F)$	locally analytic functions on $\mathbb{P}^1(F)$	22
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2

The p-adic upper half-plane

This chapter introduces p-adic analysis on the Drinfeld upper half-plane, which serves as the scaffolding for the rest of the thesis. We will define spaces of rigid analytic functions, as well as spaces of their Coleman primitives. Then, we will discuss Morita and Breuil duality, which relate functions on the upper half-plane to distributions on the boundary. Much of the material of this section is presented in more detail in [DT08b].

2.1 The Bruhat–Tits tree

Fix a prime number p. Let F be a fixed finite extension of $\mathbb{Q}_p$. Denote by $\mathcal{O}_F$ its ring of integers and $\kappa(F)$ its residue field, which has size q_F. Fix a choice of uniformizer $\pi_F \in \mathcal{O}_F$. Let e_F be the ramification index of $F/\mathbb{Q}_p$.

Definition 2.1.1 (The p-adic upper half-plane)**.** The *p-adic upper half plane* over F is the rigid analytic space, $\mathcal{H}_F$, whose E-rational points are

$$\mathcal{H}_F(E) := \mathbb{P}^1(E) \backslash \mathbb{P}^1(F),$$

for any complete extension E/F. In particular, $\mathcal{H}_F(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(F)$.

The group $\mathrm{GL}_2(F)$ acts on $\mathcal{H}_F$ by Möbius transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d}.$$

A pair (y_0, y_1) is said to be *primitive* if both y_0 and y_1 belong to $\mathcal{O}_E$ and at least one of them is a unit. For any complete extension E of F, for a primitive pair (x_0, x_1), define the closed disks in $\mathbb{P}^1(E)$ by

$$B_E\left([x_0 : x_1], r\right) = \left\{[y_0 : y_1] \mid \mathrm{ord}_p\left(x_0 \cdot y_1 - x_1 \cdot y_0\right) \geq r, \quad (y_0, y_1) \text{ is a primitive pair}\right\}.$$

Definition 2.1.2 (Bruhat–Tits tree)**.** *The Bruhat–Tits tree* over F is the graph, $\mathcal{T}_F$, whose vertex set is the set of homothety classes of $\mathcal{O}_F$-lattices in F^2. A pair of classes of lattices $[L_1]$ and $[L_2]$ are related by an edge if there exist representatives L_1 and L_2, respectively, such that

$$\pi_F L_1 \subsetneq L_2 \subsetneq L_1.$$

In this case, $[L_1]$ and $[L_2]$ are said to be *q_F-neighbors.*

Denote the set of vertices of $\mathcal{T}_F$ by $\mathcal{T}_F^0$, the set of oriented edges by $\mathcal{T}_F^1$ and the set of unoriented edges by $\overline{\mathcal{T}_F^1}$. The group $\mathrm{GL}_2(F)$ acts on $\mathcal{T}_F$ via its natural action on lattices in F^2. Denote oriented edges as an ordered pair, (v, w), where v and w are neighboring vertices.

$\mathcal{T}_F$ is a tree, and it is (q_F+1)-regular; that is, each vertex has precisely $q_F + 1 = |\mathbb{P}^1(\kappa(F))|$ neighbors. The choice of the standard basis on F^2 privileges distinguished elements of the tree. Namely, define

$$v_0 := \left[\mathcal{O}_F^2\right], \quad v_1 := \left[\mathcal{O}_F \oplus \pi_F \mathcal{O}_F\right].$$

We will refer to v_0 as the *standard vertex.* Observe that v_0 and v_1 are adjacent and denote the edge (v_0, v_1) by e_0 and refer to it as the *standard edge.*

The *level* of a vertex, $v = [L]$, is its distance from v_0 on the tree. Equivalently, the level of v is the smallest non-negative integer l such that there exists a representative, L, which satisfies

$$\pi_F^l \mathcal{O}_F^2 \subset L \subset \mathcal{O}_F^2.$$

Such a representative is called an *optimal representative*. The level of an edge is the level of the vertex which is further from v_0.

For any vertex, $v = [L]$ of $\mathcal{T}_F$ with optimal representative L, let L^{prim} denote the primitive elements of L, that is the elements of the lattice such that at least one of the two coordinates is a unit in $\mathcal{O}_F$. The image of the projection

$$L^{\mathrm{prim}} \longrightarrow \mathbb{P}^1(\mathcal{O}_F) = \mathbb{P}^1(F)$$

is a closed disk, denote by U_v. The disk U_v is independent of the choice of optimal representative.

An *end* of $\mathcal{T}_F$ is an equivalence class of infinite non-backtracking sequences of adjacent vectors,

$$(u_0, u_1, \cdots),$$

modulo tail-equivalence; that is; two sequences $(u_0, u_1, \cdots)$ and $(w_0, w_1, \cdots)$ are equivalent if there exists an integer m such that $u_i = w_{i+m}$ for all i large enough.

Proposition 2.1.3. *The set of ends of $\mathcal{T}_F$ can be naturally identified with $\mathbb{P}^1(F)$.*

Proof. Without loss of generality, we can assume that each end $(w_0, w_1, \cdots)$ begins at the standard vertex. Then, the intersection

$$\bigcap_{i \in \mathbb{N}} U_{w_i}$$

is a single element in $\mathbb{P}^1(F)$. In fact, every element $\tau \in \mathbb{P}^1(F)$ can be obtained in this way for a unique end; the choices of optimal representatives of the path from v_0 to τ can be obtained from the reduction of τ in $\mathbb{P}^1(\mathcal{O}_F/\pi_F^i \mathcal{O}_F)$. □

For any vertex v, the open set U_v is precisely the ends of $\mathcal{T}_F$ of non-backtracking paths which begin at v_0 and pass through v. Similarly, to each oriented edge, $e = (w_0, w_1)$, we let U_e be the set of ends of non-backtracking paths which begin at w_0 and pass through w_1.

Definition 2.1.4 (Reduction Map)**.** There is a $\mathrm{GL}_2(F)$-equivariant map

$$\mathrm{red}_F : \mathcal{H}_F(\mathbb{C}_p) \longrightarrow \mathcal{T}_F^0 \cup \overline{\mathcal{T}_F^1}$$

called the *reduction map*, which is defined as follows: Given $\tau \in \mathcal{H}_F(\mathbb{C}_p)$, there exists a minimal $r \in \mathbb{R}$ such that the intersection $B_{\mathbb{C}_p}(\tau, r) \cap \mathbb{P}^1(F)$ is a non-empty closed disk in $\mathbb{P}^1(F)$ and is equal to U_v for a unique vertex v.

- If $r \in \mathrm{ord}_p(F)$, then set $\mathrm{red}_F(\tau) := v$,
- otherwise, let e be the unique oriented edge starting at v and pointing towards the standard vertex. Set $\mathrm{red}_F(\tau)$ to be the unoriented edge with the same vertices as e.

Remark 2.1.5. Given an oriented edge, e, we write $\mathrm{red}_F^{-1}(e)$ to denote the preimage of the unoriented edge with the same vertices as e.

Given a vertex $v \in \mathcal{T}_F^0$, its preimage, $\mathrm{red}_F^{-1}(v)$, is an admissible affinoid of $\mathcal{H}_F$; the preimage of v_0 is referred to as the standard affinoid. The preimage of an edge is an annulus and the choice of an orientation on the edge corresponds to an orientation on the annulus.

Let $\mathcal{T}_F^{\leq l}$ be the subtree of $\mathcal{T}_F$ consisting of vertices and edges of level at most l. The preimage of this subtree, $\mathcal{H}_F^{\leq l} := \mathrm{red}_F^{-1}\left(\mathcal{T}_F^{\leq l}\right)$ is an admissible affinoid, and the collection $\left\{\mathcal{H}_F^{\leq l}\right\}_l$ is an admissible covering of $\mathcal{H}_F$.

2.2 Rigid Analysis

The ring of rigid analytic functions on $\mathcal{H}_F$ is a projective limit of affinoid algebras of entire functions on the admissible covering

$$\bigcup_{l=0}^{\infty} \mathcal{H}_F^{\leq l}.$$

We begin by giving a more explicit description of $\mathcal{H}_F^{\leq l}$. Fix a set of representatives, $\mathcal{R}_l$, of $\mathbb{P}^1(F) = \mathbb{P}^1(\mathcal{O}_F)$ modulo π_F^l such that $\infty \in \mathbb{P}^1(F)$ is the representative for $\infty \pmod{\pi_F^l}$. Define the open disks

$$B_{\mathbb{C}_p}^-\left([x_0 : x_1], r\right) = \left\{[y_0 : y_1] \in \mathbb{P}^1(\mathbb{C}_p) \mid \mathrm{ord}_p\left(x_0 \cdot y_1 - x_1 \cdot y_0\right) > r\right\}.$$

As before, all coordinates are taken to be primitive. Then the affinoids can be described as

$$\mathcal{H}_F^{\leq l} = \mathbb{P}^1(\mathbb{C}_p) \backslash \bigcup_{a \in \mathcal{R}_l} B_{\mathbb{C}_p}^-(a, l/e_F).$$

Let $\mathcal{O}(\mathcal{H}_F^{\leq l})$ be the $\mathbb{C}_p$-algebra of functions of the form

$$\sum_{i=0}^{\infty} b_{\infty,i} z^i + \sum_{a \in \mathcal{R}_l \backslash \{\infty\}} \sum_{i=1}^{\infty} \frac{b_{a,i}}{(z-a)^i},$$

with $b_{\infty,i}$, $b_{a,i} \in \mathbb{C}_p$ satisfying $\lim_{i \to \infty} \pi_F^{-il} b_{a,i} = 0$ for every $a \in \mathcal{R}_l$. These are precisely the sums of power series which converge away from the disks $B_{\mathbb{C}_p}^-(a, l/e_f)$.

The $\mathbb{C}_p$-algebra of rigid analytic functions on $\mathcal{H}_F$ is the projective limit

$$\mathcal{O}(\mathcal{H}_F) := \varprojlim_l \mathcal{O}(\mathcal{H}_F^{\leq l}),$$

where the projective morphisms are given by restriction of functions. The field of meromorphic functions $\mathcal{M}(\mathcal{H}_F)$ is the fraction field of $\mathcal{O}(\mathcal{H}_F)$.

For k any even integer and $n = k - 2$, let $\mathcal{O}(k, \mathcal{H}_F)$ be the right $\mathrm{GL}_2(F)$-module whose underlying vector space is $\mathcal{O}(\mathcal{H}_F)$, equipped with a weight k right action

$$\left(f|_k \gamma\right)(z) := \frac{\det(\gamma)^{k/2}}{(cz+d)^k} f(\gamma \cdot z), \quad \text{where } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F).$$

For a positive weight k, differentiating $(n+1)$-times is a morphism of $\mathrm{GL}_2(F)$-modules

$$\mathcal{O}(2-k, \mathcal{H}_F) \xrightarrow{d^{n+1}} \mathcal{O}(k, \mathcal{H}_F).$$

The kernel of this map is the $\mathrm{GL}_2(F)$-module of polynomials of degree at most n with coefficients in $\mathbb{C}_p$, denoted $\mathscr{P}_n(\mathbb{C}_p)$.

Following Breuil [Bre04, Bre10], define spaces of Coleman primitives of rigid-analytic functions as follows. Fix a choice of $\mathscr{L} \in \mathbb{C}_p$; this determines a branch of p-adic logarithm $\log_{\mathscr{L}} : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p$ such that

$$\log_{\mathscr{L}}(p) = \mathscr{L}.$$

For a positive weight $k > 0$, the space $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ is defined as a projective limit. For the affinoids $\mathcal{H}_F^{\leq l}$ define $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F^{\leq l})$ to be the $\mathbb{C}_p$ vector space of functions of the form

$$\sum_{i=0}^{\infty} b_{\infty,i} z^i + \sum_{a \in \mathcal{R}_l \setminus \{\infty\}} \sum_{i=1}^{\infty} \frac{b_{a,i}}{(z-a)^i} + \sum_{a \in \mathcal{R}_l \setminus \{\infty\}} P_a(z) \log_{\mathscr{L}}(z-a),$$

where P_a are polynomials in $\mathscr{P}_n(\mathbb{C}_p)$ and $b_{\infty,i}, b_{a,i} \in \mathbb{C}_p$ satisfying $\lim_{i\to\infty} \pi_F^{-ir} b_{a,i} = 0$ for every $a \in \mathcal{R}_l$. The space $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ is equipped with a weight $2-k$ action of $\mathrm{GL}_2(F)$, i.e. for $F \in \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ and $\gamma \in \Gamma$,

$$(F|\gamma)(z) = (cz+d)^{k-2} F(\gamma \cdot z).$$

Proposition 2.2.1 ([Bre04] Proposition 3.2.2)**.** *There is an exact sequence of $GL_2(F)$-modules*

$$0 \longrightarrow \mathscr{P}_n(\mathbb{C}_p) \longrightarrow \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F) \xrightarrow{d^{n+1}} \mathcal{O}(k, \mathcal{H}_F) \longrightarrow 0.$$

It will be convenient to also view rigid analytic functions as global sections of a local system on $\mathcal{H}_F$. For this we embed $\mathcal{O}(k, \mathcal{H}_F)$ inside $\mathcal{O}(2, \mathcal{H}_F) \otimes \mathscr{P}_n$ as a piece of a natural filtration, and $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ inside $\mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n$. The iterated differentiation d^{n+1} will then correspond to differentiation once on the first factor of the tensor

$$d \otimes 1 : \mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n \longrightarrow \mathcal{O}(2, \mathcal{H}_F) \otimes \mathscr{P}_n.$$

This map is roughly analogous to the Gauss–Manin connection in the setting of classical modular forms.

Define the maps

$$\gamma : \mathcal{O}(k, \mathcal{H}_F) \longrightarrow \mathcal{O}(2, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p),$$
$$g(z) \mapsto g(z)(T-z)^n$$

and

$$\beta : \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F) \longrightarrow \mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p)$$
$$G(z) \mapsto n! \sum_{i=0}^{n} \frac{G^{(i)}(z)}{i!}(T-z)^i,$$

where T is the variable for the space of polynomials $\mathscr{P}_n$.

Proposition 2.2.2. *The maps* β *and* γ *are* $\mathrm{GL}_2(F)$*-equivariant and fit into the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F) & \xrightarrow{d^{n+1}} & \mathcal{O}(k, \mathcal{H}_F) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p) & \xrightarrow{d\otimes 1} & \mathcal{O}(2, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p) \end{array}$$

Proof. This result follows from a straightforward but tedious computations. One simply expands out the desired maps and verifies that the two sides agree. □

The space $\mathcal{O}(2, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p)$ has a natural descending filtration of $\mathrm{GL}_2(F)$-modules given by

$$\mathrm{Fil}^j := \mathrm{span}_{\mathcal{O}(2,\mathcal{H}_F)} \left\langle (z-T)^i \mid j \leq i \leq n \right\rangle.$$

The map γ identifies $\mathcal{O}(k, \mathcal{H}_F)$ with Fil^n. There is also a filtration on

$$\mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p)$$

defined by

$$\mathrm{Fil}^j_{\mathscr{L}} := d^{-1}(\mathrm{Fil}^j).$$

The map β induces an isomorphism between $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ and $\mathrm{Fil}^n_{\mathscr{L}}$.

2.3 Locally Analytic Functions on the Boundary

Definition 2.3.1. Let $k \geq 2$ be an even weight, and $n = k - 2$.

- The space of locally analytic functions, $C(n, F)$, is the space of functions

  $$F \longrightarrow \mathbb{C}_p,$$

  such that for each $a \in F$, there exists a neighborhood of a on which f can be expressed as a convergent power series

  $$f(t) = \sum_{i \geq 0} c_{a,i}(t-a)^i,$$

  and in some neighborhood of ∞, the function f can be expressed as a convergent (away from ∞) power series

  $$f(t) = \sum_{i \leq n} c_{\infty,i} t^i;$$

  i.e. it has a pole at ∞ of order at most n.

- Fix a branch of the p-adic logarithm, $\log_{\mathscr{L}}$. The space $C_{\mathscr{L}}(n, F)$ is the space of locally analytic functions on F, which in a neighborhood of ∞ can be expressed in the form

  $$f(t) = \left(\sum_{i \leq n} c_{\infty,i} t^i\right) - 2P(t)\log_{\mathscr{L}}(t), \tag{2.1}$$

  where P is any polynomial of degree at most n.

The space $C(n, F)$ is a subspace of $C_{\mathscr{L}}(n, F)$, and we equip both spaces with an action of $\mathrm{GL}_2(F)$ by

$$(f|\gamma)(t) = \frac{(ct+d)^n}{\det(\gamma)^{n/2}}\left[f(\gamma \cdot t) + P(\gamma \cdot t)\log_{\mathscr{L}}\left(\frac{\det(\gamma)}{(ct+d)^2}\right)\right];$$

using the notation in (2.1).

Definition 2.3.2. Observe that $C(n,F)$ has $\mathscr{P}_n(\mathbb{C}_p)$ as a sub-module. Define $\Sigma(n,F)$ and $\Sigma_{\mathscr{L}}(n,F)$ to be the quotient modules

$$\Sigma(n,F) := C(n,F)/\mathscr{P}_n(\mathbb{C}_p),$$

and

$$\Sigma_{\mathscr{L}}(n,F) := C_{\mathscr{L}}(n,F)/\mathscr{P}_n(\mathbb{C}_p).$$

Proposition 2.3.3. *The following is an exact sequence*

$$0 \longrightarrow \Sigma(n,F) \longrightarrow \Sigma_{\mathscr{L}}(n,F) \longrightarrow \mathscr{P}_n(\mathbb{C}_p) \longrightarrow 0,$$

where the map $\Sigma_{\mathscr{L}}(n,F) \longrightarrow \mathscr{P}_n(\mathbb{C}_p)$ is given by sending to $f(t) \mapsto P(t)$ from (2.1).

We will write Σ or $\Sigma(F)$ for $\Sigma(0,F)$ and also $\Sigma_{\mathscr{L}}$ or $\Sigma_{\mathscr{L}}(F)$ for $\Sigma_{\mathscr{L}}(0,F)$. Define $\Phi^j_{\mathscr{L}}$ to be the subspace of $\Sigma_{\mathscr{L}} \otimes \mathscr{P}_n(\mathbb{C}_p)$ consisting of functions which can locally, on F, be written in the form $(t-T)^j g(t)$ where $g(t)$ is an analytic function. Let Φ^j be the intersection $\Phi^j_{\mathscr{L}} \cap \Sigma \otimes \mathscr{P}_n(\mathbb{C}_p)$. The map induced from

$$f(t) \otimes P(T) \mapsto f(t)P(t)$$

gives an isomorphism

$$\Phi^0(\Sigma_{\mathscr{L}} \otimes \mathscr{P}_n)/\Phi^1(\Sigma_{\mathscr{L}} \otimes \mathscr{P}_n) \cong \Sigma_{\mathscr{L}}(n,F). \tag{2.2}$$

2.4 Morita and Breuil Duality

The space $\mathcal{O}(2,\mathcal{H}_F)$ is best thought of as the space of differentials on $\mathcal{H}_F$, it naturally has poles on the boundary of $\mathcal{H}_F$. On the other hand, the space $C(n,F)$ of analytic functions on the boundary, $\mathbb{P}^1(F)$, is a space of functions that extend slightly beyond the boundary. By taking residues of products of differentials and analytic functions we obtain a pairing which is reminiscent of Serre duality. This duality will be used to define rigid analytic functions by identifying them as distributions on the boundary, and will also be used to evaluate rigid meromorphic cocycles.

More precisely, given $f \in \mathcal{O}(2, \mathcal{H}_F)$ and $g \in \Sigma(F)$. Pick l large enough so that $f|\mathcal{H}_F^{\leq l}$ can be written as

$$f(z) = \sum_{i=0}^{\infty} b_{\infty,i} z^i + \sum_{a \in \mathcal{R}_l \setminus \{\infty\}} \sum_{i=1}^{\infty} \frac{b_{a,i}}{(z-a)^i},$$

and for each $a \in \mathcal{R}_l$, the function g has a power series expansion of the form

$$\sum_{j=0}^{\infty} c_{a,j}(z-a)^j$$

on the disk $B_F(a, l)$ for every $a \neq \infty$, and of the form

$$\sum_{j=0}^{\infty} c_{\infty,j} z^{-j}$$

on the disc $B_F(\infty, l)$. Then the residue pairing is defined by

$$\begin{aligned} \langle f, g \rangle_M &:= \text{“} \sum_{\sigma \in \mathbb{P}^1(F)} \operatorname{res}_\sigma(f(z)g(z)dz) \text{”} \\ &:= \sum_{a \in \mathcal{R}_l \setminus \{\infty\}} \left(\sum_{i=1}^{\infty} b_{a,i} c_{a,i-1} - b_{a,1} c_{\infty,0} \right) - \sum_{i=0}^{\infty} b_{\infty,i} c_{\infty,i+1}. \end{aligned}$$

Theorem 2.4.1 (Morita Duality [Mor84])**.** *The pairing*

$$\langle \cdot, \cdot \rangle_M : \ \mathcal{O}(2, \mathcal{H}_F) \times \Sigma(F) \longrightarrow \mathbb{C}_p$$

defines a perfect bilinear pairing of GL_2-modules.

Remark 2.4.2. Given a point $\tau \in \mathcal{H}_F$, then the function $\dfrac{1}{\tau - t}$ belongs to $\Sigma(F)$ and the Morita pairing recovers values of f by

$$\left\langle f(z), \frac{1}{\tau - t} \right\rangle_M = f(\tau). \tag{2.3}$$

Further, given two points $\tau_1, \tau_2 \in \mathcal{H}_F$, the function

$$\log_{\mathscr{L}}\left(\frac{t-\tau_1}{t-\tau_2}\right)$$

also belongs to $\Sigma(F)$. We have

$$\left\langle f(z), \log_{\mathscr{L}}\left(\frac{t-\tau_1}{t-\tau_2}\right)\right\rangle_M = \int_{\tau_2}^{\tau_1} f(z)\, dz, \tag{2.4}$$

where the integral is a Coleman integral [Col82], with the same choice of branch of logarithm.

Define a pairing

$$\langle \cdot, \cdot \rangle_{\mathscr{P}_n} : \mathscr{P}_n(\mathbb{Q}) \times \mathscr{P}_n(\mathbb{Q}) \longrightarrow \mathbb{Q}$$

by

$$\left\langle T^i, T^j \right\rangle_{\mathscr{P}_n} = (-1)^i \binom{n}{i}^{-1} \delta_{n,i+j}.$$

Proposition 2.4.3. *For every n, the pairing $\langle \cdot, \cdot \rangle_{\mathscr{P}_n}$ is a perfect pairing.*

Proof. Fix an isomorphism $\mathrm{Sym}^n(\mathbb{Q}^2) \cong \mathscr{P}_n(\mathbb{Q})$ by mapping

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto 1, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto T.$$

The determinant gives a pairing on $\mathbb{Q}^2$ which induces a pairing on the symmetric product. Composing by this isomorphism, this produces $\langle \cdot, \cdot \rangle$. $\square$

It follows that for every $a \in \mathbb{Q}$,

$$\langle (t-a)^i, P(t) \rangle_{\mathscr{P}_n} = (-1)^i \frac{i!}{n!} P^{(n-i)}(a),$$

where $P^{(s)}$ denotes differentiation s times.

The combination of the Morita pairing and the pairing on polynomials produces a pairing

$$\mathcal{O}(2,\mathcal{H}_F)\otimes\mathscr{P}_n(\mathbb{C}_p)\times\Sigma(F)\otimes\mathscr{P}_n(\mathbb{C}_p)\longrightarrow\mathbb{C}_p,$$

which we also denote by $\langle\cdot,\cdot\rangle_M$. In terms of the filtration, the pairing descends to a perfect pairing ([dS09a] Corollary 2.5)

$$\mathrm{Fil}^i/\mathrm{Fil}^{i+1}\times\Phi^{n-i}/\Phi^{n+1-i}\longrightarrow\mathbb{C}_p.$$

Specializing to $i=n$, there is a pairing

$$\mathcal{O}(k,\mathcal{H}_F)\times\Sigma(n,F)\longrightarrow\mathbb{C}_p,$$

which we again denote by $\langle\,\cdot\,,\cdot\,\rangle_M$.

Similarly to (2.3), for any $f\in\mathcal{O}(k,\mathcal{H}_F)$ we have

$$\left\langle f(z),\frac{1}{\tau-t}\right\rangle_M=f(\tau).$$

Equation (2.4) relates the Morita pairing to Coleman integrals, its higher weight analogue relates to Coleman integrals on local systems $\mathscr{P}_n$; for $f\in\mathcal{O}(k,\mathcal{H}_F)$ and $\tau_1,\tau_2\in\mathcal{H}_F$ and $P\in\mathscr{P}_n(\mathbb{C}_p)$ we have

$$\left\langle f(z),P(t)\log_{\mathscr{L}}\left(\frac{t-\tau_1}{t-\tau_2}\right)\right\rangle_M=\left\langle\int_{\tau_2}^{\tau_1}\beta(f)\;dz,P(T)\right\rangle_{\mathscr{P}_n}.$$

Theorem 2.4.4 (Breuil Duality, [Bre04] Theorem 1.1.4)**.** *There is a unique pairing of GL_2-modules*

$$\langle\cdot,\cdot\rangle_B:\mathcal{O}_{\mathscr{L}}(0,\mathcal{H}_F)\times\Sigma_{\mathscr{L}}\longrightarrow\mathbb{C}_p,$$

such that

1. *If $f\in\mathcal{O}_{\mathscr{L}}(0,\mathcal{H}_F)$ and $\tau\in\mathcal{H}_F$*

$$\langle f,\log_{\mathscr{L}}(t-\tau)\rangle_B=f(\tau),$$

2. *If $g \in \Sigma \subset \Sigma_{\mathscr{L}}$, then*
$$\langle f, g \rangle_B = \left\langle \frac{d}{dz}(f), g \right\rangle_M,$$

3. *If $f \in \mathcal{O}(0, \mathcal{H}_F)$, then*
$$\langle f, g \rangle_B = -\left\langle f, \frac{d}{dt}(g) \right\rangle_M.$$

As before, combining the Breuil pairing with the pairing on polynomials we obtain a pairing
$$(\mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F) \otimes \mathscr{P}_n(\mathbb{C}_p)) \times (\Sigma_{\mathscr{L}} \otimes \mathscr{P}_n(\mathbb{C}_p)) \longrightarrow \mathbb{C}_p,$$
which we will denote by $\langle \cdot, \cdot \rangle_B$. In this case, the duality between
$$F^n_{\mathscr{L}} \times \Phi^0_{\mathscr{L}} / \Phi^1_{\mathscr{L}} \longrightarrow \mathbb{C}_p$$
defines the perfect pairing between $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ and $\Sigma_{\mathscr{L}}(n, F)$, which is also referred to as the Breuil pairing and denote by $\langle \cdot, \cdot \rangle_B$.

Proposition 2.4.5. *The Breuil pairing satisfies the following*

1. *If $\tau \in \mathcal{H}_F$ and $f \in \mathcal{O}_{\mathscr{L}}(2-k)$, then*
$$\left\langle f(z), \frac{(t-\tau)^n}{n!} \log_{\mathscr{L}}(t-\tau) \right\rangle_B = f(\tau),$$

2. *if $g(t) \in \Sigma(n)$, then*
$$\langle f(z), g(t) \rangle_B = \left\langle \frac{d^{n+1}}{dz^{n+1}}(f)(z), g(t) \right\rangle_M,$$

3. *if $f(z) \in \mathcal{O}(2-k)$, then*
$$\langle f(z), g(t) \rangle_B = \left\langle f(z), (-1)^{n+1} \frac{d^{n+1}}{dt^{n+1}}(g)(t) \right\rangle_M.$$

2.5 The Schneider–Teitelbaum Lift

Recall that every oriented edge, $e \in \mathcal{T}_F^1$, has an associated oriented annulus, $\mathrm{red}_F^{-1}(e)$. Therefore, for every $f \in \mathcal{O}(2, \mathcal{H}_F)$ there is a well-defined annular residue $\mathrm{res}_e(f(z)dz)$ at e, which is associated to the differential $f(z)dz$.

Definition 2.5.1. For an oriented edge $e = (w_1, w_2)$, let $s(e) := w_1,\ t(e) := w_2$ denote the start and terminal vertices of e, and $\overline{e} := (w_2, w_1)$ be the edge with the opposite orientation.

Every rigid analytic function $f(z) \in \mathcal{O}(2, \mathcal{H}_F)$ determines a residue function on the edges

$$\begin{aligned} c_f : \mathcal{T}_F^1 &\longrightarrow \mathbb{C}_p \\ e &\mapsto \mathrm{res}_e(f(z)dz). \end{aligned}$$

We write $\mathrm{res}_e(f)$ for $\mathrm{res}_e(f(z)\ dz)$.

The residue theorem implies that for any vertex $v \in \mathcal{H}_F$ and rigid analytic function f,

$$\sum_{s(e)=v} \mathrm{res}_e(f) = 0, \tag{2.5}$$

and for every edge e,

$$\mathrm{res}_e(f) = -\mathrm{res}_{\overline{e}}(f). \tag{2.6}$$

Similarly, for a higher weight rigid analytic function $f(z) \in \mathcal{O}(k, \mathcal{H}_F)$ its residue is given by considering the residue of $f(z)(T-z)^n\ dz$, and so we obtain a residue function

$$c_f : \mathcal{T}_F^1 \longrightarrow \mathscr{P}_n(\mathbb{C}_p),$$

which also satisfies the residue theorem.

Definition 2.5.2. For any $\mathrm{GL}_2(F)$-module M, denote by $C_F^0(M)$, the Γ-module of functions on the vertices of the tree valued in M:

$$\mathcal{T}_F^0 \longrightarrow M.$$

Similarly for edges, denote by $C_F^1(M)$ the Γ-module of functions

$$c : \mathcal{T}_F^1 \longrightarrow M$$

with the extra requirement that $c(e) = -c(\overline{e})$. When the module M is not specified then the functions take values in $\mathbb{C}_p$. Let $\nabla : C_F^1(M) \longrightarrow C_F^0(M)$ be the map taking a function $c \in C_F^1(M)$ to the function

$$\nabla(c) : v \mapsto \sum_{s(e)=v} c(e).$$

The kernel of ∇ is the space of harmonic cochains, denoted by $C_{har}^1(M)$. Since $\mathcal{T}_F$ is an infinite tree, the map ∇ is surjective, and there is an exact sequence

$$0 \longrightarrow C_{har}^1(M) \longrightarrow C_F^1(M) \longrightarrow C_F^0(M) \longrightarrow 0. \tag{2.7}$$

From (2.5) we can see that the function c_f attached to a rigid analytic function is harmonic. In fact, the map $f \mapsto c_f$ fits in the exact sequence

$$0 \to \mathscr{P}_n \longrightarrow \mathcal{O}(2-k, \mathcal{H}_F) \xrightarrow{d^{n+1}} \mathcal{O}(k, \mathcal{H}_F) \longrightarrow C_{F,har}^1(\mathscr{P}_n) \longrightarrow 0.$$

We equip these spaces of functions on the tree, $C_{har}^1(\mathscr{P}_n), C_F^1(\mathscr{P}_n)$ and $C_F^0(\mathscr{P}_n)$, with an integral structure as follows: For the standard vertex v_0, set $\mathscr{P}_{n,v_0} := \mathscr{P}_n(\mathcal{O}_{\mathbb{C}_p})$. For other vertices, define for $g \in \mathrm{GL}_2(F)$

$$\mathscr{P}_{n,g\cdot v_0} := \mathscr{P}_{n,v_0}|g^{-1}.$$

For every edge, $e = (v, v')$, let

$$\mathscr{P}_{n,e} := \mathscr{P}_{n,v} \cap \mathscr{P}_{n,v'}.$$

Let

$$C_F^{0,\mathrm{int}}(\mathscr{P}_n) := \left\{ d \in C_F^0(\mathscr{P}_n) \mid d(v) \in \mathscr{P}_{n,v} \text{ for all } v \in \mathcal{T}_F^0 \right\},$$

$$C_F^{1,\mathrm{int}}(\mathscr{P}_n) := \left\{ c \in C_F^1(\mathscr{P}_n) \mid c(e) \in \mathscr{P}_{n,e} \text{ for all } e \in \mathcal{T}_F^1 \right\},$$

and

$$C_{F,har}^{1,\mathrm{int}}(\mathscr{P}_n) := C_{F,har}^1(\mathscr{P}_n) \cap C_F^{1,\mathrm{int}}(\mathscr{P}_n). \tag{2.8}$$

Finally, the space of bounded harmonic cochains is the base-change

$$C_{F,har}^{1,b}(\mathscr{P}_n) := C_{F,har}^{1,\mathrm{int}}(\mathscr{P}_n) \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

It will be helpful to associate to harmonic cochains on $\mathcal{T}_F$ a measure on the boundary of the tree.

Definition 2.5.3. For any $\mathrm{GL}_2(F)$-module, M, let $C_F^{lc}(M)$ denote the space of locally constant functions on $\mathbb{P}^1(F)$ valued in M. Define a measure on $\mathbb{P}^1(F)$ valued in M to be a functional

$$C_F^{lc}(M^\vee) \longrightarrow \mathbb{C}_p.$$

The $\mathrm{GL}_2(F)$-module of measures is denoted by $\mathbb{M}(\mathbb{P}^1(F), M)$. A measure is said to be of *total mass zero* if it vanishes on constant functions; the submodule of measures of total mass zero is denoted by $\mathbb{M}_0(\mathbb{P}^1(F), M)$.

Given a harmonic cochain $c \in C_{F,har}^1(M)$, there is an associated measure of total mass zero defined by

$$\mu(U_e) = c(e), \quad \text{for } e \in \mathcal{T}_F^1.$$

This defines a measure since the opens sets $\{U_e\}$ form a basis for the topology of $\mathbb{P}^1(F)$ and the harmonic assumption ensures the additivity property of the measure and also implies the total mass zero criterion.

We identify $\mathscr{P}_n$ with its dual via the pairing $\langle\cdot,\cdot\rangle_{\mathscr{P}_n}$. Therefore, a measure $\mu \in \mathbb{M}_0(\mathbb{P}^1(F), \mathscr{P}_n)$ is a collection of compatible elements $\mu(U) \in \mathscr{P}_n$, for every open subset $U \subset \mathbb{P}^1(F)$, and for $P \in \mathscr{P}_n$ set,

$$\int_U P(t)d\mu(t) := \int_{\mathbb{P}^1(F)} \mathbb{1}_U(t)P(t)d\mu(t) := \langle P(T), \mu(U)\rangle_{\mathscr{P}_n}.$$

This identified $\mathbb{M}_0(\mathbb{P}^1(F), \mathscr{P}_n)$ with $C^1_{F,har}(\mathscr{P}_n)$. By Morita duality, we have identified $\mathcal{O}(k, \mathcal{H}_F)$ with the dual of

$$\Sigma(n, F) = C(n, F)/\mathscr{P}_n(\mathbb{C}_p).$$

The residue map fits into the diagram

$$\begin{tikzcd}
\mathcal{O}(k,\mathcal{H}_F) \arrow[rr, "\sim"] \arrow[drr, "\mathrm{res}"'] & & \Sigma(n,F)^\vee \arrow[d] \\
& & \big(C^{lc}(\mathscr{P}_n)/\mathscr{P}_n(\mathbb{C}_p)\big)^\vee \cong C^1_{F,har}(\mathscr{P}_n(\mathbb{C}_p)).
\end{tikzcd}$$

Where the vertical arrow is induced by the inclusion C^{lc} into $C(n, F)$.

There is no $\mathrm{GL}_2(F)$-equivariant section $C^1_{F,har} \longrightarrow \Sigma(2, F)^\vee$, this is because not all measures extend canonically to distributions on locally analytic functions. However, such a section exists if we restrict to bounded harmonic cochains.

Theorem 2.5.4 (Vishik, Amice-Velu, Schneider-Teitelbaum)**.** *There is a* $\mathrm{GL}_2(F)$*-equivariant section*

$$ST : C^{1,b}_{F,har}(\mathscr{P}_n(\mathbb{C}_p)) \longrightarrow \mathcal{O}(k, \mathcal{H}_F).$$

Proof Sketch. Below is a brief sketch. We will give a more detailed proof in the proof of Proposition 2.7.6.

A bounded harmonic cochain, ϕ, gives rise to a bounded measure

$$\mu \in \mathbb{M}_0(\mathbb{P}^1(F), \mathscr{P}_n).$$

The boundedness translates to bounds on integrals of the form

$$\int_{U(e)} (t-a)^i \, d\mu(t), \quad \text{where } a \in U_e.$$

Given a locally analytic function $g \in C(k, F)$, it can be approximated by a sequence of locally constant functions $g_i \in C_F^{lc}(\mathscr{P}_n(\mathbb{C}_p))$. The bounds on the integrals of μ imply that

$$\int_{\mathbb{P}^1(F)} g(t) \, d\mu(t) := \lim_{i\to\infty} \int_{\mathbb{P}^1(F)} g_i(t) \, d\mu(t)$$

converges. In particular,

$$ST(\phi)(z) := \int_{\mathbb{P}^1(F)} \frac{1}{z-t} \, d\mu(t)$$

converges to a rigid analytic function in z, whose annular residues recover the function ϕ.

See also [DT08b, Theorem 2.3.2], [MTT86, Chapter 1.11] and [Ort04, 3.2] for more details. □

We refer to the map ST as the Schneider–Teitelbaum map.

Definition 2.5.5. The image $ST(C_{har}^{1,b}(\mathscr{P}_n(\mathbb{C}_p)))$ is the space of bounded functions, which we denote by $\mathcal{O}(k, \mathcal{H}_F)^b$. An alternative definition of $\mathcal{O}(k, \mathcal{H}_F)^b$ can be given as follows:

Let $|\cdot|$ define a $\mathrm{GL}_2(\mathcal{O}_F)$-invariant norm on $\mathcal{O}(k, \mathcal{H}_F)$ given by

$$|f| := \max_{z \in \mathrm{red}^{-1}(v_0)} |f(z)|.$$

We extend this to a GL_2-invariant norm by

$$\|f\| := \sup_{\gamma \in \mathrm{GL}_2(F)} |(f|_k \gamma)|.$$

The space $\mathcal{O}(k, \mathcal{H}_F)^b$ is then precisely the space of all f such that $||f|| < \infty$.

Theorem 2.5.6 ([dS09a] Theorem 1.1)**.** *The Schneider-Teitelbaum map is an isomorphism*

$$ST : C^{1,b}_{F,har}(\mathscr{P}_n(\mathbb{C}_p)) \xrightarrow{\cong} \mathcal{O}^b(k, \mathcal{H}_F).$$

2.6 Computations

We record here some lemmas which will be useful later on.

Lemma 2.6.1. *Let E be any field, for any $\tau \in E$ and $P(T) \in E[T]$ a polynomial of degree n. Let*

$$d^{(n+1)}\left(P(z)\log(z-\tau)\right)$$

be the rational function defined by formally differentiating (n+1)-times the expression

$$P(z)\log(z-\tau).$$

Here $\log(z-\tau)$ is a symbol that satisfies the relation

$$d\log(z-\tau) = \frac{1}{z-\tau}.$$

Then

$$d^{(n+1)}\left(P(z)\log(z-\tau)\right) = \sum_{i=0}^{n}(-1)^i(n-i)!\frac{P^{(i)}(\tau)}{(z-\tau)^{n+1-i}}.$$

Proof. Expanding $P(z)$ in the basis $(z-\tau)^i$ and differentiating we obtain

$$d^{(n+1)}\left((P(z)\log(z-\tau)\right) = \sum_{a=0}^{n}\sum_{j=0}^{n-a}\binom{n+1}{a}\frac{P^{(j+a)}(\tau)}{j!}(z-\tau)^j \cdot \frac{(-1)^{n-a}(n-a)!}{(z-\tau)^{n+1-a}}$$

Rearranging the sums with $i = a + j$ we obtain

$$= \sum_{i=0}^{n}\left(\sum_{a=0}^{i}(-1)^{n-a}\binom{n+1}{a}\frac{(n-a)!}{(i-a)!}\right)\frac{P^{(i)}(\tau)}{(z-\tau)^{n+1-i}}$$

The equality

$$\sum_{a=0}^{i}(-1)^{n-a}\binom{n+1}{a}\frac{(n-a)!}{(i-a)!}=(-1)^i(n-i)!$$

follows from Vandermonde's identity.

□

Definition 2.6.2. Let $\mathcal{D}=\sum_i(\tau_i)\otimes P_i(t)$ be a divisor in $\mathrm{Div}(\mathbb{P}^1(F))\otimes\mathscr{P}_n(\mathbb{C}_p)$, then its associated measure is

$$\mu_{\mathcal{D}}(U):=\sum_{\tau_i\in U}P_i.$$

Lemma 2.6.3. *Let* $\mathcal{D}=\sum_i(\tau_i)\otimes P_i$ *be a degree* 0 *divisor in* $\mathrm{Div}_0(\mathbb{P}^1(F))\otimes\mathscr{P}_n(\mathbb{C}_p)$. *Then the measure* $\mu_{\mathcal{D}}$ *is a bounded measure on* $\mathbb{P}^1(F)$, *whose Schneider-Teitelbaum lift is*

$$ST(\mu_{\mathcal{D}})(z)=\frac{1}{n!}\sum_i d^{n+1}(P_i(z)\log(z-\tau_i)).$$

Proof. Let $\{e_i\}_i$ be any set of edges on $\mathcal{T}_F$ such that $U(e_i)$ are all pairwise disjoint and $\tau_i\in U(e_i)$.

$$\begin{aligned}
ST(\mu_{\mathcal{D}})(z)&=\int_{\mathbb{P}^1(F)}\frac{1}{z-t}\,d\mu_{\mathcal{D}}(t)\\
&=\sum_i\int_{U(e_i)}\frac{1}{z-t}\,d\mu_{\mathcal{D}}(t)\\
&=\sum_i\sum_{j=0}^{\infty}\frac{\int_{U(e_i)}(t-\tau_i)^j\,d\mu_{\mathcal{D}}(t)}{(z-\tau_i)^{j+1}}\\
&=\sum_i\sum_{j=0}^{n}(-1)^j\frac{j!}{n!}\frac{P_i^{n-j}(\tau_i)}{(z-\tau_i)^{j+1}}\\
&=\frac{1}{n!}\sum_i d^{n+1}\left(P_i(z)\log(z-\tau_i)\right).
\end{aligned}$$

□

We will record here also some helpful identities regarding the Breuil pairing. First we prove the following special case of Theorem 2.4.4.

Lemma 2.6.4. *For $\tau \in \mathcal{H}_F$ and $G \in \mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F)$,*

$$\langle G(z), \log_{\mathscr{L}}(t-\tau)\rangle_B = G(\tau).$$

Proof. Let L be the subspace of $\mathcal{O}_{\mathscr{L}}(0, \mathcal{H}_F)$ spanned by constant functions and functions of the form $\log_{\mathscr{L}}(z-\alpha)$ for $\alpha \in F$, by [Bre04, Lemma 3.5.1], L is a dense subspace and so it suffices to prove this identity for G of this form. Let U be a neighborhood of $\infty \in \mathbb{P}^1(F)$ not containing α. Define $H(t) = \log_{\mathscr{L}}(t-\tau) \in \Sigma_{\mathscr{L}}(0, F)$ and

$$H_U(t) = \begin{cases} \log_{\mathscr{L}}(t-\tau), & t \in U \\ 0, & t \notin U. \end{cases}$$

Then

$$\langle \log_{\mathscr{L}}(z-\alpha), H(t)\rangle_B = \left\langle \frac{1}{z-\alpha}, H(t) - H_U(t) \right\rangle_M + \langle \log_{\mathscr{L}}(z-\alpha), H_U(t)\rangle_B .$$

By [Bre04, Definition 3.4.1], $\langle \log_{\mathscr{L}}(z-\tau), H_U(t)\rangle_B = 0$. So

$$\begin{aligned} \langle \log_{\mathscr{L}}(z-\alpha), H(t)\rangle_B &= \left\langle \frac{1}{z-\alpha}, H(t) - H_U(t) \right\rangle_M \\ =&\mathrm{Res}_{z=\tau} \frac{\log_{\mathscr{L}}(z-\tau)}{z-\alpha} = \log_{\mathscr{L}}(\tau-\alpha). \end{aligned}$$

□

Corollary 2.6.5. *For $\tau \in \mathcal{H}_F$ and $G \in \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$,*

$$\left\langle G(z), (t-\tau)^i \log_{\mathscr{L}}(t-\tau)\right\rangle_B = (-1)^{n-i} i! G^{(n-i)}(\tau).$$

Proof. Recall that the Breuil pairing was defined by identifying $\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F)$ (resp.

$\Sigma_{\mathscr{L}}(n,F)$) as a graded piece of $\mathcal{O}_{\mathscr{L}}(0,\mathcal{H}_F)\otimes\mathscr{P}_n$ (resp. $\Sigma_{\mathscr{L}}(0,F)\otimes\mathscr{P}_n$). So we find that

$$\left\langle G(z),(t-\tau)^i\log_{\mathscr{L}}(t-\tau)\right\rangle_B=\left\langle n!\sum_{j=0}^{n}\frac{G^{(j)}(z)}{j!}(T-z)^j,(T-\tau)^i\log_{\mathscr{L}}(t-\tau)\right\rangle_B$$
$$=n!\sum_{j=0}^{n}\frac{G^{(j)}(\tau)}{j!}\left\langle(T-\tau)^j,(T-\tau)^i\right\rangle_{\mathscr{P}_n}=(-1)^{n-i}i!G^{(n-i)}(\tau).$$

□

2.7 Rigid Meromorphic Functions

Let E/F be a finite field extension with ramification index $e_{E/F}$.

Definition 2.7.1. Denote by $\mathcal{M}(k,\mathcal{H}_F)$, the $\mathrm{GL}_2(F)$-module of rigid meromorphic functions on $\mathcal{H}_F$ equipped with a weight k action, and $\mathcal{M}(k,\mathcal{H}_F,E)$ the submodule of meromorphic functions whose poles all lie in E. For any $f\in\mathcal{M}(k,\mathcal{H}_F)$, write S_f for its set of poles in $\mathcal{H}_F$.

Remark 2.7.2. For any meromorphic function f, the set S_f is locally finite, i.e. its intersection with every affinoid is finite.

Lemma 2.7.3. *Let $f\in\mathcal{M}(k,\mathcal{H}_F,E)$ be a rigid meromorphic function whose poles lie in E. Then, the restriction $f|_{\mathcal{H}_E}$ is a rigid analytic function on $\mathcal{H}_E$.*

Proof. Notice the containment

$$\mathcal{H}_E^{\le(e_{E/F}l)}\subset\mathcal{H}_F^{\le l}.$$

By the Weierstrass preparation theorem, the restriction of f to $\mathcal{H}_F^{\le l}$ can be expressed as a sum of convergent power series

$$\sum_{i=0}^{\infty}b_{\infty,i}z^i+\sum_{a\in\mathcal{R}_l\setminus\{\infty\}}\sum_{i=1}^{\infty}\frac{b_{a,i}}{(z-a)^i}+\sum_{j\in J}\frac{Q_j(z)}{(z-\alpha_j)^{e_j}},\tag{2.9}$$

where $\{\alpha_j\}_{j\in J}$ is the finite set of poles of f that lie in $\mathcal{H}_F^{\le l}$, the exponents e_j are the order of the poles, and $Q_j(z)$ are polynomials.

Since the poles all lie in E, the expansion above is the Mittag-Leffler expansion of a rigid analytic function on $\mathcal{H}_E^{\leq l}$ and the result follows. $\square$

Our aim now is to describe a subspace of $C^1_{E,har}(\mathscr{P}_n)$ for which there is a splitting of the residue map

$$\mathcal{M}(k, \mathcal{H}_F, E) \longrightarrow C^1_{E,har}(\mathscr{P}_n).$$

Notice that in (2.9), the Mittag-Leffler expansion in the directions towards $\mathbb{P}^1(E)\backslash\mathbb{P}^1(F)$ are a finite sum of rational functions. We will translate this to a property on harmonic cochains on the Bruhat-Tits tree.

The base-change of lattices produces a map on vertices of the Bruhat-Tits trees

$$\mathcal{T}_F^0 \longrightarrow \mathcal{T}_E^0.$$

This map does not preserve edges if the extension E/F is ramified; every edge in $\mathcal{T}_F^1$ splits into a path whose length is the ramification index. We obtain instead a map of real trees. Denote the associated real trees by $|\mathcal{T}_F|$; there is a natural inclusion

$$|\mathcal{T}_F| \longrightarrow |\mathcal{T}_E|.$$

Definition 2.7.4. A vertex (edge) in $\mathcal{T}_E$ is called an F-vertex (F-edge) if it lies in the image of real trees $|\mathcal{T}_F| \longrightarrow |\mathcal{T}_E|$. For any F-vertex $v \in \mathcal{T}_E^0$, denote by $\mathcal{P}(v, E/F)$ the set of infinite non-backtracking paths beginning at v and not passing through any F-edge. Let $U(v, E/F)$ denote the set of ends in $\mathcal{P}(v, E/F)$.

Definition 2.7.5. Define $C^{1,int}_{E,har}(\mathscr{P}_n, F)$ to be the subspace of $C^1_{E,har}$ of functions

$$\phi : \mathcal{T}_E^1 \longrightarrow \mathscr{P}_n$$

that satisfy the following conditions

- For every F-edge e, we have $\phi(e) \in \mathscr{P}_{n,e}$,
- for every F-vertex $v \in \mathcal{T}_E^0$, the restriction of the measure associated to ϕ to $U(v, E/F)$

is of the form $\mu_{\mathcal{D}_v}$ (Definition 2.6.2) for some divisor

$$\mathcal{D}_v \in \mathrm{Div}(U(v, E/F)) \otimes \mathscr{P}_n,$$

- for every F-vertex, $v \in \mathcal{T}_E^0$, if

$$\mathcal{D}_v = \sum_i (\tau_i) \otimes P_i,$$

then

$$P_i \in \bigcap_{t(e)=v} \mathscr{P}_{n,e}$$

for every F-edge $e \in \mathcal{T}_E^1$.

If the second condition is satisfied, and $\mathcal{D}_v = \sum_i (\tau_i) \otimes P_i$, write

$$\phi(\tau_i) := P_i.$$

Further, define

$$C^{1,b}_{E,har}(\mathscr{P}_n, F) := C^{1,int}_{E,har}(\mathscr{P}_n, F) \otimes_{\mathcal{O}_{\mathbb{C}_p}} \mathbb{C}_p.$$

Proposition 2.7.6. *There is a* $\mathrm{GL}_2(F)$*-equivariant section*

$$\mathrm{ST} : C^{1,b}_{E,har}(\mathscr{P}_n; F) \longrightarrow \mathcal{M}(k, \mathcal{H}_F, E).$$

The idea of the proof below is that the measure associated to a bounded harmonic cochain ϕ extends to a distribution pairing against locally analytic functions. We will show this by defining the distribution via Riemann sums, the boundedness guarantees estimates on the Riemann sums that guarantee their convergence.

Proof. Assume without loss of generality that $\phi \in C^{1,int}_{E,har}(\mathscr{P}_n, F)$. We will prove in fact that the trick of Vishik and Amice-Velu applies to extend the measure associated to ϕ to a distribution on $\mathbb{P}^1(E)$. We follow the proof presented in [MTT86]. Fix $G \in C(k, E)$.

Define for $\alpha \in E$ the disk

$$D(\alpha, \nu) = \{x \in E \mid |x - \alpha| \leq |\pi_E|^{\nu}\}$$

and

$$D(\infty, \nu) = \{x \in E \mid |x| \geq |\pi_E|^{-\nu}\}.$$

For each pair α and ν, write

$$h = -\min(0, \mathrm{ord}_{\pi_E}(\alpha)),$$

and

$$\nu' = \nu + h.$$

When $\nu' > 0$, the pair (α, ν) has an associated edge $e(\alpha, \nu)$, which is the edge that points towards $D(\alpha, \nu)$, i.e.

$$U(e(\alpha, \nu)) = D(\alpha, \nu).$$

Note that the distance of $e(\alpha, \nu)$ from the standard vertex in $\mathcal{T}_E$ is $h + \nu'$.

Lemma 2.7.7. *If the edge $e \in \mathcal{T}_E^1$ is an F-edge that points towards $D(\alpha, \nu)$, then*

$$\langle \phi(e), (t - \alpha)^i \rangle_{\mathscr{P}_n} \in \pi_E^{\nu(i-n/2)} \mathcal{O}_{\mathbb{C}_p}.$$

Let $v = t(e)$. If $\tau \in U(v, E/F)$, then

$$\langle \phi(\tau), (t - \tau)^i \rangle \in \pi_E^{\nu(i-n/2)} \mathcal{O}_{\mathbb{C}_p}.$$

Proof. Let $\gamma = \begin{pmatrix} \pi_E^{\nu} & \alpha \\ 0 & 1 \end{pmatrix}$, then $\gamma \cdot D(0,0) = D(\alpha, \nu)$. We compute that

$$(t - \alpha)^i | \gamma = \pi_E^{\nu(i-n/2)} t^i.$$

Since ϕ is integral, there exists a polynomial $P \in \mathscr{P}_n(\mathcal{O}_{\mathbb{C}_p})$ such that $\phi(e) = P(t)|\gamma^{-1}$.

So we have

$$\begin{aligned}&\langle\phi(e),(t-\alpha)^i\rangle_{\mathscr{P}_n}\\=&\langle P(t)|\gamma^{-1},\pi_E^{\nu(i-n/2)}t^i|\gamma^{-1}\rangle_{\mathscr{P}_n}\\=&\pi_E^{\nu(i-n/2)}\langle P(t),t^i\rangle_{\mathscr{P}_n}\in\pi_E^{\nu(i-n/2)}\mathcal{O}_{\mathbb{C}_p}.\end{aligned}$$

The proof for the second part proceeds in exactly the same way. □

- If G can be written as a convergent power series

  $$G(z)=\sum_{i\geq 0}c_{\alpha,i}(z-\alpha)^i$$

  on $D(\alpha,\nu)$, define the ideal

  $$I_{\alpha,\nu}=\pi_E^{-h}\pi_E^{-\nu'(n+1)}\sum_{i>n}c_{\alpha,i}\pi_E^{i\nu}\mathcal{O}_{\mathbb{C}_p}.$$

  In this case, write

  $$\mathrm{Trun}_\alpha(G)=\sum_{i=0}^{n}c_{\alpha,i}(z-\alpha)^i.$$

- and if G can be written as a convergent power series

  $$G(z)=\sum_{i\geq -n}c_{\infty,i}z^{-i}$$

  on $D(\infty,\nu)$, define the ideal

  $$I_{\infty,\nu}=\pi_E^{-\nu}\sum_{i>0}c_{\infty,i}\pi_E^{i\nu}\mathcal{O}_{\mathbb{C}_p}.$$

  In this case, write

  $$\mathrm{Trun}_\infty(G)=\sum_{i=-n}^{0}c_{\infty,z}z^{-i}.$$

The next few lemmas are estimates on the terms appearing in the Riemann sums.

Lemma 2.7.8. *If* $D(\alpha', \nu_1) \subset D(\alpha, \nu_2)$*, then* $I_{\alpha',\nu_1} \subset I_{\alpha,\nu_2}$.

Proof. For $\alpha, \alpha' \neq \infty$, the condition implies that $\nu_1 \geq \nu_2$. The result is clear for

$$D(\alpha, \nu_1) \subset D(\alpha, \nu_2)$$

and therefore it remains to prove it for the case $D(\alpha', \nu) = D(\alpha, \nu)$.

If we write

$$G(z) = \sum_{i \geq 0} c_{\alpha,i}(z-\alpha)^i$$

then by the binomial theorem we obtain

$$G(z) = \sum_{j\geq 0} c_{\alpha',j}(z-\alpha')^j = \sum_{j \geq 0}\left(\sum_{i \geq j}\binom{i}{j}(\alpha'-\alpha)^{i-j}c_{\alpha,i}\right)(z-\alpha')^j$$

Since $|\alpha' - \alpha| \leq \pi_E^{\nu}$, we get

$$\pi_E^{j\nu} c_{\alpha',j} \in \sum_{i \geq j} \pi_E^{i\nu} c_{\alpha,i}\mathcal{O}_{\mathbb{C}_p} = \pi_E^h \pi_E^{\nu'(n+1)} I_{\alpha,\nu}.$$

Now for the case of $D(\alpha, \nu_2) \subset D(\infty, \nu_1)$, it suffices to prove this statement for $D(\alpha, -\nu+1) \subset D(\infty, \nu)$.

Write

$$G(z) = \sum_{i \geq -n} c_{\infty,i} z^{-i}.$$

By the binomial theorem

$$\sum_{j \geq 0} c_{\alpha,j}(z-\alpha)^j = \sum_{j=0}^{n}\left(\sum_{i=j}^{n}\binom{i}{j}c_{\infty,-i}\alpha^{i-j}\right)(z-\alpha)^j + \sum_{j\geq 0}\left(\sum_{i>0}\binom{-i}{j}c_{\infty,i}\alpha^{-(i+j)}\right)(z-\alpha)^j$$

Therefore, for $j > n$ we get

$$c_{\alpha,j}\pi_E^{j(-\nu+1)} \in \sum_{i>0} \pi_E^{j(-\nu+1)}\pi_E^{(i+j)\nu} c_{\infty,i}\mathcal{O}_{\mathbb{C}_p}$$

$$= \pi_E^j \sum_{i>0} \pi_E^{i\nu} c_{\infty,i}\mathcal{O}_{\mathbb{C}_p} \subset \pi_E^{n+1} I_{\infty,\nu}.$$

□

Following from this lemma, by scaling G, assume without loss of generality that the ideals $I_{\alpha,\nu}$ and $I_{\infty,\nu}$ are all contained in $\mathcal{O}_{\mathbb{C}_p}$.

Lemma 2.7.9. *If $D(\alpha,\nu) = D(\alpha',\nu)$ and we write*

$$\mathrm{Trun}_\alpha(G) - \mathrm{Trun}_{\alpha'}(G) = \sum_{i=0}^{n} b_i(z-\alpha')^i$$

then

$$\pi_E^{i\nu} b_i \in \pi_E^h \pi_E^{(n+1)\nu'} I_{\alpha,\nu},$$

and further if e points towards $D(\alpha',\nu+r)$, then

$$\langle \phi(e), \mathrm{Trun}_\alpha(G) - \mathrm{Trun}_{\alpha'}(G)\rangle_{\mathscr{P}_n} \in \pi_E^{(h+\nu')(n/2+1)}\pi_E^{r(i-n/2)} I_{\alpha,\nu}.$$

If $\tau \in U(t(e), E/F)$, then

$$\langle \phi(\tau), \mathrm{Trun}_\alpha(G) - \mathrm{Trun}_{\tau}(G)\rangle_{\mathscr{P}_n} \in \pi_E^{(h+\nu')(n/2+1)}\pi_E^{r(i-n/2)} I_{\alpha,\nu}.$$

Proof. We have

$$\mathrm{Trun}_{\alpha'}(G) - \mathrm{Trun}_\alpha(G) = \sum_{i=0}^{n}\left(\sum_{j\geq n}\binom{j}{i} c_{\alpha,j}(\alpha'-\alpha)^{j-i}\right)(z-\alpha')^i$$

and so

$$\pi_E^{i\nu}\binom{j}{i} c_{\alpha,j}(\alpha'-\alpha)^{j-i} \in \pi_E^{j\nu} c_{\alpha,j}\mathcal{O}_{\mathbb{C}_p} \subset \pi_E^h \pi_E^{(n+1)\nu'} I_{\alpha,\nu}.$$

The first claim follows. Now consider

$$\langle \phi(e), b_i(z-\alpha')^i \rangle_{\mathscr{P}_n}.$$

By Lemma 2.7.7 this pairing lies in

$$\begin{aligned} b_i \pi_E^{\nu(i-n/2)} \mathcal{O}_{\mathbb{C}_p} &\subset \pi_E^h \pi_E^{(n+1)\nu'} \pi_E^{-i\nu} \pi_E^{\nu(i-n/2)} I_{\alpha,\nu} \\ &= \pi_E^h \pi_E^{(n+1)\nu'} \pi_E^{-\nu(n/2)} I_{\alpha,\nu} \end{aligned}$$

Recall that $\nu = \nu' - h$ to get

$$= \pi_E^{(h+\nu')(1+n/2)} I_{\alpha,\nu}.$$

The proof for $\langle \phi(\tau), \mathrm{Trun}_\tau(G) - \mathrm{Trun}_{\alpha'}(G) \rangle_{\mathscr{P}_n}$ is the same as above. $\square$

Lemma 2.7.10. *If $\alpha \in D(\infty, h)$ and*

$$\mathrm{Trun}_\alpha(G) - \mathrm{Trun}_\infty(G) = \sum_{i=0}^{n} b_i (z-\alpha)^i$$

then

$$b_i \in \pi_E^{h(i+1)} I_{\infty,h}$$

and further if e is the edge pointing towards $D(\alpha, \nu' - h)$, then

$$\langle \phi(e), \mathrm{Trun}_\alpha(G) - \mathrm{Trun}_\infty(G) \rangle_{\mathscr{P}_n} \in \pi_E^{h(1+n/2)} \pi_E^{\nu'(i-n/2)} I_{\infty,h}.$$

If $\tau \in U(t(e), E/F)$, then

$$\langle \phi(\tau), \mathrm{Trun}_\tau(G) - \mathrm{Trun}_\infty(G) \rangle_{\mathscr{P}_n} \in \pi_E^{h(1+n/2)} \pi_E^{\nu'(i-n/2)} I_{\infty,h}.$$

Proof. Again we can write

$$\mathrm{Trun}_\alpha(G) - \mathrm{Trun}_{\alpha'}(G) = \sum_{i=0}^{n}\left(\sum_{j>1}\binom{-j}{i}\alpha^{-(i+j)}c_{\infty,j}\right)(z-\alpha)^i$$

and compute

$$\binom{-j}{i}\alpha^{-(i+j)}c_{\infty,j} \in \pi_E^{h(i+j)}c_{\infty,j} \in \pi_E^{h(i+1)}I_{\infty,h}$$

The first claim follows, and now we compute

$$\langle \phi(e), b_i(z-\alpha)^i\rangle_{\mathscr{P}_n}$$

by Lemma 2.7.7, this lies in

$$\pi_E^{h(i+1)}\pi_E^{\nu(i-n/2)}I_{\infty,h} = \pi_E^{h(i+1)}\pi_E^{(\nu'-h)(i-n/2)}I_{\infty,h} = \pi^{h(1+n/2)}\pi_E^{\nu'(i-n/2)}I_{\infty,h}$$

□

For every l define the following two sets $\mathfrak{R}_l$ is a choice of representatives for $\mathbb{P}^1(F)$ modulo π_F^l. For every F-vertex of, the measure associated to ϕ is of the form $\mu_{\mathcal{D}_v}$ for some divisor $\mathcal{D}_v$ supported on $U(v, E/F)$. Let $\mathfrak{T}_l$ be the union of the supports of $\mathcal{D}_v$ for F-vertices v of distance at most $le_{E/F}$ steps away from the standard vertex. Then define the Riemann sum

$$\mathfrak{S}_l = \sum_{\alpha\in\mathfrak{R}_l}\langle\phi(e_\alpha), \mathrm{Trun}_\alpha(G)\rangle_{\mathscr{P}_n} + \sum_{\tau\in\mathfrak{T}_l}\langle\phi(\tau), \mathrm{Trun}_\tau(G)\rangle_{\mathscr{P}_n}.$$

Lemma 2.7.11. *The sum $\mathfrak{S}_l$ is well-defined, independent of the choice of $\mathfrak{R}_l$, modulo*

$$\pi_F^{l(1+n/2)}\sum_{\alpha\in\mathfrak{R}_l} I_{\alpha,\nu}.$$

In the sum above, for each $\alpha\in\mathfrak{R}_l$, its associated ν is defined such that $l = \nu' + h$.

Proof. Given a different choice $\mathfrak{R}'_l$ defining a Riemann sum $\mathfrak{S}'$, the difference $\mathfrak{S} - \mathfrak{S}'$ will involve a sum of the form

$$\langle \phi(e_{\alpha,l}), \mathrm{Trun}_{\alpha}(G) - \mathrm{Trun}_{\alpha'}(G) \rangle_{\mathscr{P}_n},$$

where $D(\alpha, \nu) = D(\alpha', \nu)$ and where $l = \nu' + h$. The lemma follows from Lemma 2.7.9. ☐

Lemma 2.7.12. *For any choice of $\mathfrak{R}_l$ and $\mathfrak{R}_{l+1}$ we have that*

$$|\mathfrak{S}_{l+1} - \mathfrak{S}_l| \in \pi_F^{l(1+n/2)-n/2} \sum_{\alpha \in \mathfrak{R}_l} I_{\alpha,\nu}.$$

In the sum above, for each $\alpha \in \mathfrak{R}_l$, its associated ν is defined such that $l = \nu' + h$.

Proof. The summand $\langle \phi(e_{\alpha,l}), \mathrm{Trun}_{\alpha}(G) \rangle_{\mathscr{P}_n}$ gets "refined" into a sum

$$\sum_{\substack{\alpha' \in \mathfrak{R}_{l+1} \\ \alpha' \equiv \alpha (\mathrm{mod} \pi_F^l)}} \langle \phi(e_{\alpha',l+1}), \mathrm{Trun}_{\alpha'}(G) \rangle_{\mathscr{P}_n} + \sum_{\substack{\tau \in \mathfrak{T}_{l+1} \setminus \mathfrak{T}_l \\ \tau \equiv \alpha (\mathrm{mod} \pi_F^l)}} \langle \phi(\tau), \mathrm{Trun}_{\tau}(G) \rangle_{\mathscr{P}_n} \tag{2.10}$$

Since ϕ is harmonic,

$$\phi(e_{\alpha,l}) = \sum_{\substack{\alpha' \in \mathfrak{R}_{l+1} \\ \alpha' \equiv \alpha (\mathrm{mod} \pi_F^l)}} \phi(e_{\alpha',l+1}) + \sum_{\substack{\tau \in \mathfrak{T}_{l+1} \setminus \mathfrak{T}_l \\ \tau \equiv \alpha (\mathrm{mod} \pi_F^l)}} \phi(\tau).$$

Therefore, the difference between $\langle \phi(e_{\alpha}), \mathrm{Trun}_{\alpha}(G) \rangle_{\mathscr{P}_n}$ and (2.10) can be expressed as a sum of elements of the form

$$\langle \phi(e_{\alpha',l+1}), \mathrm{Trun}_{\alpha'}(G) - \mathrm{Trun}_{\alpha}(G) \rangle_{\mathscr{P}_n}$$

or

$$\langle \phi(\tau), \mathrm{Trun}_{\tau}(G) - \mathrm{Trun}_{\alpha}(G) \rangle_{\mathscr{P}_n}.$$

The bound then follows from Lemmas 2.7.9 and 2.7.10. ☐

We have therefore shown that for any locally analytic function $G \in C(k, E)$, the Riemann sums $\mathfrak{S}_l$ converge; denote their limit by

$$\int_{\mathbb{P}^1(E)} G(t)\ d\mu(t).$$

Similarly, for any open set $U \in \mathbb{P}^1(E)$, write

$$\int_U G(t)\ d\mu(t) := \int_{\mathbb{P}^1(E)} G(t)\mathbb{1}_U(t)\ d\mu(t).$$

Lemma 2.7.13. *For every $i \geq 0$, if e is the edge F-edge pointing towards $D(\alpha, \nu)$, then we have*

$$\int_{D(\alpha,\nu)} (z-\alpha)^i\ d\mu \in \pi_F^{-n/2}\pi_F^{\nu(i-n/2)}\mathcal{O}_{\mathbb{C}_p}$$

Proof. From Lemma 2.7.11 and truncating at α, the first Riemann sum is equal to 0 modulo $\pi_F^{\nu(i-n/2)}$, and so the estimate follows from Lemma 2.7.12. □

The proof of Lemma 2.6.3 shows that if the restriction of μ to $D(\alpha, \nu)$ is discrete of the form $\sum P_i\delta_{\tau_i}$, then

$$\int_{D(\alpha,\nu)} \frac{1}{t-z}\ d\mu(t) = \frac{1}{n!}d^{n+1}\left(P_i(z)\log(z-\tau_i)\right).$$

For any level l, we calculate

$$\int_{\mathbb{P}^1(F)} \frac{1}{t-z}\ d\mu(t) = \sum_{\alpha\in\mathfrak{R}_l}\int_{D(\alpha,\nu)} \frac{1}{t-z}\ d\mu(t) + \sum_{\tau\in\mathfrak{T}_l}\frac{1}{n!}d^{n+1}\left(\phi(\tau_i)\log(z-\tau)\right), \quad (2.11)$$

where ν is chosen such the pair (α, ν) satisfies $\nu' + h = l$.

For $\alpha \neq \infty$, we have

$$\int_{D(\alpha,\nu)} \frac{1}{t-z}\ d\mu(t) = -\sum_{i=0}^{\infty}\frac{\int_{D(\alpha,\nu)}(t-\alpha)^i\ d\mu(t)}{(z-\alpha)^{i+1}}.$$

and
$$\int_{D(\infty,\nu)} \frac{1}{t-z}\, d\mu(t) = \sum_{i=0}^{\infty} \left(\int_{D(\infty,\nu)} t^{-(i+1)}\, d\mu(t) \right) z^i$$

This infinite sum converges on $\mathcal{H}_F^{<l}$ by Lemma 2.7.13 and therefore the expression in (2.11) is a rigid meromorphic function on $\mathcal{H}_F^{<l}$.

□

Definition 2.7.14. Let $\mathcal{M}(k, \mathcal{H}_F, E)^b$ denote the image of

$$\mathrm{ST} : C^{1,b}_{E,har}(\mathscr{P}_n; F) \longrightarrow \mathcal{M}(k, \mathcal{H}_F, E).$$

Definition 2.7.15. Denote by $\mathfrak{Div}_n^\dagger(E\backslash F)$ the set of formal (possibly infinite) sums of points in $E\backslash F$ of the form
$$D = \sum_{i\in I} (z_i) P_i(t),$$

with $P_i(t) \in \mathscr{P}_n$ and $z_i \in E\backslash F$, such that the support of D intersects each affinoid $\mathcal{H}_F^{\leq l}$ in a finite set.

Given a function $f \in \mathcal{M}(k, \mathcal{H}_F, E)$ we define its residual divisor in $\mathfrak{Div}_n^\dagger(E\backslash F)$ by

$$\mathrm{rdiv}_k(f) := \sum_{w\in\mathcal{H}_F} (w)\mathrm{res}_w(f(z)(T-z)^n\, dz).$$

This defines a $\mathrm{GL}_2(F)$-equivariant map

$$\mathrm{rdiv}_k : \mathcal{M}(k, \mathcal{H}_F, E) \longrightarrow \mathfrak{Div}_n^\dagger(E/F).$$

Lemma 2.7.16. *If $\phi \in C^{1,b}_{E,har}(\mathscr{P}_n, F)$, then*

$$\mathrm{rdiv}_k(\mathrm{ST}(\phi)) = \sum_{v \text{ an } F\text{-vertex}} \mathcal{D}_v,$$

where $\mathcal{D}_v$ is as in Definition 2.7.5.

Proof. □

2.8 The Ihara group

For the remainder of this thesis, let p be an odd prime. Now consider the case where $F = \mathbb{Q}_p$ and $E = K :=$ a biquadratic extension of $\mathbb{Q}_p$ (unique up to isomorphism). To simplify notation, write $\mathcal{H}_p$ for $\mathcal{H}_{\mathbb{Q}_p}$, $\mathcal{T}_p$ for $\mathcal{T}_{\mathbb{Q}_p}$ and $C^*_{p,har}$ and C^*_p for $C^*_{\mathbb{Q}_p,har}$ and $C^*_{\mathbb{Q}_p}$. We note that the methods of this thesis apply also for $p = 2$, however one would have to work with a degree 8 extension.

Let $\Gamma := \mathrm{SL}_2(\mathbb{Z}[1/p])$ be the Ihara group. The action of Γ on vertices of $\mathcal{T}_p$ has two orbits; two vertices are in the same orbit if and only if the distance between them is even. The stabilizer of the vertex v_0 is $\mathrm{SL}_2(\mathbb{Z})$ and the stabilizer of any other vertex is a conjugate of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{GL}_2(\mathbb{Z}[1/p])$. In particular, the stabilizer of $v_1 = P^{-1}v_0$ is

$$P^{-1} \cdot \mathrm{SL}_2(\mathbb{Z}) \cdot P,$$

where

$$P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}.$$

The action of Γ on non-oriented edges is transitive, and the stabilizer of the standard edge is

$$\Gamma_0(p) = P^{-1}\mathrm{SL}_2(\mathbb{Z})P \cap \mathrm{SL}_2(\mathbb{Z}).$$

The action preserves the orientation of edges.

There is a natural inclusion of $\mathbb{R}$-trees

$$i : |\mathcal{T}_p| \hookrightarrow |\mathcal{T}_K|,$$

which is given by base-changing $\mathbb{Z}_p$ lattices to $\mathcal{O}_K$ lattices. This is not an inclusion of combinatorial trees since every edge is "split" into two edges. For example, the p-neighboring lattices $\mathbb{Z}_p^2$ and $\mathbb{Z}_p \oplus p\mathbb{Z}_p$ do not base change to π_K-neighboring lattices. Instead, their bases changes, $\mathcal{O}_K^2$ and $\mathcal{O}_K \oplus p\mathcal{O}_K$, share the π_K-neighbor $\mathcal{O}_K \oplus \pi_K\mathcal{O}_K$.

Definition 2.8.1. Write $\mathcal{T}_{K,\mathbb{Q}_p}$ for the subtree of $\mathcal{T}_K$ consisting of $\mathbb{Q}_p$-vertices and $\mathbb{Q}_p$-edges. Similarly, write $C^*_{K,\mathbb{Q}_p}$ for functions on vertices and edges of $\mathcal{T}_{K,\mathbb{Q}_p}$.

There are therefore three Γ-orbits of $\mathbb{Q}_p$-vertices in $\mathcal{T}_{K,\mathbb{Q}_p}$:

$$\Gamma \cdot v_0,\ \Gamma \cdot v_1 \text{ and } \Gamma \cdot v_K := [\mathcal{O}_K \oplus \pi_K \mathcal{O}_K]\,.$$

Their stabilizers in Γ are

$$\mathrm{SL}_2(\mathbb{Z}),\ P^{-1}\mathrm{SL}_2(\mathbb{Z})P \text{ and } \Gamma_0(p),$$

respectively. There are also two Γ-orbits of non-oriented $\mathbb{Q}_p$-edges in $\mathcal{T}_{K,\mathbb{Q}_p}$:

$$\Gamma \cdot (v_0, v_K) \text{ and } \Gamma \cdot (v_K, v_1),$$

whose stabilizers in Γ are both $\Gamma_0(p)$.

3

Higher Weight Rigid Cocycles

The aim of this chapter is to define and classify rigid meromorphic cocycles in higher weight. We will begin with a brief introduction to group homology and cohomology in order to fix notation, and introduce modular symbols and their relation to group cohomology. In 3.2 we will recall the connection between group cohomology valued in rigid analytic cocycles and modular forms on congruence subgroups. In 3.3 we will define certain modular symbols associated to real quadratic points on $\mathcal{H}_p$ that will later be used to construct rigid meromorphic cocycles. In 3.4 we will recast Darmon–Vonk's construction of rigid analytic cocycles on $\mathcal{H}_p$ as rigid analytic cocycles on $\mathcal{H}_K$. Finally, in 3.5 and 3.6 we will provide definitions and the classification of higher weight rigid meromorphic cocycles. We recall that p was set to be a fixed odd prime.

3.1 (Co)homology and Modular Symbols

The definitions and results of this section on homology and cohomology can be found in [Bro94].

Definition 3.1.1 (Group Homology)**.** For any group G and right G-module, M, the group $Z_1(G,M)$ of 1-cycles in M is the subgroup of $\mathbb{Z}[G] \otimes M$ of elements

$$\sum_i g_i \otimes m_i$$

that satisfy

$$\sum_i m_i \cdot g_i^{-1} = \sum_i m_i.$$

The homology group $H_1(G,M)$ is the quotient of $Z_1(G,M)$ by the subgroup, $B_1(G,M)$, generated by elements of the form

$$g_1 g_2 \otimes m - g_1 \otimes m \cdot g_2^{-1} - g_2 \otimes m,$$

for any $g_1, g_2 \in G$ and $m \in M$.

The group $Z_2(G,M)$ of 2-cycles in M is the subgroup of $\mathbb{Z}[G] \otimes \mathbb{Z}[G] \otimes M$ of elements

$$\sum_i g_i \otimes h_i \otimes m_i,$$

that satisfy

$$\sum_i g_i h_i \otimes m = \sum_i g_i \otimes m \cdot h_i^{-1} + \sum_i h_i \otimes m.$$

The homology group $H_2(G,M)$ is the quotient of $Z_2(G,M)$ by the subgroup, $B_2(G,M)$, generated by elements of the form

$$g_1 \otimes g_2 \otimes m \cdot g_3^{-1} + g_1 g_2 \otimes g_3 \otimes m - g_2 \otimes g_3 \otimes m - g_1 \otimes g_2 g_3 \otimes m,$$

for any $g_1, g_2, g_3 \in G$ and $m \in M$.

Definition 3.1.2 (Group Cohomology)**.** For any group G and right G-module M, a 1-cocycle valued in M is an element of the group, $Z^1(G,M)$, of functions $J : G \longrightarrow M$ that satisfy

$$J(g_1 g_2) = J(g_1) \cdot g_2 + J(g_2), \quad \text{for all } g_1, g_2 \in G.$$

The cohomology group $H^1(G, M)$ is the quotient of $Z^1(G, M)$ by the subgroup $B^1(G, M)$ generated by co-boundaries; those are functions of the form

$$g \mapsto f \cdot g - f,$$

for some fixed $f \in M$.

A 2-cocycle valued in M is an element of the group $Z^2(G, M)$ of functions

$$\varphi : G \times G \longrightarrow M$$

that satisfy

$$\varphi(g_1, g_2) \cdot g_3 + \varphi(g_1 g_2, g_3) = \varphi(g_2, g_3) + \varphi(g_1, g_2 g_3), \text{ for all } g_1, g_2, g_3 \in G.$$

The cohomology group $H^2(G, M)$ is the quotient of $Z^2(G, M)$ by the subgroup $B^2(G, M)$ generated by co-boundaries of the form

$$\varphi(g_1, g_2) = F(g_1 g_2) - F(g_1) \cdot g_2 - F(g_2),$$

for $F : G \longrightarrow M$.

Remark 3.1.3. The homology and cohomology groups are more generally defined as the derived functors of the functors of invariants and co-invariants, respectively. This also leads to their associated long exact sequence. We include the definitions above for concreteness because we will be concerned with specific elements in first and second homology and cohomology groups. We will also use the definition of group homology and cohomology as derived functors. See [Bro94, III.1] for a comparison between the explicit definition described here and the more abstract ones.

Let F be any field, and assume now that M and N are right $F[G]$-modules with a perfect bilinear G-equivariant pairing

$$\langle\ ,\rangle : M \times N \longrightarrow F.$$

Observe that from the definitions above, there are natural pairings

$$H_1(G, M) \times H^1(G, N) \longrightarrow F,$$

given on a choice of representatives by

$$\left(\sum_i g_i \otimes m_i \ , J\right) \mapsto \sum_i \langle m_i, J(g_i)\rangle,$$

and similarly

$$H_2(G, M) \times H^2(G, N) \longrightarrow F,$$

given on a choice of representatives by

$$\left(\sum_i g_i \otimes h_i \otimes m_i, \varphi\right) \mapsto \sum_i \langle m_i, \varphi(g_i, h_i)\rangle.$$

Proposition 3.1.4 (Universal Coefficient Theorem)**.** *For M, N as above, the pairings between $H_i(G, M)$ and $H^i(G, N)$ are perfect pairings.*

Proof. We will follow the proof of Proposition 7.1 in Chapter VI of [Bro94]. Using the pairing on M and N, identify M with $N^\vee := \mathrm{Hom}_F(N, F)$. Take $R^\bullet$ to be a projective resolution of F in the category of $F[G]$-modules. Note that the cohomology groups $H^i(G, M)$ are computed by the homology of the complex

$$\mathrm{Hom}_G(R^\bullet, M),$$

and the homology groups are computed by the homology of the complex

$$R^\bullet \otimes_{F[G]} M.$$

So then we compute that

$$\begin{aligned}\mathrm{Hom}_G(R^\bullet, M^\vee) &= \mathrm{Hom}_G(R^\bullet, \mathrm{Hom}_F(M, F)) = \mathrm{Hom}_G(R^\bullet \otimes_F M, F) \\ &= \mathrm{Hom}_F(R^\bullet \otimes_G M, F) = (R^\bullet \otimes_G M)^\vee .\end{aligned}$$

Since the functor $(\cdot)^\vee$ is exact, by taking cohomology we obtain that

$$H^i(G, N) \cong H^i(G, M^\vee) \cong H_i(G, M)^\vee.$$

□

Definition 3.1.5 (Modular Symbols)**.** Let $\mathbb{D}_0$ be the group of degree-zero divisors in $\mathbb{P}^1(\mathbb{Q})$, which is defined by the exact sequence

$$0 \longrightarrow \mathbb{D}_0 \longrightarrow \mathbb{Z}[\mathbb{P}^1(\mathbb{Q})] \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0. \tag{3.1}$$

Let G be any group which acts on $\mathbb{P}^1(\mathbb{Q})$. For any G-module, M, an M-valued *modular symbol* is a homomorphism of abelian groups

$$\mathbb{D}_0 \longrightarrow M.$$

Denote by $\mathrm{MS}(M)$ the group of M-valued modular symbols. As r, s range over $\mathbb{P}^1(\mathbb{Q})$, divisors of the form $(r) - (s)$ span the group $\mathbb{D}_0$. Denote such a divisor by $\{r, s\}$. The right action of G on $\mathrm{MS}(M)$ is described by

$$(m \cdot \gamma)\{r, s\} = m\{\gamma r, \gamma s\} \cdot \gamma.$$

Let $\mathrm{MS}^G(M)$ be the subgroup of $\mathrm{MS}(M)$ fixed by G.

The short exact sequence (3.1) yields the exact sequence of G-modules

$$0 \longrightarrow M \cong \mathrm{Hom}(\mathbb{Z}, M) \longrightarrow \mathrm{Hom}\left(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], M\right) \longrightarrow \mathrm{Hom}(\mathbb{D}_0, M) \longrightarrow 0. \tag{3.2}$$

By taking G-cohomology of (3.2), the first δ-function is a map

$$\delta : (\mathrm{Hom}(\mathbb{D}_0, M))^G \longrightarrow H^1(G, M).$$

This map can be described as follows: Chose an arbitrary $r \in \mathbb{P}^1(\mathbb{Q})$. Given a G-invariant modular symbol m, define

$$J_m(\gamma) := m\{r, \gamma r\}, \quad \text{for all } \gamma \in G.$$

The cohomology class of J_m is independent of the choice of r and is equal to $\delta(m)$.

Let M be an $\mathrm{SL}_2(\mathbb{Z})$-module. Let $M^\dagger$ denote the set of elements $h \in M$ that satisfy the two-term relation

$$h \cdot S + h = 0,$$

and the three-term relation

$$h \cdot U^2 + h \cdot U + h = 0,$$

where $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and $U = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$.

Proposition 3.1.6. *For any* $\mathrm{SL}_2(\mathbb{Z})$*-module* M*, the map*

$$m \mapsto m\{0, \infty\}$$

is a bijection between $\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(M)$ *and* $M^\dagger$*.*

Proof. A proof of this statement can be found in [DV21] Proposition 1.4. This statement holds because $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on the set of pairs $\{\frac{a}{c}, \frac{b}{d}\} \in \mathbb{D}_0$ such that $ad - bc = 1$ (with the convention $\infty = \frac{1}{0}$), and the set of all such pairs generates $\mathbb{D}_0$ (by the Euclidean algorithm). □

3.2 Rigid Analytic Cocycles

We will now consider modular symbols and group cohomology valued in spaces of rigid analytic functions. As before, Γ denotes the group $\mathrm{SL}_2(\mathbb{Z}[1/p])$. It is crucial here that the rigid analytic functions which arise as values of cocycles or modular symbols are necessarily bounded (Proposition 3.2.1). This is due to the Γ-invariance condition on cocycles and modular symbols, along with the fact that Γ is finitely generated. These conditions imply that the data of specifying a cocycle amounts to the information at a fundamental domain for the action of Γ and is therefore bounded.

Proposition 3.2.1. *The homomorphisms induced by the natural inclusion*

$$H^1(\Gamma, C^{1,b}_{har}(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$$

and

$$MS^\Gamma(C^{1,b}_{har}(\mathscr{P}_n)) \longrightarrow \mathrm{MS}^\Gamma(C^1_{har}(\mathscr{P}_n))$$

are isomorphisms.

Proof. Both of these statements follow from [RS12, Proposition 3.7]. □

Corollary 3.2.2. *The residue map induces isomorphisms*

$$H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b) \longrightarrow H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$$

and

$$MS^\Gamma(\mathcal{O}(k, \mathcal{H}_p)^b) \longrightarrow MS^\Gamma(C^1_{har}(\mathscr{P}_n)).$$

Corollary 3.2.2 reduces the construction of rigid analytic modular symbols and cohomology classes to constructing harmonic cochains on the Bruhat–Tits tree, where it is possible to exploit the relatively simple combinatorics of the tree and the action of Γ on it.

After taking Γ-cohomology, the short exact sequence (2.7) yields

$$H^0(\Gamma, C^0_p(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_p(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_p(\mathscr{P}_n))$$

Since the action of Γ on non-oriented edges is transitive and the stabilizer of an edge is conjugate to $\Gamma_0(p)$, there is an isomorphism

$$C^1_p(\mathscr{P}_n) \cong \mathrm{Ind}^{\Gamma}_{\Gamma_0(p)} \mathscr{P}_n.$$

Therefore, Shapiro's lemma provides us with an isomorphism

$$H^1(\Gamma, C_p^1(\mathscr{P}_n)) \cong H^1(\Gamma_0(p), \mathscr{P}_n). \tag{3.3}$$

Similarly, Γ has two orbits of vertices with stabilizers conjugate to $\mathrm{SL}_2(\mathbb{Z})$, so

$$H^1(\Gamma, C_p^0(\mathscr{P}_n)) \cong H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)^2. \tag{3.4}$$

Notice that for $k > 2$ the group $H^0(\Gamma, C_p^0(\mathscr{P}_n))$ is trivial. It follows from (3.3) and (3.4) that we can identify

$$H^1(\Gamma, C_{har}^1(\mathscr{P}_n)) \cong \mathrm{Ker}\left[H^1(\Gamma_0(p), \mathscr{P}_n) \longrightarrow H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)^2\right], \tag{3.5}$$

where the maps $H^1(\Gamma_0(p), \mathscr{P}_n) \to H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)$ come from the two trace maps of modular curves

$$X_0(p) \longrightarrow X_0(1).$$

Theorem 3.2.3 (Shimura, Eichler–Shimura)**.** *For all $k \geq 2$,*

(1) Let f be a p-new normalized eigenform of weight k for $\Gamma_0(p)$, then there exists transcendental periods $\Omega_f^{\pm}$ such that for every $\gamma \in \Gamma_0(p)$, the cocycle

$$p_f^{\pm}(\gamma) := \frac{1}{\Omega_f^{\pm}} \int_{\eta}^{\gamma\eta} \frac{\omega_f \pm \overline{\omega}_f}{2}, \quad \omega_f = f(z)(T-z)^n \, dz$$

takes values in polynomials whose coefficients lie in a certain finite extension of $\mathbb{Q}$, where η is a fixed choice of base-point and ω_f is the differential associated to f on the modular curve $X_0(p)$,

(2) $p_f^{\pm} \in Z^1(\Gamma_0(p), \mathscr{P}_n)$ is a cocycle whose cohomology class lies in

$$\mathrm{Ker}\left(H^1(\Gamma_0(p), \mathscr{P}_n) \longrightarrow H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)^2\right),$$

(3) If $k > 2$, as f ranges over all p-new normalized eigenforms, the cocycles $p_f^{\pm}$ form a

basis of

$$\mathrm{Ker}\left(H^1(\Gamma_0(p), \mathscr{P}_n) \longrightarrow H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)^2\right).$$

Proof. It is well-known (e.g. see [AS86] Theorem 2.3) that for any N, there is an isomorphism of Hecke-modules

$$H^1(\Gamma_0(N), \mathscr{P}_n) \cong S_k(\Gamma_0(N)) \oplus \overline{S_k(\Gamma_0(N))} \oplus \mathcal{E}_k(\Gamma_0(N)),$$

where $\mathcal{E}_k(\Gamma_0(N))$ is the space of Eisenstein series for the congruence subgroup $\Gamma_0(N)$. (2) follows from the condition that f is p-new. (3) holds because the Eisenstein series of $\Gamma_0(p)$ are not in the kernel of the two trace maps

$$M_k(\Gamma_0(p)) \longrightarrow M_k(\mathrm{SL}_2(\mathbb{Z})).$$

□

Corollary 3.2.4. *For $k > 2$, the map*

$$\mathrm{MS}^\Gamma(\mathcal{O}(k, \mathcal{H}_p)^b) \longrightarrow H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b)$$

is an isomorphism.

Proof. As Hecke-modules both modules are isomorphic to $S_k(\Gamma_0(p))^{p-\mathrm{new}}$, and the map is Hecke-equivariant and the modular symbols associated to an eigenform are sent to its associated cocycles. The claim

$$H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b) \cong S_k(\Gamma_0(p)^{p-\mathrm{new}})$$

follows from Theorem 3.2.3 combined with the isomorphism (3.5). For modular symbols, we similarly have the isomorphisms

$$\mathrm{MS}^\Gamma(\mathcal{O}(k, \mathcal{H}_p)^b) \cong \mathrm{MS}^\Gamma(C^1_{p,har}(\mathscr{P}_n)) \cong \mathrm{Ker}\left[\mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \longrightarrow \left(\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n)\right)^2\right].$$

It is well-known that (e.g. see [BD15])

$$\mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \cong \mathrm{BMS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus S_k(\Gamma_0(p))$$

and

$$\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \cong \mathrm{BMS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \oplus S_k(\mathrm{SL}_2(\mathbb{Z})).$$

The co-restriction maps yield an isomorphism

$$\mathrm{BMS}^{\Gamma_0(p)}(\mathscr{P}_n) \longrightarrow \left(\mathrm{BMS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n)\right)^2,$$

and the kernel of

$$S_k(\Gamma_0(p)) \longrightarrow S_k(\mathrm{SL}_2(\mathbb{Z}))^2$$

is precisely the p-new part, $S_k(\Gamma_0(p))^{p-\mathrm{new}}$. □

Lemma 3.2.5. *If Γ_0 is any congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then*

1. $H^2(\Gamma_0, \mathscr{P}_n) = 0$,
2. $H^1(\Gamma_0, \mathrm{MS}(\mathscr{P}_n)) = 0$.

Proof. 1. follows from [Hid93] p.162 Proposition 1.

2. follows from the same proposition after identifying $H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n))$ with Hida's $H^2_P(\Gamma, \mathscr{P}_n)$ for example by combining [AS86] Proposition 4.2 and p. 363 Proposition 5 of [Hid93].

□

Lemma 3.2.6. *We have*

1. $H^1(\Gamma, \mathscr{P}_n) = 0$,
2. $\mathrm{MS}^{\Gamma}(\mathscr{P}_n) = 0$.

Proof. 1. is precisely [RS12, Lemma 3.10].

2. can be proven with the same method as *loc. cit.* or as a consequence as follows: the long exact sequence in Γ-cohomology from (3.2), with $M = \mathscr{P}_n$, yields

$$\mathscr{P}_n^\Gamma \longrightarrow \mathrm{Hom}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})], \mathscr{P}_n)^\Gamma \longrightarrow \mathrm{MS}^\Gamma(\mathscr{P}_n) \longrightarrow H^1(\Gamma, \mathscr{P}_n).$$

Note that $\mathscr{P}_n^\Gamma = 0$. Further, since Γ acts transitively on $\mathbb{P}^1(\mathbb{Q})$, then

$$\mathrm{Hom}(\mathbb{Z}[\mathbb{P}^1(\mathbb{Q})])^\Gamma \cong \mathscr{P}_n^{\Gamma_\infty}.$$

To complete the proof, it suffices to show that $\mathscr{P}_n^{\Gamma_\infty} = 0$.

If

$$P(T) = \sum_{i=0}^n a_i T^i \in \mathscr{P}_n^{\Gamma_\infty},$$

then for $b \in \mathbb{Z}$,

$$P(T) = P| \begin{pmatrix} p^b & 0 \\ 0 & p^{-b} \end{pmatrix} (T) = \sum_{i=0}^n a_i p^{2bi-n} T^i.$$

This implies that $P(T)$ is a scalar multiple of $T^{n/2}$, but this polynomial is not fixed by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

□

The map

$$\Delta : C^0(\mathscr{P}_n) \longrightarrow C^1(\mathscr{P}_n)$$

given by

$$f \mapsto (e = (v_1, v_2) \mapsto f(v_1) - f(v_2))$$

yields a short exact sequence

$$0 \to \mathscr{P}_n \to C^0(\mathscr{P}_n) \to C^1(\mathscr{P}_n) \to 0. \tag{3.6}$$

Let $C^0_{p,har}(\mathscr{P}_n)$ denote the kernel of

$$\nabla \circ \Delta : C^0_p(\mathscr{P}_n) \longrightarrow C^0_p(\mathscr{P}_n).$$

Then the map

$$\Delta : C^0_{p,har}(\mathscr{P}_n) \longrightarrow C^1_{p,har}(\mathscr{P}_n)$$

is surjective and fits in a short exact sequence

$$0 \longrightarrow \mathscr{P}_n \longrightarrow C^0_{p,har}(\mathscr{P}_n) \longrightarrow C^1_{p,har}(\mathscr{P}_n) \longrightarrow 0.$$

The δ-maps in Γ-cohomology produce maps

$$H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n)) \longrightarrow H^2(\Gamma, \mathscr{P}_n)$$

and

$$H^0(\Gamma, \mathrm{MS}(C^1_{p,har}(\mathscr{P}_n))) \longrightarrow H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)).$$

Lemma 3.2.7. *For $k > 2$, the maps*

$$H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b) \longrightarrow H^2(\Gamma, \mathscr{P}_n)$$

and

$$H^0(\Gamma, \mathrm{MS}(\mathcal{O}(k, \mathcal{H}_p)^b)) \longrightarrow H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n))$$

are isomorphisms.

Proof. The associated long exact sequence in Γ-cohomology to (3.6) is

$$H^1(\Gamma, C^0(\mathscr{P}_n)) \to H^1(\Gamma, C^1(\mathscr{P}_n)) \to H^2(\Gamma, \mathscr{P}_n) \to H^2(\Gamma, C^0(\mathscr{P}_n)). \tag{3.7}$$

Similarly, there is an associated long exact sequence with modular symbols given by

$$\mathrm{MS}^\Gamma(C^0_p(\mathscr{P}_n)) \longrightarrow \mathrm{MS}^\Gamma(C^1_p(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, \mathrm{MS}(C^0(\mathscr{P}_n))). \tag{3.8}$$

Shapiro's lemma and Lemma 3.2.5 imply that $H^2(\Gamma, C^0(\mathscr{P}_n))$ is trivial. Further, the inclusion of $H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$ into $H^1(\Gamma, C^1(\mathscr{P}_n))$ is complementary to the image of $H^1(\Gamma, C^0(\mathscr{P}_n))$, and therefore $H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$ is isomorphic to $H^2(\Gamma, \mathscr{P}_n)$.

To see that $H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$ is complementary to the image of Δ, denote by $C^0_{har}(\mathscr{P}_n)$ the kernel of $\nabla \circ \Delta$. The map $\nabla \circ \Delta$ defines a short exact sequence

$$0 \longrightarrow C^0_{har}(\mathscr{P}_n) \longrightarrow C^0(\mathscr{P}_n) \longrightarrow C^0(\mathscr{P}_n) \longrightarrow 0.$$

The map Δ produces an exact sequence

$$0 \longrightarrow \mathscr{P}_n \longrightarrow C^0_{har}(\mathscr{P}_n) \longrightarrow C^1_{har}(\mathscr{P}_n) \longrightarrow 0.$$

For which the associated long exact sequence is

$$0 \longrightarrow H^1(\Gamma, \mathscr{P}_n) \longrightarrow H^1(\Gamma, C^0_{har}(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_{har}(\mathscr{P}_n)) \longrightarrow \cdots$$

But the map

$$H^1(\Gamma, C^0_{har}(\mathscr{P}_n)) \longrightarrow H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$$

is Hecke-equivariant, and the target is spanned by cocycles associated to p-new cusp forms, while the domain embeds into

$$H^1(\Gamma, C^0(\mathscr{P}_n)) \cong H^1(\mathrm{SL}_2(\mathbb{Z}), \mathscr{P}_n)^2$$

which as a Hecke-module is spanned by the image of oldforms and Eisenstein series, therefore this map must be zero. This implies that $H^1(\Gamma, C^0_{har}(\mathscr{P}_n))$ is equal to the image of $H^1(\Gamma, \mathscr{P}_n)$. That is, every cohomology class in $H^1(\Gamma, C^0_{har}(\mathscr{P}_n))$ has a representative cocycle which takes values in constant functions on the tree. These cocycles all vanish under the map Δ. So

$$\mathrm{Ker}(\Delta) = \mathrm{Ker}(\nabla \circ \Delta) = H^1(\Gamma, C^0_{har}(\mathscr{P}_n)) = H^1(\Gamma, \mathscr{P}_n) = 0.$$

The last equality is Lemma 3.2.6. This implies that $H^1(\Gamma, C^1_{har}(\mathscr{P}_n))$ and the image of

Δ are disjoint. Now, since $H^1(\Gamma, C^0_{p,har}) = 0$, the map $\nabla \circ \Delta$ is an injective linear map of finite dimensional vector spaces and so is an isomorphism. This implies that ∇ is surjective, and hence

$$H^1(\Gamma, C^1(\mathscr{P}_n)) = H^1(\Gamma, C^1_{har}(\mathscr{P}_n)) \oplus \operatorname{im}(\nabla).$$

The same argument proves the second assertion for modular symbols.

The argument can be summarized by the following diagram

$$\begin{tikzcd}
 & H^2(\Gamma, \mathscr{P}_n) & \\
H^1(\Gamma, C^1_{har}(\mathscr{P}_n)) \arrow[r] \arrow[ur] & H^1(\Gamma, C^1(\mathscr{P}_n)) \arrow[r, "\nabla"] \arrow[u] & H^1(\Gamma, C^0(\mathscr{P}_n)) \\
 & H^1(\Gamma, C^0(\mathscr{P}_n)) \arrow[u, "\Delta"] \arrow[ur, "\nabla\circ\Delta"'] & \\
H^1(\Gamma, C^0_{har}(\mathscr{P}_n)) \arrow[uu] \arrow[ur] & & \\
H^1(\Gamma, \mathscr{P}_n) \arrow[u] & &
\end{tikzcd}$$

and the analogous diagram for modular symbols

$$\begin{tikzcd}
 & H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)) & \\
MS^\Gamma(C^1_{har}(\mathscr{P}_n)) \arrow[r] \arrow[ur] & MS^\Gamma(C^1(\mathscr{P}_n)) \arrow[r, "\nabla"] \arrow[u] & MS^\Gamma(C^0(\mathscr{P}_n)) \\
 & MS^\Gamma(C^0(\mathscr{P}_n)) \arrow[u, "\Delta"] \arrow[ur, "\nabla\circ\Delta"'] & \\
MS^\Gamma(C^0_{har}(\mathscr{P}_n)) \arrow[uu] \arrow[ur] & & \\
MS^\Gamma(\mathscr{P}_n) \arrow[u] & &
\end{tikzcd}$$

□

Proposition 3.2.8. *For $k > 2$, the map*

$$H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)) \longrightarrow H^2(\Gamma, \mathscr{P}_n),$$

obtained from the long exact sequence in Γ-cohomology of (3.2), is an isomorphism.

Proof. This follows from the commutative diagram

$$\begin{array}{ccc} H^0(\Gamma, \mathrm{MS}(\mathcal{O}(k, \mathcal{H}_p)^b)) & \longrightarrow & H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b) \\ \downarrow & & \downarrow \\ H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)) & \longrightarrow & H^2(\Gamma, \mathscr{P}_n). \end{array}$$

Lemma 3.2.7 shows that the vertical arrows are isomorphisms and Corollary 3.2.4 shows that the top horizontal arrow is an isomorphism. □

We will also require the following results concerning a bounded subspace of $\mathcal{M}(k, \mathcal{H}_p; K)$.

Definition 3.2.9. Let $\mathcal{M}(k, \mathcal{H}_p; K)^b$ denote the image of $C^{1,b}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)$ under the Schneider-Teitelbaum map of Proposition 2.7.6.

Lemma 3.2.10. *The natural maps*

$$H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K)^b) \cong H^1(\Gamma, C^{1,b}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)) \longrightarrow H^1(\Gamma, C^{1}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)),$$

and

$$\mathrm{MS}^\Gamma(\mathcal{M}(k, \mathcal{H}_p; K)^b) \cong \mathrm{MS}^\Gamma(C^{1,b}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)) \longrightarrow \mathrm{MS}^\Gamma(C^{1}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p))$$

are isomorphisms.

Proof. As before, these statements follow from [RS12, Section 3]. We explain here how Shaprio's lemma is applied to the module $C^1_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)$.

For every $\mathbb{Q}_p$-vertex v, let $\mathrm{div}_{n,v}$ be the group of finite formal sums of the form

$$\sum_i (\tau_i) \otimes P_i(T),$$

where $\tau_i \in U(v, K/\mathbb{Q}_p)$ (see Definition 2.7.4), and $P_i \in \mathscr{P}_n$. Let $\mathrm{div}_{n,v}^{int}$ be the subgroup of $\mathrm{div}_{n,v}$ satisfying that

$$P_i \in \bigcap_{\substack{e \text{ an } F\text{-edge} \\ t(e)=v}} \mathscr{P}_{n,e}.$$

For every $\mathbb{Q}_p$-vertex v, let $\mathcal{T}_v$ be the subtree of $\mathcal{T}_K$ of vertices and edges which can be connected to v via a path that does not contain any $\mathbb{Q}_p$-edges. Define $C^1_K(\mathscr{P}_n, \mathbb{Q}_p)$ to be the submodule of $C^1_K(\mathscr{P}_n)$ of functions ϕ such that for every $\mathbb{Q}_p$-vertex v, there exists a divisor

$$D_v = \sum_i (\tau_i) \otimes P_i \in \mathrm{div}_{v,n}$$

and for each $e \in \mathcal{T}_v$,

$$\phi(e) = \sum_{\tau_i \in U(e)} P_i.$$

The natural inclusion of $C^1_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)$ into $C^1_K(\mathscr{P}_n)$ fits in the short exact sequence

$$0 \longrightarrow C^1_{K,har}(\mathscr{P}_n, \mathbb{Q}_p) \longrightarrow C^1_K(\mathscr{P}_n, \mathbb{Q}_p) \longrightarrow C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n) \longrightarrow 0.$$

See Definition 2.8.1 for $C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n)$, and Section 2.8 for some of the notation that follows.

The Γ-module $C^1_K(\mathscr{P}_n, \mathbb{Q}_p)$ is naturally identified with

$$\mathrm{Func}(\Gamma \cdot (v_0, v_K), \mathscr{P}_n) \oplus \mathrm{Func}(\Gamma \cdot (v_K, v_1), \mathscr{P}_n) \oplus \prod_{v \text{ an } F\text{-vertex}} \mathrm{div}_{n,v}.$$

Similarly, $C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n)$ is isomorphic to

$$\mathrm{Func}(\Gamma \cdot v_0, \mathscr{P}_n) \oplus \mathrm{Func}(\Gamma \cdot v_1, \mathscr{P}_n) \oplus \mathrm{Func}(\Gamma \cdot v_K, \mathscr{P}_n).$$

Finally, to view these modules as co-induced modules from congruence subgroups it remains to see that

$$\prod_{v \text{ an } F\text{-vertex}} \mathrm{div}_{n,v} = \prod_{v \in \Gamma \cdot v_0} \mathrm{div}_{n,v} \oplus \prod_{v \in \Gamma \cdot v_1} \mathrm{div}_{n,v} \oplus \prod_{v \in \Gamma \cdot v_K} \mathrm{div}_{n,v},$$

and

$$\prod_{v \in \Gamma \cdot v_i} \mathrm{div}_{n,v_i} \cong \mathrm{coInd}_{\Gamma_{v_i}}^{\Gamma} \mathrm{div}_{n,v_i}.$$

With the setup above, the proof of [RS12, Theorem] applies almost exactly.

□

3.3 RM-points and Binary Quadratic Forms

Fix once and for all embeddings of $\overline{\mathbb{Q}}$ into $\mathbb{C}$ and $\mathbb{C}_p$.

Definition 3.3.1. An RM-point is a point $\tau \in \mathcal{H}_p$, such that $\mathbb{Q}(\tau)$ is a real-quadratic field. This necessarily implies that p is either inert or ramified in $\mathbb{Q}(\tau)$. If p is inert (ramified) in $\mathbb{Q}(\tau)$, then we say τ is an inert (ramified) RM-point.

To each RM-point, τ, there is a unique binary quadratic form

$$Q_\tau(x, y) = ax^2 + bxy + cy^2,$$

with $a, b, c \in \mathbb{Z}$ co-prime, such that $Q(\tau, 1) = 0$ and τ is the "positive root"; i.e.

$$\tau = \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Remark 3.3.2. Here the sign of the roots is determined via the fixed embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

The discriminant of τ is

$$D_\tau = \mathrm{Disc}(Q_\tau) = b^2 - 4ac.$$

If τ is an inert RM-point, the discriminant factorizes as $D_\tau = D_0 p^{2l}$, where D_0 is co-prime to p and l is the level of $\mathrm{red}_p(\tau)$. If τ is a ramified RM-point, the discriminant is of the form $D_0 p^{2(l-1/2)}$.

The discriminant D_τ can be written as

$$D_\tau = c^2 D_{\mathbb{Q}(\tau)},$$

where $D_{\mathbb{Q}(\tau)}$ is the discriminant of the ring of integers $\mathcal{O}_{\mathbb{Q}(\tau)}$. The order $\mathcal{O}_\tau \subset \mathbb{Q}(\tau)$ is defined as

$$\mathcal{O}_\tau := \mathbb{Z} + c\mathcal{O}_{\mathbb{Q}(\tau)}.$$

The stabilizer subgroup of τ in $\mathrm{SL}_2(\mathbb{Z})$ is isomorphic to the group of norm-1 units in $\mathcal{O}_\tau$. There is a unique stabilizer, $\gamma_\tau \in \mathrm{SL}_2(\mathbb{Z})$, which corresponds to a positive generator of $\mathcal{O}_\tau^\times$ (equivalently, $c\tau + d > 0$), and such that the iterated action of the matrix γ_τ on $\mathbb{C}$ has τ as a stable fixed point. Denote by $(\tau, \overline{\tau})$ the geodesic in the complex upper half-plane $\mathcal{H}$ whose endpoints are τ and $\overline{\tau}$. Note that this geodesic is fixed by the action of γ_τ.

Given two oriented geodesics g_1 and g_2 in $\mathcal{H}$, denote by $g_1 \cdot g_2$ their signed topological intersection number, as defined in [DV22a, p. 2]. For a pair of cusps $r, s \in \mathbb{P}^1(\mathbb{Q})$, denote by (r, s) the oriented geodesic in $\mathcal{H}$ whose endpoints are r and s.

Definition 3.3.3. For a fixed pair $(r, s) \in \mathbb{P}^1(\mathbb{Q})^2$, define the set

$$\Sigma_\tau\{r, s\} = \{w \in \Gamma \cdot \tau \mid (w, \overline{w}) \cdot (r, s) \neq 0\},$$

and for every l, the set $\Sigma_\tau^{\leq l}\{r, s\}$ is the subset of $\Sigma_\tau\{r, s\}$ of RM-points whose reduction to $\mathcal{T}_p$ is of level at most l.

Lemma 3.3.4 ([DV21] p.41)**.** *For each $\{r, s\}$, the set $\Sigma_\tau^{\leq l}\{r, s\}$ is finite.*

Proof. Finiteness of $\Sigma_\tau^{\leq l}\{0, \infty\}$ follows from the fact that $(\tau, \overline{\tau}) \cdot (0, \infty)$ is non-zero if and only if $ac < 0$ where $Q_\tau(x, y) = ax^2 + bxy + cy^2$, and there are only finitely many choices of such a, b, c for a fixed discriminant. For any pair (r, s), the set $\Sigma_\tau^{\leq l}\{r, s\}$ can be written as a finite combination of translates of $\Sigma_\tau^{\leq l}\{0, \infty\}$, and is therefore finite. $\square$

Definition 3.3.5 (RM-divisors)**.** An RM-divisor $\mathcal{D}$ is a finite formal sum of Γ-orbits of real quadratic points on $\mathcal{H}_p$, we will write such an RM-divisor as

$$\sum_i m_i \cdot [\tau_i].$$

Denote by $\mathcal{RM}$ the group of RM-divisors. Denote by $\mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)$ the set of formal (possibly infinite) sums of points in $K\backslash\mathbb{Q}_p$ of the form

$$D = \sum_{i\in I}(z_i)P_i(t),$$

with $P_i(t) \in \mathscr{P}_n$ and $z_i \in K\backslash\mathbb{Q}_p$, such that the support of D intersects each affinoid $\mathcal{H}_p^{\leq l}$ in a finite set.

For every RM-divisor $\mathcal{D}$, we can associate a formal divisor

$$\mathrm{Div}_{k,\mathcal{D}} \in \mathrm{MS}^\Gamma(\mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p))$$

by

$$\mathrm{Div}_{k,\mathcal{D}}\{r,s\} = \sum_i \sum_{w\in[\tau_i]} m_i \cdot (w) \otimes [(w,\overline{w})\cdot(r,s)]\,\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}.$$

The divisor $\mathrm{Div}_{k,\mathcal{D}}\{r,s\}$ indeed belongs to $\mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)$ by Lemma 3.3.4.

Recall (Definition 2.8.1) that $\mathcal{T}_{K,\mathbb{Q}_p}$ denotes the subtree of $\mathcal{T}_K$ of $\mathbb{Q}_p$-vertices and $\mathbb{Q}_p$-edges. For every $\mathbb{Q}_p$-vertex, $v \in \mathcal{T}^0_{K,\mathbb{Q}_p}$, the intersection of a divisor with $\mathrm{red}^{-1}_{\mathbb{Q}_p}(v)$ is a finite divisor. The collection of the degrees of all such finite divisors can be assembled into a map

$$\deg : \mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p) \longrightarrow C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n(\mathbb{C}_p)).$$

The image of $\mathrm{Div}_{k,\mathcal{D}}$ under the degree map is

$$\mathrm{Deg}_{k,\mathcal{D}}\{r,s\}(v) := \sum_i \sum_{\substack{w\in[\tau_i]\\ \mathrm{red}_p(w)=v}} m_i\,[(w,\overline{w})\cdot(r,s)]\,\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}.$$

Proposition 3.3.6. *The symbols* $\mathrm{Div}_{k,\mathcal{D}}$ *and* $\mathrm{Deg}_{k,\mathcal{D}}$ *are* Γ*-invariant modular symbols.*

Proof. It suffices to prove this proposition for $\mathrm{Div}_{k,\mathcal{D}}$ since $\mathrm{Deg}_{k,\mathcal{D}}$ is its image under the degree map, which is Γ-equivariant. The fact that $\mathrm{Div}_{k,\mathcal{D}}$ defines a modular symbol follows from the observation that the map

$$\{r, s\} \mapsto (w, \overline{w}) \cdot (r, s)$$

is a modular symbol.

To show the Γ-invariance, a simple computation shows that

$$\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \mid \gamma = \frac{(T-\gamma^{-1}w)^{n/2}(T-\gamma^{-1}\overline{w})^{n/2}}{(\gamma^{-1}w-\gamma^{-1}\overline{w})^{n/2}}.$$

So for $\gamma \in \Gamma$,

$$\begin{aligned}
\mathrm{Div}_{k,\mathcal{D}}\{r, s\} \mid \gamma &= \sum_i \sum_{w \in \Gamma \cdot \tau_i} m_i \cdot (\gamma^{-1}w) \cdot [(w, \overline{w}) \cdot (r, s)] \\
&\qquad \times \frac{(T-\gamma^{-1}w)^{n/2}(T-\gamma^{-1}\overline{w})^{n/2}}{(\gamma^{-1}w-\gamma^{-1}\overline{w})^{n/2}} \\
&= \sum_i \sum_{w \in \Gamma \cdot \tau_i} m_i \cdot (\gamma^{-1}w) \cdot \left[(\gamma^{-1}w, \gamma^{-1}\overline{w}) \cdot (\gamma^{-1}r, \gamma^{-1}s)\right] \\
&\qquad \times \frac{(T-\gamma^{-1}w)^{n/2}(T-\gamma^{-1}\overline{w})^{n/2}}{(\gamma^{-1}w-\gamma^{-1}\overline{w})^{n/2}} \\
&= \sum_i \sum_{w \in \Gamma \cdot \tau_i} m_i \cdot (w) \cdot \left[(w, \overline{w}) \cdot (\gamma^{-1}r, \gamma^{-1}s)\right] \\
&\qquad \times \frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \\
&= \mathrm{Div}_{k,\mathcal{D}}\{\gamma^{-1}r, \gamma^{-1}s\}.
\end{aligned}$$

□

Lemma 3.3.7. *For* $k = 2$.

1. *If* τ *is inert, the modular symbol* $\mathrm{Deg}_{2,\tau}$ *is trivial.*

2. *If τ is ramified, for each vertex v in the Γ-orbit of v_0 or v_1 (i.e. coming from the base change of a $\mathbb{Z}_p$-lattice, see Definition 2.8.1) let N_v denote the set of neighbors of v that are $\mathbb{Q}_p$-vertices. For every r, s we have*

$$\sum_{v' \in N_v} \mathrm{Deg}_{2,\tau}\{r,s\}(v') = 0,$$

and

$$\mathrm{Deg}_{2,\tau}\{r,s\}(v) = 0.$$

Proof. Details of the proof can be found in [DV21, Lemma 1.16] and [DV22b, Lemma 4.5]. It is related to the fact that the space of cusp forms of level 1 and weight 2 is trivial. □

3.4 Weight 2 Rigid Meromorphic Cocycles

In this section, we recall Darmon–Vonk's definition of multiplicative rigid meromorphic cocycles and study their boundary distribution when restricted to $\mathcal{H}_K$. For this section, denote by $\mathcal{M}$ the fraction field of $\mathcal{O}(\mathcal{H}_p)$ and $\mathcal{M}^\times$ its multiplicative group.

Definition 3.4.1. A rigid meromorphic theta symbol is a modular symbol in

$$\mathrm{MS}^\Gamma(\mathcal{M}^\times/\mathbb{C}_p^\times).$$

Recall the construction of rigid meromorphic theta symbols in [DV21]. First define for $w \in \mathcal{H}_p$, the choice of rational function with a zero at w:

$$t_w(z) = \begin{cases} z - w & \text{if } |w| \leq 1, \\ \frac{z}{w} - 1 & \text{if } |w| > 1. \end{cases} \tag{3.9}$$

We start with the case where τ is inert.

Theorem 3.4.2 (Darmon-Vonk,[DV21][DV22b])**.** *For τ an inert RM-point, The symbol defined by*

$$J_\tau^\times\{r,s\}(z) = \prod_{w \in \Gamma\cdot\tau} t_w(z)^{(r,s)\cdot(w,\overline{w})}$$

is a modular symbol in $MS^{\Gamma}\left(\mathcal{M}^{\times}/\mathbb{C}_p^{\times}\right)$ *whose logarithmic derivative is*

$$J_{2,\tau}\{r,s\}(z) = \sum_{w\in\Gamma\cdot\tau} (r,s)\cdot(w,\overline{w})\frac{1}{z-w}.$$

Similarly, if τ *is ramified*

$$J_{\tau}^{\times,+}\{r,s\} = \lim_{d\to\infty} \prod_{w\in\Sigma_{\tau}^{\leq 2d}\{r,s\}} t_w(z)^{(r,s)\cdot(w,\overline{w})}$$

and

$$J_{\tau}^{\times,-}\{r,s\} = \lim_{d\to\infty} \prod_{w\in\Sigma_{\tau}^{\leq 2d+1}\{r,s\}} t_w(z)^{(r,s)\cdot(w,\overline{w})}$$

belong to $MS^{\Gamma}\left(\mathcal{M}^{\times}/\mathbb{C}_p^{\times}\right)$ *with logarithmic derivatives denoted* $J_{2,\tau}^{\pm}$.

The set of rigid analytic modular symbols $\mathrm{MS}^{\Gamma}(\mathcal{O}(2,\mathcal{H}_p))$ *along with the modular symbols* $\{J_{2,\tau}^{*}\}_{\tau}$, *where* τ *ranges over all RM-points, span the vector space* $\mathrm{MS}^{\Gamma}(\mathcal{M}(2,\mathcal{H}_p))$.

Remark 3.4.3. For the definition of $J_{\tau}^{\times,\pm}$ see [DV22b, Lemma 4.4].

Lemma 3.4.4. *Let* τ *be an RM-point and* $r,s\in\mathbb{P}^1(\mathbb{Q})$. *For an edge* $e\in\mathcal{T}_K$ *such that* $U_e\cap\mathbb{P}^1(\mathbb{Q}_p)=\emptyset$, *then the intersection*

$$U_e\cap\Sigma_{\tau}\{r,s\}$$

is finite.

Proof. This follows from Lemma 3.3.4, since as a subset of $\mathcal{H}_p$, the set U_e satisfies that

$$\mathrm{red}_p(U_e)$$

is either a single vertex or a single edge, and is certainly contained in $\mathcal{H}_p^{\leq l}$ for some l. □

Proposition 3.4.5. *Let* τ *be an inert RM-point. When restricted to* $\mathcal{H}_K$, *the function* $J_{2,\tau}\{r,s\}|_{\mathcal{H}_K}$ *is rigid analytic and its associated harmonic cochain,*

$$\Phi_{2,\tau}\in\mathrm{MS}^{\Gamma}(C^1_{K,har}(\mathbb{C}_p)),$$

can be described as follows:

- *For a $\mathbb{Q}_p$-edge, e,*
$$\Phi_{2,\tau}\{r,s\}(e) = 0,$$
- *for an edge $e \in \mathcal{T}_K$ which is not a $\mathbb{Q}_p$-edge and that is pointing away from the standard vertex*
$$\Phi_{2,\tau}\{r,s\}(e) = \sum_{w \in \Sigma_\tau\{r,s\} \cap U_e} \left[(w,\overline{w}) \cdot (r,s)\right].$$

Proof. Define the rational function on $\mathcal{H}_K$ given by

$$J_{2,\tau}^{\leq d}\{r,s\} = \sum_{w \in \Sigma_\tau^{\leq d}\{r,s\}} (r,s) \cdot (w,\overline{w}) \frac{1}{z-w}.$$

Let $\Phi_{2,\tau}^{\leq d}\{r,s\}$ be the residue cochain of the function $J_{2,\tau}^{\leq d}\{r,s\}$.

Given an oriented edge $e \in \mathcal{T}_K^1$ of level l and which is pointing away from v_0. For $d > l$, the only elements of the sum defining $J_{2,\tau}^{\leq d}$ which contribute to the residue at e are the RM-points

$$w \in \Sigma_\tau^{\leq d}\{r,s\} \cap U_e.$$

Lemma 3.3.7 implies that if $e \in \mathcal{T}_{K,\mathbb{Q}_p}^1$, then the sum of those residues contributing at e is 0, and so $\Phi_{2,\tau}^{\leq d}\{r,s\}(e) = 0$. If instead $U_e \cap \mathbb{P}^1(\mathbb{Q}_p) = \emptyset$, then by Lemma 3.3.4 the set $\Sigma_\tau\{r,s\} \cap U_e$ is finite and the residue at e is a finite sum

$$\Phi_{2,\tau}^{\leq d}\{r,s\}(e) = \sum_{w \in \Sigma_\tau^{\leq d}\{r,s\} \cap U_e} \left[(w,\overline{w}) \cdot (r,s)\right].$$

The sequence of functions

$$\left(\Phi_{2,\tau}^{\leq d}\{r,s\}\right)_d$$

converge to $\Phi_{2,\tau}\{r,s\}$ as d tends to ∞. Also, $J_{2,\tau}^{\leq d}\{r,s\}$ converges to $J_{2,\tau}\{r,s\}$; thus the residue of $J_{2,\tau}$ is equal to $\Phi_{2,\tau}$ $\square$

Now lets take τ to be ramified. We might define a function $\Phi_{2,\tau}\{r,s\} \in C_K^1(\mathbb{C}_p)$

as given in Proposition 3.4.5. This function will not be harmonic because $\mathrm{Deg}_{2,\tau}\{r,s\}$ no longer necessarily vanishes. It fails to be harmonic only on $\mathbb{Q}_p$-vertices. To remedy this, we have the flexibility of altering the definition of $\Phi_{2,\tau}\{r,s\}$ only at the $\mathbb{Q}_p$-edges. Darmon–Vonk's functions $J_\tau^\pm$ correspond to two different choices.

Proposition 3.4.6. *When restricted to $\mathcal{H}_K$, the function $J_{2,\tau}^+$ is rigid analytic and its associated harmonic cochain,*

$$\Phi_{2,\tau}^+ \in \mathrm{MS}^\Gamma(C^1_{K,har}(\mathbb{C}_p)),$$

can be described as follows:

- *For a $\mathbb{Q}_p$-edge, e, in the Γ-orbit of (v_0, v_K), oriented to point towards v_K, we have*

$$\Phi_{2,\tau}^+\{r,s\}(e) = \mathrm{Div}_{k,\tau}\{r,s\}(v_K),$$

- *for a $\mathbb{Q}_p$-edge, e, in the Γ-orbit of (v_K, v_1) with any orientation*

$$\Phi_{2,\tau}^+\{r,s\}(e) = 0,$$

- *for any edge e that is not a $\mathbb{Q}_p$-edge and is oriented to point away from the standard vertex*

$$\Phi_{2,\tau}^+\{r,s\} \sum_{w\in\Sigma_\tau\{r,s\}\cap U(e)} [(w,\overline{w})\cdot(r,s)].$$

The analogous statement holds for $J_{2,\tau}^-$ and $\Phi_{2,\tau}^-$.

Proof. The proof is the same as in Proposition 3.4.5. □

For an RM-divisor, $\mathcal{D} = \sum_i m_i[\tau_i]$, write

$$J_{\mathcal{D}}^\times = \prod_i \left(J_{\tau_i}^\times\right)^{m_i}.$$

Given a rigid meromorphic theta cohomology class in $H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times)$, we consider lifting the class to a rigid meromorphic cohomology class in $H^1(\Gamma, \mathcal{M}^\times)$. It is not always possible to find such a lift. The RM-divisors for which such a lift exists are analogous to principal divisors on an algebraic curve.

Definition 3.4.7. Given a rigid meromorphic cohomology class

$$[\mathfrak{J}] \in H^1(\Gamma, \mathcal{M}^\times),$$

let $\mathfrak{J}$ be a representative meromorphic cocycle $\mathfrak{J}$. Let σ be an RM-point which is not a pole of $\mathfrak{J}(\gamma_\sigma)$. The value of $\mathfrak{J}$ at σ is defined by

$$\mathfrak{J}[\sigma] = \mathfrak{J}(\gamma_\sigma)(\sigma).$$

This value is independent of the choice of representative.

3.5 Higher Weight Cocycles

We generalize this construction of meromorphic cocycles from the previous section to higher weight. In doing so, we replace the weight-2 module with $\mathcal{O}(k, \mathcal{H}_K)$.

Fix an even weight $k > 2$.

Definition 3.5.1. For every RM-divisor

$$\mathcal{D} = \sum_i m_i \otimes [\tau_i],$$

where $[\tau_i] = \Gamma\tau_i$, and any pair (r, s) define the function $\Phi_{k,\mathcal{D}}\{r, s\} \in C^1_K(\mathscr{P}_n)$ as follows:

- For any edge $e \in \mathcal{T}^1_K$ which is not a $\mathbb{Q}_p$-edge and oriented to point away from the standard vertex v_0, set

$$\Phi_{k,\mathcal{D}}\{r, s\}(e) = \sum_i \sum_{w \in \Sigma_{\tau_i}\{r,s\} \cap U_e} m_i \left[(w, \overline{w}) \cdot (r, s)\right] \frac{(T - w)^{n/2}(T - \overline{w})^{n/2}}{(w - \overline{w})^{n/2}},$$

- for any $\mathbb{Q}_p$-edge, e, set

$$\Phi_{k,\mathcal{D}}(e) = 0.$$

- Otherwise, the function is defined by $\Phi_{k,\mathcal{D}}\{r,s\}(\bar{e}) = -\Phi_{k,\mathcal{D}}\{r,s\}(e)$.

Proposition 3.5.2. *For an RM-divisor $\mathcal{D}$, the map*

$$\{r,s\} \mapsto \Phi_{k,\mathcal{D}}\{r,s\}$$

is a Γ-invariant modular symbol with values in $C^1_K(\mathscr{P}_n)$.

Proof. The proof is exactly the same as for proposition 3.3.6. □

As opposed to the situation in weight 2, these functions are not necessarily harmonic, but $\nabla\Phi_{k,\mathcal{D}}$ is supported only on $\mathbb{Q}_p$-vertices.

Proposition 3.5.3.

$$\nabla(\Phi_{k,\mathcal{D}}) = \mathrm{Deg}_{k,\mathcal{D}}.$$

Proof. For any vertex $v \in \mathcal{T}^0_K$ which is not a $\mathbb{Q}_p$-vertex, there is a unique edge, $e_{v,0}$, which starts at v and points towards the standard vertex and the remaining edges, $e_{v,1}, \cdots, e_{v,p^2}$ that start at v satisfy $U_{e_{v,i}} \cap \mathbb{P}^1(\mathbb{Q}_p) = \emptyset$, and similarly $U_{\overline{e_{v,0}}} \cap \mathbb{P}^1(\mathbb{Q}_p) = \emptyset$. Since

$$U(\overline{e_{v,0}}) = \bigcup_{i=1}^{p^2} U_{e_i},$$

we see that

$$\nabla(\Phi_{k,\mathcal{D}})(v) = \left(\sum_{i=1}^{p^2} \Phi_{k,\mathcal{D}}(e_{v,i})\right) - \Phi_{k,\mathcal{D}}(\overline{e_{v,0}}) = 0.$$

If instead, v is a $\mathbb{Q}_p$-edge, then there are exactly $p^2 - p$ edges $e_{v,1}, \cdots, e_{v,p^2-p}$ starting at v and satisfying $U_{e_{v,i}} \cap \mathbb{P}^1(\mathbb{Q}_p) = \emptyset$. The result follows since an RM-point, $\tau \in \mathcal{H}_p$, reduces to v:

$$\mathrm{red}_p(\tau) = v$$

if and only if $v \in U_{e_{v,i}}$ for some i.

□

Proposition 3.5.4. *For every RM-divisor $\mathcal{D}$ and pair (r, s), the function $\Phi_{k,\mathcal{D}}\{r, s\}$ is integral; i.e.*

$$\Phi_{k,\mathcal{D}}\{r,s\} \in C_K^{1,int}(\mathscr{P}_n(\mathbb{C}_p))$$

(see p. 30 for the integral structure).

Proof. Let w be an RM-point, and $e \in \mathcal{T}_K^1$ an edge such that $U_e \cap \mathbb{P}^1(\mathbb{Q}_p) = \emptyset$ and $w \in U_e$. It suffices to show that

$$\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \in \mathscr{P}_{n,e}.$$

Let $v_w \in \mathcal{T}_p^0$ be the reduction of w in $\mathcal{T}_p$. Then e lies on the infinite path $(v_w, v_{w,1}, v_{w,2}, \cdots)$ in $\mathcal{T}_K$ which begins at v_w and ends at w. We will show that for every vertex $v_{w,a}$, we have

$$\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \in \mathscr{P}_{n,v_{w,a}}.$$

First, consider the matrix $\begin{pmatrix} 1 & -\overline{w} \\ 1 & -w \end{pmatrix}$; this matrix sends $w \mapsto \infty$, $\overline{w} \mapsto 0$ and maps $v_{w,i}$ to some vertex on the path from ∞ to 0. Since

$$T^{n/2}| \begin{pmatrix} p^a & 0 \\ 0 & 1 \end{pmatrix} = T^{n/2},$$

every vertex v in the path connected 0 to ∞ has $T^{n/2} \in \mathscr{P}_{n,v}$. It follows that

$$T^{n/2}| \begin{pmatrix} 1 & -\overline{w} \\ 1 & -w \end{pmatrix} = (-1)^{n/2}\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}$$

belongs to $\mathscr{P}_{n,v_{w,i}}$. □

Definition 3.5.5. If $\mathrm{Deg}_{k,\mathcal{D}}$ vanishes, then we say that $\mathcal{D}$ is of *strong degree zero in weight* k. In this case, by Proposition 3.5.3, $\Phi_{k,\mathcal{D}}$ defines a modular symbol valued in bounded harmonic cochains and applying the Schneider-Teitelbaum lift, we obtain a modular symbol

$$J_{k,\mathcal{D}} := \mathrm{ST}(\Phi_{k,\mathcal{D}}) \in \mathrm{MS}^{\Gamma}(\mathcal{O}(k, \mathcal{H}_K)).$$

In fact, $J_{k,\mathcal{D}}$ belongs to $\mathrm{MS}^{\Gamma}(\mathcal{M}(k,\mathcal{H}_p,K)^b)$.

Proposition 3.5.6. *Let $\mathcal{D} = \sum_i m_i[\tau_i]$ be an RM-divisor of strong degree zero in weight k. Then*

$$J_{k,\mathcal{D}}\{r,s\}(z) = \sum_i \sum_{w\in\Sigma_{\tau_i}\{r,s\}} m_i\left[(w,\overline{w})\cdot(r,s)\right] \times d^{n+1}\left(\frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}\log(z-w)\right).$$

Proof. Truncate $\Phi_{k,\mathcal{D}}$ to define $\Phi_{k,\mathcal{D}}^{\leq l}$ exactly as in Definition 3.5.1, except that for e not a $\mathbb{Q}_p$-edge that is pointing away from the standard vertex, set

$$\Phi_{k,\mathcal{D}}^{\leq l}\{r,s\}(e) = \sum_i \sum_{w\in\Sigma_{\tau_i}^{\leq l}\{r,s\}\cap U_e} m_i\left[(w,\overline{w})\cdot(r,s)\right]\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}.$$

The boundedness of $\Phi_{k,\mathcal{D}}$ implies that

$$\lim_{l\to\infty}\Phi_{k,\mathcal{D}}^{\leq l}\{r,s\} = \Phi_{k,\mathcal{D}}\{r,s\}.$$

Following the proof of Proposition 3.5.3, one can show in general that

$$\nabla(\Phi_{k,\mathcal{D}}^{\leq l}) = \mathrm{Deg}_{k,\mathcal{D}}\cap\mathcal{H}_p^{\leq l},$$

and so in this case $\Phi_{k,\mathcal{D}}^{\leq l}$ is harmonic. By Lemma 2.6.3 we have

$$\mathrm{ST}\left(\Phi_{k,\mathcal{D}}^{\leq l}\{r,s\}\right) = \sum_i \sum_{w\in\Sigma_{\tau_i}^{\leq l}\{r,s\}} m_i\left[(w,\overline{w})\cdot(r,s)\right] \times d^{n+1}\left(\frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}\log(z-w)\right),$$

and since Schneider-Teitelbaum lift is a topological isomorphism, the limit of the sum above converges and equals $\mathrm{ST}(\Phi_{k,\mathcal{D}})$ as l tends to ∞.

□

Remark 3.5.7. The divisor $\mathcal{D} = [\tau] + (-1)^{n/2}[\overline{\tau}]$ is of strong degree zero in all weights since for every RM-point, $\mathrm{red}_p(w) = \mathrm{red}_p(\overline{w})$. It follows from the identity

$$d^{n+1}\left(\frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}\log\left(\frac{z-w}{z-\overline{w}}\right)\right) = (n!)^2\frac{(w-\overline{w})^{k/2}}{(z-w)^{k/2}(z-\overline{w})^{k/2}}$$

that $J_{k,\mathcal{D}}$ is equal to the meromorphic symbols denoted by Φ_τ in Negrini's thesis [Neg22] and in [Neg23], Section 1.

Theorem 3.5.8. *1. Let $J \in \mathrm{MS}^\Gamma(\mathcal{M}(k,\mathcal{H}_p))$ be a weight k rigid meromorphic modular symbol. For each r, s, the divisor of $J\{r,s\}$ is supported on RM-points, and the residues*

$$\mathrm{res}_w J\{r,s\}(z)(T-z)^n\ dz$$

are multiples of

$$\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}.$$

The divisor of J is equal to a linear combination of $\mathrm{Div}_{k,\mathcal{D}}$ for some RM-divisors $\mathcal{D}$.

2. For any RM-divisor $\mathcal{D}$, there exists a rigid meromorphic modular symbol

$$J \in \mathrm{MS}^\Gamma(\mathcal{M}(k,\mathcal{H}_p,K)^b)$$

such that

$$\mathrm{rdiv}_k(J) = \mathrm{Div}_{k,\mathcal{D}}.$$

(See Definition 2.7.15 for rdiv_k).

Proof. The first part follows from Choie–Zagier [CZ93] Lemma 5. We note that even though Choie–Zagier's proof is over $\mathbb{C}$, this assumption is not crucial and with small adjustments applies to our setting, e.g. see [DV21, Lemma 1.19].

For the second part, note that $\Phi_{k,\mathcal{D}}$ is automatically harmonic except at the $\mathbb{Q}_p$-vertices. We only need to "fix" its harmonicity at the $\mathbb{Q}_p$-vertices. It suffices to find some $\chi \in$

$\mathrm{MS}^\Gamma(C^1_{K,\mathbb{Q}_p}(\mathscr{P}_n))$ such that $\Phi_{k,\mathcal{D}} + \chi$ is harmonic. That is,

$$\nabla\chi\{r,s\} = -\mathrm{Div}_{k,\mathcal{D}}\{r,s\}.$$

By definition, there exists such a χ if and only if $\mathrm{Div}_{k,\mathcal{D}}$ is in the image of

$$\mathrm{MS}^\Gamma(C^1_{K,\mathbb{Q}_p}(\mathscr{P}_n(\mathbb{C}_p))) \xrightarrow{\nabla} \mathrm{MS}^\Gamma(C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n(\mathbb{C}_p))).$$

But ∇ is in fact surjective. This can be seen as follows:

By Shapiro's lemma, since $\mathcal{T}^1_{K,\mathbb{Q}_p}$ has two Γ-orbits, we can write

$$\mathrm{MS}^\Gamma(C^1_{K,\mathbb{Q}_p}(\mathscr{P}_n)) \cong \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n).$$

Similarly,

$$\mathrm{MS}^\Gamma(C^0_{K,\mathbb{Q}_p}(\mathscr{P}_n)) \cong \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \oplus \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n).$$

There are two co-restriction maps

$$r_1, r_2 : \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \longrightarrow \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n).$$

It is also known that as a Hecke-module (see for example [BD15]), there are decompositions

$$\mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \cong \mathrm{BMS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus S_k(\Gamma_0(p))^2,$$

and

$$\mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \cong \mathrm{BMS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \oplus S_k(\mathrm{SL}_2(\mathbb{Z}))^2,$$

where BMS is the space of boundary modular symbols. In this case $\mathrm{BMS}^{\Gamma_0(p)}(\mathscr{P}_n)$ is 2-dimensional, while $\mathrm{BMS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n)$ is 1-dimensional. The co-restriction maps r_1 and r_2 preserve the decompositions above. The map

$$r_1 \oplus r_2 : \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \longrightarrow \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \oplus \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n)$$

is surjective with kernel precisely equal to the p-new part of $\mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n)$.

Therefore, the map

$$\nabla : \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \longrightarrow \mathrm{MS}^{\Gamma_0(p)}(\mathscr{P}_n) \oplus \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n) \oplus \mathrm{MS}^{\mathrm{SL}_2(\mathbb{Z})}(\mathscr{P}_n),$$

given by

$$(f,g) \mapsto (f-g, r_1(f), r_2(g)),$$

is surjective.

We conclude that such a $\chi \in \mathrm{MS}^{\Gamma}(C^1_{K,\mathbb{Q}_p}(\mathscr{P}_n))$ exists. Notice that Γ acts on $\mathcal{T}^1_{K,\mathbb{Q}_p}$ with finitely many orbits, we have that

$$\mathrm{MS}^{\Gamma}(C^1_{K,\mathbb{Q}_p}(\mathscr{P}_n)) = \mathrm{MS}^{\Gamma}(C^{1,b}_{K,\mathbb{Q}_p}(\mathscr{P}_n)).$$

Since $\Phi_{k,\mathcal{D}}$ is bounded (Proposition 3.5.4), the modular symbol $\Phi_{k,\mathcal{D}} + \chi$ is bounded. By 2.7.6, the Schneider–Teitelbaum lift defines a modular symbol

$$J := \mathrm{ST}(\Phi_{k,\mathcal{D}} + \chi) \in \mathrm{MS}^{\Gamma}(\mathcal{M}(k, \mathcal{H}_p, K)^b).$$

By Lemma 2.7.16, we have

$$\mathrm{rdiv}_{\mathrm{k}}(J) = \mathrm{Div}_{k,\mathcal{D}}.$$

□

3.6 A Cohomological Approach

Following the approach of [DV25] we will describe here a slightly different approach to constructing rigid meromorphic cocycles. In particular, we describe a map

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p)).$$

Definition 3.6.1. Given a divisor $D \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n$ define an associated element

$$\Phi^0(D) \in C^1_K(\mathscr{P}_n)$$

as follows: Write

$$D = \sum_i \tau_i \otimes P_i(T).$$

For every edge $e \in \mathcal{T}_K^1$, set

$$\Phi^0(D)(e) = \sum_i \mathbb{1}_{U(e)}(\tau_i) P_i(T).$$

Lemma 3.6.2. *For every $D \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n$, the associated function $\Phi^0(D)$ is harmonic.*

Proof. Given a vertex v of $\mathcal{T}_K$, enumerate the set of edges starting at v by $\{e_j\}_j$. Since the sets $\{U(e_j)\}_j$ are pairwise disjoint and their union is all of $\mathbb{P}^1(K)$, we see that

$$\sum_j \Phi^0(D)(e_j) = \sum_i P_i(T) = 0,$$

where the last equality follows from D having degree 0. □

Now given an element $[\mathfrak{D}] \in H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$, pick a representative cycle $\mathfrak{D} \in Z_1(\Gamma, \mathrm{Div}_0(K/\mathbb{Q}_p) \otimes \mathscr{P}_n)$, which is written as

$$\mathfrak{D} = \sum_a \gamma_a \otimes D_a$$

for $\gamma_a \in \Gamma$ and $D_a \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n$, and which satisfies

$$\sum_a \gamma_a D_a = \sum_a D_a. \tag{3.10}$$

Pick two base-points $\eta_0, \xi_0 \in \mathcal{H}$ that are in generic position relative to Γ. That is for every $\gamma_1, \gamma_2 \in \Gamma$, the point η_0 does not lie on the geodesic $(\gamma_1\xi_0, \gamma_2\xi_0)$ and also the point ξ_0 does not lie on the geodesic $(\gamma_1\eta_0, \gamma_2\eta_0)$. By [DV25, Lemma 10], there exists a choice of such points.

Define a function

$$\Phi_{\mathfrak{D}} : \Gamma \longrightarrow C_K^1(\mathscr{P}_n)$$

by

$$\Phi_{\mathfrak{D}}(\gamma)(e) = \sum_a \sum_{b\in\Gamma} [(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\right)(e). \tag{3.11}$$

Lemma 3.6.3. *The sum in (3.11) is a finite sum, and therefore converges.*

Proof. For each a, even though $\Phi^0(D_a)$ is infinitely supported, it is finitely supported on every Γ-orbit. Therefore, it suffices to prove that for each a, and for each edge e with stabilizer in Γ denoted Γ_e, the sum

$$\sum_{b\in\Gamma_e} [(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\right)(e)$$

is finite. Notice that the paths $(\eta_0, \gamma\eta_0)$ and $(\xi_0, \gamma_a\xi_0)$ are both contained in compact subsets of $\mathcal{H}$, and since Γ_e is a conjugate of a congruence subgroup, it acts properly discontinuously on $\mathcal{H}$ and so the intersection $[(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_a\xi_0)]$ is non-zero for only finitely many $b \in \Gamma_e$. □

Lemma 3.6.4. *The function $\Phi_{\mathfrak{D}}(\gamma) \in C^1_K(\mathscr{P}_n)$ is harmonic.*

Proof. The sum defining $\Phi_{\mathfrak{D}}$ is an infinite sum of harmonic cochains. For each vertex v, the terms contributing to the neighborhood of v is a finite sum of harmonic cochains, and therefore their sum is also harmonic. □

Lemma 3.6.5. *The function $\Phi_{\mathfrak{D}}(\gamma)$ is bounded, i.e.*

$$\Phi_{\mathfrak{D}}(\gamma) \in C^{1,b}_{K,har}(\mathscr{P}_n; \mathbb{Q}_p).$$

Proof. Exactly as in the proof of Lemma 3.6.3, one can show that for every affinoid domain in $\mathcal{H}_p$ and for every $\gamma \in \Gamma$, there are only finitely many $b \in \Gamma$ such that the support of the divisor

$$(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_a\xi_0)\,(b\cdot D_a)$$

intersects the affinoid domain. This implies property (1) in Definition 2.7.5.

The other two properties follow from the fact that for each a, the divisor D_a is a finite sum and the function $\Phi^0(D_a)$ restricted to $\mathcal{T}_{\mathbb{Q}_p}$ has finite support. Therefore, we can assume without loss of generality that

$$\Phi^0(D_a) \in C^{1,int}_{K,har}(\mathscr{P}_n, \mathbb{Q}_p)$$

is integral. In which case, all of its translates $\Phi^0(D_a)|b^{-1}$ are also integral, and so the sum defining $\Phi_{\mathfrak{D}}(\gamma)$ must also be integral. □

Lemma 3.6.6. *The function*

$$\Phi_{\mathfrak{D}} : \Gamma \longrightarrow C^1_{K,har}(\mathscr{P}_n)$$

is a cocycle.

Proof. We have to show that

$$\left(\Phi_{\mathfrak{D}}(\gamma\sigma) - \Phi_{\mathfrak{D}}(\gamma) - \gamma \cdot \Phi_{\mathfrak{D}}(\sigma)\right)(e) = 0$$

for every edge. So, we calculate

$$\begin{aligned}
&\gamma \cdot \left(\Phi_{\mathfrak{D}}(\sigma)\right)(e) = \Phi_{\mathfrak{D}}(\sigma)(\gamma e)|\gamma^{-1} \\
&= \sum_a \sum_{b\in\Gamma} [(\eta_0, \sigma\eta_0) \cdot (b\xi_0, b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\gamma^{-1}\right)(e) \\
&= \sum_a \sum_{b\in\Gamma} [(\eta_0, \sigma\eta_0) \cdot (\gamma^{-1}b\xi_0, \gamma^{-1}b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\right)(e) \\
&= \sum_a \sum_{b\in\Gamma} [(\gamma\eta_0, \gamma\sigma\eta_0) \cdot (b\xi_0, b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\right)(e).
\end{aligned}$$

We would be done if the intersection of geodesics on the upper half-plane had similar relations to that of modular symbols; namely, if

$$[((\eta_0, \gamma\sigma\eta_0) - (\eta_0, \gamma\eta_0) - (\gamma\eta_0, \gamma\sigma\eta_0)) \cdot (b\xi_0, b\gamma_a\xi_0)] = 0.$$

However, this equality does not hold. Instead, let $\mathfrak{B}$ be the interior of the triangle in $\mathcal{H}$

spanned by the geodesics

$$(\eta_0, \gamma\sigma\eta_0), (\eta_0, \gamma\eta_0), (\gamma\eta_0, \gamma\sigma\eta_0).$$

Observe that

$$\begin{aligned}
&[((\eta_0, \gamma\sigma\eta_0) - (\eta_0, \gamma\eta_0) - (\gamma\eta_0, \gamma\sigma\eta_0)) \cdot (b\xi_0, b\gamma_a\xi_0)] \\
&= \mathbb{1}_{\mathfrak{B}}(b\xi_0) - \mathbb{1}_{\mathfrak{B}}(b\gamma_a\xi_0) \\
&= \mathbb{1}_{b^{-1}\mathfrak{B}}(\eta_0) - \mathbb{1}_{b^{-1}\mathfrak{B}}(\gamma_a\xi_0)
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(\Phi_{\mathfrak{D}}(\gamma\sigma) - \Phi_{\mathfrak{D}}(\gamma) - \gamma \cdot \Phi_{\mathfrak{D}}(\sigma))(e) \\
&= \sum_a \sum_{b \in \Gamma} [((\eta_0, \gamma\sigma\eta_0) - (\eta_0, \gamma\eta_0) - (\gamma\eta_0, \gamma\sigma\eta_0)) \cdot (b\xi_0, b\gamma_a\xi_0)] \left(\Phi^0(D_a)|b^{-1}\right)(e) \\
&= \sum_a \sum_{b \in \Gamma} (\mathbb{1}_{b^{-1}\mathfrak{B}}(\eta_0) - \mathbb{1}_{b^{-1}\mathfrak{B}}(\gamma_a\eta_0)) \left(\Phi^0(D_a)|b^{-1}\right)(e)
\end{aligned}$$

$$\begin{aligned}
&= \sum_a \sum_{b \in \Gamma} \mathbb{1}_{\mathfrak{B}}(b\eta_0) \left(\Phi^0(D_a)|b^{-1}\right)(e) - \sum_a \sum_{b \in \Gamma} \mathbb{1}_{\mathfrak{B}}(b\gamma_a\eta_0) \left(\Phi^0(D_a)|b^{-1}\right)(e) \\
&= \sum_a \sum_{b \in \Gamma} \mathbb{1}_{\mathfrak{b}}(b\eta_0) \left(\Phi^0(D_a)|b^{-1}\right)(e) - \sum_a \sum_{b \in \Gamma} \mathbb{1}_{\mathfrak{B}}(b\eta_0) \left(\Phi^0(D_a)|\gamma b^{-1}\right)(e) \\
&= \sum_{b \in \Gamma} \mathbb{1}_{\mathfrak{B}}(b\eta_0) \left[\left(\sum_a \Phi^0(D_a)\right) - \left(\sum_a \Phi^0(\gamma^{-1}D_a)\right)\right] |b^{-1}(e) \\
&= 0.
\end{aligned}$$

$\square$

Lemma 3.6.7. *For a cycle class $[\mathfrak{D}] \in H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$, the cohomology class*

$$[\Phi_{\mathfrak{D}}] \in H^1(\Gamma, C^1_{K,har}(\mathscr{P}_n))$$

is well-defined.

Proof. We show that a boundary in $B_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$ maps to a coboundary in $B^1(\Gamma, C^{1,b}_{K,har}(\mathscr{P}_n; \mathbb{Q}_p))$.

Boundaries are linear combinations of the form

$$\mathfrak{D} = \gamma\sigma \otimes D - \gamma \otimes D - \sigma \otimes \gamma^{-1} D,$$

where $D \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n$ and $\gamma, \sigma \in \Gamma$. The proof is similar to that of Lemma 3.6.6. Let $\mathfrak{B}$ be the interior of the triangle spanned by the geodesics

$$(\xi_0, \gamma\sigma\xi_0), (\xi_0, \gamma\xi_0), (\gamma\xi_0, \gamma\sigma\xi_0).$$

We compute that

$$\begin{aligned}
\Phi_{\mathfrak{D}}(\alpha)(e) &= \sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)(e) - \sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\alpha\eta_0) \left(\Phi^0(D)|b^{-1}\right)(e) \\
&= \sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)(e) - \sum_{b\in\Gamma} \mathbb{1}_{\alpha^{-1}b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)(e) \\
&= \sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)(e) - \sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\alpha^{-1}\right)(e) \\
&= \left(\sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)\right)(e) - \left(\sum_{b\in\Gamma} \mathbb{1}_{b\mathfrak{B}}(\eta_0) \left(\Phi^0(D)|b^{-1}\right)\right)|\alpha^{-1}(e).
\end{aligned}$$

Observe that this last expression is a coboundary.

□

We have therefore shown that

Proposition 3.6.8. *The map*

$$[\mathfrak{D}] \mapsto [\Phi_{\mathfrak{D}}]$$

is a well-defined linear map

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow H^1(\Gamma, C^{1,b}_{K,har}(\mathscr{P}_n; \mathbb{Q}_p)).$$

If we write $J_{\mathfrak{D}} \in H^1(\Gamma, \mathcal{M}(k, \mathbb{Q}_p))$ *for the associated cocycle valued in meromorphic functions, then* $J_{\mathfrak{D}}$ *has poles supported on the* Γ*-orbit of the support of* $\mathfrak{D}$.

For each $[\mathfrak{D}] \in H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$, denote by $\mathfrak{D}^\dagger$ the map

$$\Gamma \longrightarrow \mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)$$

given by

$$\mathfrak{D}^\dagger(\gamma) = \sum_a \sum_{b\in\Gamma} [(\eta_0, \gamma\eta_0) \cdot (b\xi_0, b\gamma_a\xi_0)](b \cdot D_a).$$

From the same proofs as above we can see that for any $\gamma \in \Gamma$ the formal sum $\mathfrak{D}^\dagger(\gamma)$ indeed belongs to $\mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)$, and that the map $\mathfrak{D}^\dagger$ is a cocycle whose class in cohomology is independent of the choice of representative $\mathfrak{D}$. We will also use the same notation for a class $[\mathfrak{D}] \in H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$ without requiring the degree 0 assumption in this case.

We have the following commutative diagram

$$\begin{array}{ccc} H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) & \longrightarrow & H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \\ \downarrow & & \downarrow \\ H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K)) & \longrightarrow & H^1(\Gamma, \mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)). \end{array}$$

Both horizontal lines in this commutative diagram extend to long exact sequences

$$H_2(\Gamma, \mathscr{P}_n) \longrightarrow H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n) \longrightarrow H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n) \longrightarrow H_1(\Gamma, \mathscr{P}_n)$$

and

$$H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)) \longrightarrow H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K)) \longrightarrow H^1(\Gamma, \mathfrak{Div}_n^\dagger(K\backslash\mathbb{Q}_p)) \longrightarrow H^1(\Gamma, \mathscr{P}_n).$$

Therefore, this diagram induces a map

$$H_2(\Gamma, \mathscr{P}_n) \longrightarrow H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)).$$

Proposition 3.6.9. *For $\mathfrak{D} = \gamma_\tau \otimes (\tau) \otimes P_\tau$, we have that*

$$[\mathfrak{D}^\dagger] = [\mathrm{Div}_{k,\tau}]$$

in $H^1(\Gamma, \mathfrak{Div}_n^\dagger(K/\mathbb{Q}_p))$.

Proof. Notice that

$$\Gamma_\tau := \mathrm{Stab}_\Gamma(\tau) = \gamma_\tau^{\mathbb{Z}}.$$

From the definition of $\mathfrak{D}^\dagger$ we write

$$\begin{aligned}\mathfrak{D}^\dagger(\gamma) &= \sum_{b\in\Gamma}[(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_\tau\xi_0)](b.\tau)P_{b\tau}(T)\\ &= \sum_{b\in\Gamma/\Gamma_\tau}\left(\sum_{i\in\mathbb{Z}}[(\eta_0, \gamma\eta_0), (b\gamma_\tau^i\xi_0, b\gamma_\tau^{i+1}\xi_0)]\right)(b\cdot\tau)P_{b\tau}(T).\end{aligned}$$

Notice that the segments $\{(b\gamma_\tau^i\xi_0, b\gamma_\tau^{i+1\xi_0})\}$ form a path going from $\overline{b\tau}$ to $b\tau$, and therefore the inner sum is just equal

$$(\eta_0, \gamma\eta_0)\cdot(\overline{b\tau}, b\tau).$$

Hence,

$$\mathfrak{D}^\dagger(\gamma) = \sum_{w\in\Gamma\cdot\tau}(\eta_0, \gamma\eta_0)\cdot(\overline{w}, w)(w)P_w(T)$$

$\square$

Proposition 3.6.10. *Let $\tau \in \mathcal{H}_p$ be an RM-point and $\gamma_\tau \in \Gamma$ its fundamental stabilizer. Then*

$$\mathfrak{D} = \gamma_\tau \otimes (\tau - \overline{\tau}) \otimes P_\tau(T)$$

defines a class in $H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$.

The associated cocycle class $[J_\mathfrak{D}] \in H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K))$ is equal to $[J_\mathcal{D}]$ where $\mathcal{D} = [\tau] + (-1)^{n/2}[\overline{\tau}]$ (Definition 3.5.5, see also Remark 3.5.7).

Proof. Since τ and $\overline{\tau}$ both have the same reduction to $\mathcal{T}_p$, the function $\Phi^0((\tau - \overline{\tau})\otimes P_\tau(t))$

has value 0 on all $\mathbb{Q}_p$-edges, and so do all of its translates by elements of Γ. Therefore, $\Phi_{\mathfrak{D}}$ always vanishes on $\mathbb{Q}_p$-edges.

The difference $J_{\mathcal{D}} - J_{\mathfrak{D}}$ has trivial divisor, and so is actually rigid analytic, but also both cocycles have no residues on $\mathbb{Q}_p$-edges. Since cocycles take values in bounded functions this implies that the two cocycles are equal.

□

4

Obstruction to lifting and the Green's functions

4.1 Obstruction to Lifting

In the multiplicative setting there is an exact sequence

$$H^1(\Gamma, \mathcal{M}^\times) \longrightarrow H^1(\Gamma, \mathcal{M}^\times/\mathbb{C}_p^\times) \longrightarrow H^2(\Gamma, \mathbb{C}_p^\times),$$

where the map to $H^2(\Gamma, \mathbb{C}_p^\times)$ is the obstruction to lifting a theta cocycle class. In higher weight, we work instead with the additive counterpart to this obstruction map. Every homomorphism

$$\rho : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p,$$

defines a map in Γ-cohomology and so by composition we obtain

$$\mathrm{Obs}_\rho : H^1(\Gamma, \mathcal{M}^\times) \longrightarrow H^2(\Gamma, \mathbb{C}_p).$$

In what follows, we will define homomorphisms

$$\mathrm{Obs}_{\mathscr{L}} : H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K)) \longrightarrow H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p))$$

for each $\mathscr{L} \in \mathbb{C}_p$ corresponding to the homomorphism

$$\log_{\mathscr{L}} : \mathbb{C}_p^\times \longrightarrow \mathbb{C}_p,$$

and a homomorphism Obs_o corresponding to the p-adic valuation

$$\mathrm{Obs}_o : H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p; K)) \longrightarrow H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)).$$

For the following, let F denote a finite extension of $\mathbb{Q}_p$. Later, we will take F to be either K or $\mathbb{Q}_p$.

Definition 4.1.1 ($\log_{\mathscr{L}}$ obstruction)**.** For every $\mathscr{L} \in \mathbb{C}_p$ the Γ-cohomology of the short exact sequence

$$0 \longrightarrow \mathscr{P}_n(\mathbb{C}_p) \longrightarrow \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_F) \longrightarrow \mathcal{O}(k, \mathcal{H}_F) \longrightarrow 0 \tag{4.1}$$

gives rise to a δ-map

$$H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_F)) \longrightarrow H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)),$$

which we denote by $\mathrm{Obs}_{\mathscr{L}}$.

Definition 4.1.2 (ord_p obstruction)**.** Define $C^0_{F,har}(\mathscr{P}_n(\mathbb{C}_p))$ to be the set of functions

$f : \mathcal{T}_F^0 \longrightarrow \mathscr{P}_n(\mathbb{C}_p)$ such that for every vertex, v, we have

$$\sum_{\substack{v' \in \mathcal{T}_F^0 \\ (v,v') \in \mathcal{T}_F^1}} (f(v) - f(v')) = 0.$$

There is an analogous short exact sequence to (4.1) for the p-adic valuation:

$$0 \longrightarrow \mathscr{P}_n(\mathbb{C}_p) \longrightarrow C^0_{F,har}(\mathscr{P}_n(\mathbb{C}_p)) \longrightarrow C^1_{F,har}(\mathscr{P}_n(\mathbb{C}_p)) \longrightarrow 0, \tag{4.2}$$

where the map $C^0_{F,har}(\mathscr{P}_n(\mathbb{C}_p)) \longrightarrow C^1_{F,har}(\mathscr{P}_n(\mathbb{C}_p))$ is given by

$$f \mapsto ((v, v') \mapsto f(v') - f(v)) \,. \tag{4.3}$$

The long exact sequence in group cohomology from (4.2) gives a map

$$\delta : H^1(\Gamma, C^1_{F,har}(\mathscr{P}_n(\mathbb{C}_p))) \longrightarrow H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)).$$

Precomposing with the residue map provides us with the obstruction

$$\mathrm{Obs}_o : H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_F)) \longrightarrow H^2(\Gamma, \mathscr{P}_n).$$

The map Obs_o can be described more explicitly as follows: Given a cocycle

$$\varphi \in Z^1(\Gamma, C^1_{F,har}(\mathscr{P}_n(\mathbb{C}_p))),$$

a particular choice of a lift to $\mathrm{Func}\left(\Gamma, C^0_{F,har}(\mathscr{P}_n)\right)$ is given by the map

$$\gamma \in \Gamma \mapsto \left(v \mapsto \sum_{e \in \mathrm{path}(v_0, v)} \varphi(\gamma)(e) \right),$$

which is not necessarily a cocycle. The δ-map to $H^2(\Gamma, \mathscr{P}_n)$ with this choice of lift yields the representative 2-cocycle

$$\mathrm{Obs}_o(\varphi)(\gamma_1,\gamma_2) = \sum_{e\in \mathrm{path}(v_0,\gamma_2^{-1}v_0)} \varphi(\gamma_1)(e)|\gamma_2.$$

Proposition 4.1.3. *For any $\mathscr{L}_1,\mathscr{L}_2 \in \mathbb{C}_p$ the various obstruction maps are related by*

$$\mathrm{Obs}_{\mathscr{L}_1} - \mathrm{Obs}_{\mathscr{L}_2} = (\mathscr{L}_1 - \mathscr{L}_2)\mathrm{Obs}_o.$$

We will apply the following lemma of de Shalit:

Lemma 4.1.4 ([dS89] Lemma 4.3)**.** *For any $f \in \mathcal{O}(k,\mathcal{H}_F)$, let $e = (v_1,v_2) \in \mathcal{T}_F^1$ be an edge. Fix two lifts $F^{\mathscr{L}_1} \in \mathcal{O}_{\mathscr{L}_1}\otimes \mathscr{P}_n$ and $F^{\mathscr{L}_2} \in \mathcal{O}_{\mathscr{L}_2}\otimes \mathscr{P}_n$ of $f(z)(T-z)^n dz$ to $\mathcal{O}_{\mathscr{L}_i}(k,\mathcal{H}_F)$, and let $\varphi_f \in C^1_{F,har}(\mathscr{P}_n)$ be the harmonic cochain associated to f. Then for any $z_i \in \mathrm{red}^{-1}(v_i)$*

$$(F^{\mathcal{L}_1}(z_1) - F^{\mathcal{L}_1}(z_2)) - (F^{\mathcal{L}_2}(z_1) - F^{\mathcal{L}_2}(z_2)) = (\mathscr{L}_1 - \mathscr{L}_2)\varphi_f(e).$$

Proof of Proposition 4.1.3. Following de Shalit's proof [dS89, Proposition 4.2], fix $\tau \in \mathcal{H}_F$ with $\mathrm{red}_F(\tau) = v_0$ and fix two lifts

$$J^{\mathscr{L}_i} : \Gamma \longrightarrow \mathcal{O}_{\mathcal{L}_i}\otimes \mathscr{P}_n(\mathbb{C}_p),\ i\in\{1,2\},$$

of the cocycle $\gamma \mapsto J(\gamma)(z)(T-z)^n\, dz$. The value of the 2-cocycle

$$\mathrm{Obs}_{\mathscr{L}}(J)(\gamma_1,\gamma_2) - \mathrm{Obs}_{\mathscr{L}}(J)(\gamma_1,\gamma_2)$$

is given by

$$\begin{aligned}&\left(J^{\mathscr{L}_1}(\gamma_1\gamma_2)(\tau) - \left(J^{\mathscr{L}_1}(\gamma_1)\mid\gamma_2\right)(\tau) - J^{\mathscr{L}_1}(\gamma_2)(\tau)\right)\\ -&\left(J^{\mathscr{L}_2}(\gamma_1\gamma_2)(\tau) - \left(J^{\mathscr{L}_2}(\gamma_1)\mid\gamma_2\right)(\tau) - J^{\mathscr{L}_2}(\gamma_2)(\tau)\right).\end{aligned}$$

The function $G : \Gamma \longrightarrow \mathscr{P}_n(\mathbb{C}_p)$ defined by

$$G(\gamma) := J^{\mathscr{L}_1}(\gamma)(\tau) - J^{\mathscr{L}_2}(\gamma)(\tau)$$

produces a co-boundary

$$\gamma_1, \gamma_2 \mapsto G(\gamma_1\gamma_2)(\tau) - G(\gamma_1)(\tau)|\gamma_2 - G(\gamma_2)(\tau).$$

By subtracting this co-boundary, we obtain a co-homologous cocycle given by

$$\gamma_1, \gamma_2 \mapsto \left[\left(J^{\mathscr{L}_1}(\gamma_1)(\gamma_2\tau) - J^{\mathscr{L}_1}(\gamma_1)(\tau)\right) - \left(J^{\mathscr{L}_2}(\gamma_1)(\gamma_2\tau) - J^{\mathscr{L}_2}(\gamma_1)(\tau)\right)\right] \mid \gamma_2.$$

Let $(w_1, \cdots, w_h)$ be the path on $\mathcal{T}_K$ connecting w_1 and γw_h and pick a collection of points $(z_1, \cdots, z_h)$ such that $z_1 = \tau$, $z_h = \gamma\tau$ and for all i, $\mathrm{red}_K(z_i) = w_i$. Applying Lemma 4.1.4 to the telescoping sum

$$\sum_{i=1}^{h-1} \left[\left(J^{\mathscr{L}_1}(\gamma_1)(z_{i+1}) - J^{\mathscr{L}_1}(\gamma_1)(z_i)\right) - \left(J^{\mathscr{L}_2}(\gamma_2)(z_{i+1}) - J^{\mathscr{L}_2}(\gamma_1)(z_i)\right)\right] \mid \gamma_2,$$

we obtain

$$\begin{aligned} &\left(\mathrm{Obs}_{\mathscr{L}_1} - \mathrm{Obs}_{\mathscr{L}_2}\right)(J)(\gamma_1, \gamma_2) + \text{ (co-boundary)} \\ &= \sum_{i=1}^{h-1} \varphi(\gamma_1)((v_i, v_{i+1}))|\gamma_2 = (\mathscr{L}_1 - \mathscr{L}_2)\mathrm{Obs}_o(J)(\gamma_1, \gamma_2). \end{aligned}$$

□

Lemma 4.1.5. *Let $\mathcal{D}$ be an RM-divisor. If $\mathcal{D}$ is of strong degree zero in weight k, then* $\mathrm{Obs}_o(J_{k,\mathcal{D}})$ *is trivial.*

Proof. By definition of $J_{k,\mathcal{D}}$ (Definition 3.5.5), its associated harmonic cochain $\Phi_{k,\mathcal{D}}$ vanishes on every $\mathbb{Q}_p$-edge. For any $\gamma \in \Gamma$, the path $(v_0, \gamma v_0)$ passes only through $\mathbb{Q}_p$-edges, so from the definition of Obs_o as a sum over the path $(v_0, \gamma v_0)$ it is follows that $\mathrm{Obs}_o(J_{k,\mathcal{D}})$ is trivial. □

Lemma 4.1.6. *For every RM-divisor $\mathcal{D}$, there exists a unique cohomology class*

$$J_{k,\mathcal{D}} \in H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p, K)^b),$$

that satisfies $\mathrm{rdiv}_k(J_{k,\mathcal{D}}) = \mathrm{Div}_{k,\mathcal{D}}$ *and* $\mathrm{Obs}_o(J_{k,\mathcal{D}}) = 0$. *(See Definition 2.7.15 for the definition of the residual divisor map* rdiv_k*.)*

Proof. By Theorem 3.5.8, pick an arbitrary cohomology class $J'_{k,\mathcal{D}}$ which has divisor $\mathcal{D}$. The map $\mathrm{Obs}_o : H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b) \longrightarrow H^2(\Gamma, \mathscr{P}_n)$ is an isomorphism by Lemma 3.2.7. Therefore, there exists a unique cohomology class $J \in H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)^b)$ such that $\mathrm{Obs}_o(J) = \mathrm{Obs}_o(J'_{k,\mathcal{D}})$. Therefore,

$$\mathrm{Obs}_o(J'_{k,\mathcal{D}} - J) = 0,$$

and the divisor of $J'_{k,\mathcal{D}} - J$ is $\mathrm{Div}_{k,\mathcal{D}}$. □

Lemma 4.1.7. *If $\mathcal{D}$ is a divisor of strong degree zero in weight k, then for each r, s define a function $J^{\mathscr{L}}_{k,\mathcal{D}}$ by*

$$J^{\mathscr{L}}_{k,\mathcal{D}}\{r,s\}(z) = \langle J_{k,\mathcal{D}}\{r,s\}, l_z(t)\rangle_M\,,$$

with

$$l_z(t) = \begin{cases} -\frac{1}{n!}(t-z)^n \log_{\mathscr{L}}(t-z) & \textit{if } |t| \le 1 \\ -\frac{1}{n!}(t-z)^n \log_{\mathscr{L}}\left(1 - \frac{z}{t}\right) & \textit{if } |t| > 1. \end{cases}$$

Then $J^{\mathscr{L}}_{k,\mathcal{D}}\{r,s\}$ belongs to $\mathcal{M}_{\mathscr{L}}(2-k, \mathcal{H}_p, K)$ and satisfies

$$d^{n+1} J^{\mathscr{L}}_{k,\mathcal{D}}\{r,s\} = J_{k,\mathcal{D}}\{r,s\}. \tag{4.4}$$

Proof. Let $\mu_{k,\mathcal{D}}$ be the measure associated to $J_{k,\mathcal{D}}$, then the Morita pairing can be computed by

$$\langle J_{k,\mathcal{D}}\{r,s\}, l_z(t)\rangle_M = \int_{\mathbb{P}^1(K)} l_z(t) d\mu_{k,\mathcal{D}}.$$

We will expand this integral just as in the proof of Proposition 2.7.6. Let $\mathfrak{R}_l$ be a set of representatives for $\mathbb{P}^1(\mathbb{Q}_p)$ modulo p^l and let $\mathfrak{T}_l$ be the finite set of RM-points in $K \backslash \mathbb{Q}_p \cap$

$\mathcal{H}_p^{<l}$ which lie in the support of $\mathrm{Div}_{k,\mathcal{D}}\{r,s\}$. For each $w \in \mathfrak{T}_l$, let m_w be the multiplicity of $\Gamma \cdot w$ in $\mathcal{D}$ and let U_w be an open subset of $K\backslash\mathbb{Q}_p \cap \mathcal{H}_p^{<l}$ which intersects $\mathfrak{T}_l$ exactly at w. For every $\alpha \in \mathfrak{R}_l$ let ν be such that $\nu' + h = l$. Then we have

$$\int_{\mathbb{P}^1(K)} l_z(t) d\mu_{k,\mathcal{D}} = \sum_{\alpha \in \mathfrak{R}_l} \int_{D(\alpha,\nu)} l_z(t) d\mu_{k,\mathcal{D}}(t) + \sum_{\tau \in \mathfrak{T}_l} \int_{U_\tau} l_z(t)\ d\mu_{k,\mathcal{D}}(t).$$

For $t \in \mathbb{P}^1(K)$ close enough to some point $\alpha \in \mathbb{P}^1(K)$ we can write,

$$\begin{aligned}(t-z)^n \log_{\mathscr{L}}(t-z) &= (t-z)^n \log_{\mathscr{L}}(z-\alpha) + (t-z)^n \log_{\mathscr{L}}\left(1 - \frac{t-\alpha}{z-\alpha}\right)\\ &= (t-z)^n \log_{\mathscr{L}}(z-\alpha) - (t-z)^n \sum_{i=1}^{\infty} \frac{(t-\alpha)^i}{i(z-\alpha)^i}.\end{aligned}$$

For $w \in \mathcal{H}_p^{<l}$ in the support of $\mathrm{Div}_{k,\mathcal{D}}\{r,s\}$ with multiplicity m_w and $U_w \subset K\backslash\mathbb{Q}_p$ which intersects the support of $\mathrm{Div}_{k,\mathcal{D}}\{r,s\}$ only at w, it follows that

$$\begin{aligned}&\int_{U_w} l_z(t)\ d\mu_{k,\mathcal{D}}(t)\\ &= \frac{-m_w}{n!}[(w,\overline{w})\cdot(r,s)] \left\langle (T-z)^n, \frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \right\rangle_{\mathscr{P}_n} \log_{\mathscr{L}}(z-w) + (\text{ a polynomial in } z)\\ &= \frac{-m_w}{n!}[(w,\overline{w})\cdot(r,s)] \frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \log_{\mathscr{L}}(z-w) + (\text{ a polynomial in } z)\,.\end{aligned}$$

Similarly, for $\alpha \in \mathfrak{R}_l$,

$$\begin{aligned}&\int_{D(\alpha,\nu)} l_z(t) d\mu_{k,\mathcal{D}}(t)\\ =&\frac{1}{n!}\sum_{i=1}^{\infty} \frac{\int_{D(\alpha,\nu)}(t-z)^n(t-\alpha)^i d\mu_{k,\mathcal{D}}(t)}{i(z-\alpha)^i} + (\text{ a polynomial in } z)\,.\end{aligned}$$

We have used above that $\int_{D(\alpha,\nu)} P(t)\ d\mu_{k,\mathcal{D}}(t) = 0$ for any P any polynomial of degree at most n. This shows that $J_{k,\mathcal{D}}^{\mathscr{L}}\{r,s\}(z)$ belongs to $\mathcal{M}_{\mathscr{L}}(2-k,\mathcal{H}_p,K)$.

The equality (4.4) follows from the fact that one can move the derivative inside the

Morita duality, i.e.

$$\frac{d^{n+1}}{dz^{n+1}} \langle J_{k,\mathcal{D}}\{r,s\}, l_z(t)\rangle_M = \left\langle J_{k,\mathcal{D}}\{r,s\}, \frac{d^{n+1}}{dz^{n+1}} l_z(t) \right\rangle_M,$$

and we have that

$$\frac{d^{n+1}}{dz^{n+1}}((t-z)^n \log(t-z)) = \frac{-n!}{t-z}.$$

□

Remark 4.1.8. The collection of functions $J^{\mathscr{L}}_{k,\mathcal{D}}\{r,s\}$ does not form a modular symbol, and certainly not a Γ-invariant one.

Remark 4.1.9. From its expression as an infinite sum, one is tempted to find a primitve of $J_{k,\mathcal{D}}$ term by term and add them in a sum of the form

$$\sum_{w\in\Sigma_\tau\{r,s\}} [(w,\overline{w})\cdot(r,s)] \frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \log_{\mathscr{L}}(z-w).$$

However, this sum does not converge. It only converges "up to polynomials". That is, on every affinoid, the coefficients of a Mittag-Leffler expansion of this sum converge except for the terms of the form $c_i z^i$ where $0 \le i \le n$.

Lemma 4.1.10. *If $\mathcal{D} = \sum_i m_i \tau_i$ is of strong degree zero in weight k, then for each r,s the infinite sum*

$$\frac{1}{n!}\sum_i \sum_{w\in\Sigma_{\tau_i}\{r,s\}} m_i[(w,\overline{w}).(r,s)] \left(\frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \log_{\mathscr{L}}(t_w(z)) + p_w(z) \right)$$

converges to $J^{\mathscr{L}}_{k,\mathcal{D}}\{r,s\}(z)$. Where $t_w(z)$ is as in (3.9) *and if $|w| \le 1$ then*

$$p_w(z) = -\sum_{r=1}^{n/2}\left(\sum_{i=1}^{r-1}\left(\frac{(-1)^{n-i}\binom{n}{i}}{(r-i)} \right) \frac{(-1)^r \binom{n-r}{n/2-r}(w-\overline{w})^{r-n/2}}{\binom{n}{n/2}}(z-w)^{n-r} \right),$$

and otherwise

$$p_w(z) = -\sum_{r=1}^{n/2}\left(\sum_{i=1}^{r-1}\left(\frac{(-1)^{n-i}\binom{n}{i}}{(r-i)}\right)\frac{(-1)^r\binom{n-r}{n/2-r}(w-\overline{w})^{r-n/2}}{\binom{n}{n/2}}(z-w)^{n-r}\right)$$

$$+\sum_{i=n/2+1}^{n}(-1)^i\binom{n}{i}\left(\sum_{j=1}^{i-n/2}\frac{(-1)^j(-1)^{n-i+j}\binom{i-j}{i-j-n/2}(w-\overline{w})^{n/2-i+j}}{\binom{n}{n/2}jw^j}\right)(z-w)^i.$$

Proof. The proof of this lemma is very similar to that of Lemma 4.1.7 except one has to write the polynomial part explicitly. We will follow the notation of the proof of Lemma 4.1.7. From the definition of the integral

$$\int_{\mathbb{P}^1(K)} l_z(t)d\mu_{k,\mathcal{D}}(t),$$

it can be written as a limit of Riemann sums

$$\lim_{l\to\infty}\sum_{w\in\in\mathfrak{T}_k} m_w[(w,\overline{w})\cdot(r,s)]\left\langle\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}, \mathrm{Trunc}_w(l_z(t))\right\rangle.$$

The claim follows by expanding out the truncation as in Lemma 4.1.7. □

4.2 Duality Pairing of Cocycles

We would like to evaluate higher weight cocycles at certain divisors. In the following, we present the analogue of the evaluation of a multiplicative rigid meromorphic cocycle at a point away from the poles. Because the higher weight theory is additive, we will define a collection of related pairings associated to the functionals $\log_{\mathscr{L}}$ and ord_p. Morita and Breuil duality can be extended to evaluate at divisors which lie in the boundary, and are away from the poles of the cocycle. We will use these pairings to define the higher Green's function.

Fix $\mathscr{L}\in\mathbb{C}_p$. The short exact sequence

$$0\longrightarrow\Sigma(n,\mathbb{Q}_p)\longrightarrow\Sigma_{\mathscr{L}}(n,\mathbb{Q}_p)\longrightarrow\mathscr{P}_n(\mathbb{C}_p)\longrightarrow 0$$

leads to a long exact sequence in Γ-homology

$$\cdots \leftarrow H_1(\Gamma, \Sigma_{\mathscr{L}}(n, \mathbb{Q}_p)) \leftarrow H_1(\Gamma, \Sigma(n, \mathbb{Q}_p)) \leftarrow H_2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)) \leftarrow \cdots . \tag{4.5}$$

Let $i_{\mathscr{L}}$ be the inclusion

$$\begin{aligned} i_{\mathscr{L}} : \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n &\to \Sigma_{\mathscr{L}}(n, \mathbb{Q}_p) \\ (x) \otimes P &\mapsto P(t) \log_{\mathscr{L}}(t - x) . \end{aligned}$$

The map $i_{\mathscr{L}}$ sends $\mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n$ to $\Sigma(n, \mathbb{Q}_p)$. The combination of $i_{\mathscr{L}}$ with the Morita pairing defines a pairing

$$\langle\ ,\ \rangle_M : H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)) \times H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow \mathbb{C}_p.$$

The Breuil pairing extends this pairing to

$$\langle\ ,\ \rangle_B : H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_p)) \times H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow \mathbb{C}_p.$$

Further, there is also a pairing for the p-adic valuation

$$\langle\ ,\ \rangle_o : H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p)) \times H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \longrightarrow \mathbb{C}_p,$$

which is induced from a pairing

$$\mathcal{O}(k, \mathcal{H}_p) \times (\mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow \mathbb{C}_p,$$

that is defined as follows: Given $f \in \mathcal{O}(k, \mathcal{H}_p)$ with associated harmonic cochain $\phi_f \in C^1_{p,har}(\mathscr{P}_n)$. There is a canonical inclusion

$$C^1_{p,har}(\mathscr{P}_n) \longrightarrow C^1_{K,har}(\mathscr{P}_n)$$

obtained from the inclusion of $\mathbb{R}$-trees

$$|\mathcal{T}_p| \longrightarrow |\mathcal{T}_K|$$

and which is described as follows: For $\varphi \in C^1_{p,har}(\mathscr{P}_n)$, let $\tilde{e} \in \mathcal{T}^1_K$ be an oriented $\mathbb{Q}_p$-edge whose geometric realization is a subset of the realization of $e \in \mathcal{T}^1_p$ with the same orientation, then set

$$\tilde{\varphi}(\tilde{e}) = \frac{1}{2}\phi(e).$$

If $\tilde{e}$ is not a $\mathbb{Q}_p$-edge then set $\tilde{\varphi}(\tilde{e}) = 0$. It follows that $\tilde{\varphi}$ belongs to $C^1_{K,har}(\mathscr{P}_n)$. For

$$D = \sum_i \tau_i \otimes p_i \in \mathrm{Div}_0(K \backslash \mathbb{Q}_p) \otimes \mathscr{P}_n,$$

the pairing between f and D is equal to the sum

$$\langle f, D\rangle_o := \sum_i \sum_{e \in \mathrm{path}(v_0, \tau_i)} \langle \tilde{\phi}_f(e), p_i\rangle_{\mathscr{P}_n}.$$

Notice that this sum is finite. The degree zero assumption ensures that the pairing above does not depend on the choice of base vertex v_0.

There is also a pairing

$$\langle \cdot, \cdot \rangle_o : C^0_{p,har}(\mathscr{P}_n(\mathbb{C}_p)) \times (\mathrm{Div}(K \backslash \mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow \mathbb{C}_p$$

defined as follows: For $\phi \in C^0_{p,har}(\mathscr{P}_n(\mathbb{C}_p))$ and $D \in \mathrm{Div}(K \backslash \mathbb{Q}_p) \otimes \mathscr{P}_n$, write $\mathrm{red}_p(D) = \sum_i w_i \otimes p_i$ and set

$$\langle \phi, D\rangle_o := \sum_i \langle \phi(w_i), p_i\rangle_{\mathscr{P}_n}.$$

If in the above a point τ_i reduces to an edge $w_i \in \mathcal{T}^1_p$ and not to a vertex, then if the edge w_i joins the vertices v and v', set $\phi(w_i)$ to be the average

$$\frac{\phi(v) + \phi(v')}{2}.$$

Definition 4.2.1. For σ an RM-point whose fundamental stabilizer is $\gamma_\sigma \in \Gamma$. The Stark-Heegner cycle associated to σ is

$$y_{k,\sigma} = \gamma_\sigma \otimes (\sigma) \otimes \frac{(T-\sigma)^{n/2}(T-\overline{\sigma})^{n/2}}{(\sigma-\overline{\sigma})^{n/2}} \in H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)).$$

This indeed defines an element in the homology since γ_σ fixes both σ and the polynomial $(T-\sigma)^{n/2}(T-\overline{\sigma})^{n/2}$. Given a cocycle $\mathfrak{J} \in H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_p))$, denote the pairing $\langle \mathfrak{J}, y_{k,\sigma}\rangle_B$ by

$$\mathfrak{J}[\sigma].$$

This pairing is given by

$$\left\langle \mathfrak{J}(\gamma_\sigma)(z), \frac{(t-\sigma)^{n/2}(t-\overline{\sigma})^{n/2}}{(\sigma-\overline{\sigma})^{n/2}} \log_{\mathscr{L}}(t-\sigma) \right\rangle_B.$$

Proposition 4.2.2. *The pairings described above are related by the following commutative diagrams*

$$\begin{tikzcd}
H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k))^\vee & H^1(\Gamma, \mathcal{O}(k))^\vee \arrow[l, "(d^{n+1})^\vee"'] & H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p))^\vee \arrow[l, "\mathrm{Obs}_{\mathscr{L}}^\vee"'] \\
H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \arrow[u] & H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \arrow[l] \arrow[u] & H_2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)) \arrow[l] \arrow[u]
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^1(\Gamma, C^0_{p,har}(\mathscr{P}_n(\mathbb{C}_p)))^\vee & H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n(\mathbb{C}_p)))^\vee \arrow[l] & H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p))^\vee \arrow[l, "\mathrm{Obs}_o^\vee"'] \\
H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \arrow[u] & H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \arrow[l] \arrow[u] & H_2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)) \arrow[l] \arrow[u]
\end{tikzcd}$$

Proof. The horizontal lines in cohomology are obtained from taking the Γ-cohomology of the short exact sequences

$$0 \longrightarrow \mathscr{P}_n \longrightarrow \mathcal{O}_{\mathscr{L}}(2-k) \longrightarrow \mathcal{O}(k) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathscr{P}_n \longrightarrow C^0_{p,har}(\mathscr{P}_n) \longrightarrow C^1_{p,har}(\mathscr{P}_n) \longrightarrow 0.$$

The horizontal lines in homology are obtained from taking the Γ-homology of the short exact sequence

$$0 \longrightarrow \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n \longrightarrow \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n \longrightarrow \mathscr{P}_n \longrightarrow 0.$$

The vertical arrows are the pairings described above in this section. □

Proposition 4.2.3. *The map* $H_2(\Gamma, \mathscr{P}_n(\mathbb{C}_p)) \longrightarrow H^2(\Gamma, \mathscr{P}_n(\mathbb{C}_p))^\vee$ *is an isomorphism.*

Proof. This follows from the Universal Coefficient Theorem. □

Lemma 4.2.4. *The map*

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \longrightarrow H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p))$$

is surjective.

Proof. The co-kernel of this map is contained in $H_1(\Gamma, \mathscr{P}_n)$. The result is a consequence of the vanishing of $H_1(\Gamma, \mathscr{P}_n(\mathbb{C}_p))$, which follows since $H^1(\Gamma, \mathscr{P}_n)$ vanishes by [RS12] Lemma 3.10 and the Universal Coefficient Theorem provides a perfect pairing between $H_1(\Gamma, \mathscr{P}_n)$ and $H^1(\Gamma, \mathscr{P}_n)$. □

Definition 4.2.5. Let $\mathcal{U}$ be the kernel of

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) \longrightarrow H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n(\mathbb{C}_p)))^\vee.$$

A divisor in $\mathcal{U}$ is called a *unitary divisor.*

Corollary 4.2.6. *The homology of degree zero divisors has the decomposition*

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p)) = \delta(H_2(\Gamma, \mathscr{P}_n(\mathbb{C}_p))) \oplus \mathcal{U}.$$

Proof. To see that the two spaces intersect trivially, given a non-zero class $d \in H_2(\Gamma, \mathscr{P}_n)$, its image in $H^2(\Gamma, \mathscr{P}_n)^\vee$ is non-trivial by Proposition 4.2.3. By Lemma 3.2.7, the image of d in $H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n(\mathbb{C}_p)))^\vee$ is also non-zero. Therefore, $\delta(d)$ does not belong to $\mathcal{U}$.

Given a class $c \in H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n)$ there exists a unique $d \in H_2(\Gamma, \mathscr{P}_n)$ such that the images of c and d in $H^1(\Gamma, C^1_{p,har}(\mathscr{P}_n))$ are equal. This implies that $c-\delta(d) \in \mathcal{U}$. The claim follows. $\square$

Definition 4.2.7. It follows from Corollary 4.2.6 that for each $D \in H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p))$ there exists a unique unitary divisor $D^\natural \in \mathcal{U}$ which maps to D.

We would like to slightly extend the pairings described above to pair a meromorphic cocycle $\mathfrak{J} \in H^1(\Gamma, \mathcal{M}_{\mathscr{L}}(2-k, \mathcal{H}_p; K))$ with a divisor that is supported away from the poles of $\mathfrak{J}$. This generalizes the evaluation of the logarithm of Darmon–Vonk's multiplicative rigid cocycles at RM-points.

Definition 4.2.8. Given $f \in \mathcal{M}(k, \mathcal{H}_p, K)$ and $\tau \in \mathbb{P}^1(K)\backslash$. We say that f has no residues near τ if there exists an open neighborhood, $U \subset \mathbb{P}^1(K)$, of τ such that for every edge $e \in \mathcal{T}^1_K$ with $U_e \subset U$, the residue vanishes:

$$\mathrm{red}_e(f(z)(T-z)^n dz) = 0.$$

We say in this case that f has no residues in U. For $G \in \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)$ we say G has no residues near τ if $d^{n+1}G$ has no residues near τ. We extend these definitions to finite divisors, to say that f has no residues on a divisor D on $\mathrm{Div}(\mathbb{P}^1(K))$ if f has no residues near each point in the support of D.

Proposition 4.2.9. *If $f \in \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)$ with*

$$d^{n+1} f \in \mathcal{M}(k, \mathcal{H}_p, K),$$

$\tau \in \mathbb{P}^1(K)\backslash\{\infty\}$ and $P \in \mathscr{P}_n(\mathbb{C}_p)$ such that f has no residue near τ, then the pairing

$$\langle f, \alpha_{\tau\otimes P,U}(t)\rangle_B$$

does not depend on the choice of U. Where U is any neighborhood of τ on which f has no residues and $\alpha_{\tau\otimes P,U} \in \Sigma_{\mathscr{L}}(n, K)$ is defined by

$$\alpha_{\tau\otimes P,U}(t) = \begin{cases} 0 & \textit{if } t \in U, \\ P(t)\log_{\mathscr{L}}(t-\tau) & \textit{otherwise.} \end{cases}$$

Proof. Given any two such neighborhoods, U_1, and U_2, the difference

$$\alpha_{\tau\otimes P,U_1} - \alpha_{\tau\otimes P,U_2}$$

belongs to $\Sigma(n, K)$ and is supported on $U_1 \cup U_2$, then by (2) of Proposition 2.4.5

$$\langle f, \alpha_{\tau\otimes P,U_1} - \alpha_{\tau\otimes P,U_2}\rangle_B = \langle d^{n+1} f, \alpha_{\tau\otimes P,U_1} - \alpha_{\tau\otimes P,U_2}\rangle_M = 0.$$

The last equality follows since $d^{n+1}f$ has associated measure zero on $U_1 \cup U_2$, and the Morita pairing is computed by integrating against the measure. □

We will still use the notation $\langle f, P(t)\log_{\mathscr{L}}(t-\tau)\rangle_B$ to refer to $\langle f, \alpha_{\tau\otimes P,U}\rangle_B$ for some choice of open set U.

The p-adic valuation pairing can also be similarly extended: If $f \in \mathcal{O}(k, \mathcal{H}_K)$ has no residues near a divisor,

$$D = \sum_i (\tau_i) \otimes p_i \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n(\mathbb{C}_p).$$

Then for each i, the sum

$$\sum_{e\in \mathrm{path}(v_0,\tau_i)} \langle \phi_f(e), p_i\rangle_{\mathscr{P}_n}$$

is finite since $\phi_f(e)$ is eventually 0 as the edges approach τ_i. So we set

$$\langle f, D\rangle_o := \sum_i \sum_{e \in \text{path}(v_0,\tau_i)} \langle \phi_f(e), p_i\rangle_{\mathscr{P}_n}.$$

Definition 4.2.10. For a function $\phi \in C^0_{K,har}(\mathscr{P}_n)$, we say that it has no residues near τ if there exists an open neighborhood $U \subset \mathbb{P}^1(K)$ of τ, such that for every $w \in U$, and for every path $(w_0, w_1, \cdots)$ which tends to w the value, $\phi(w_i)$, is eventually constant; denote this constant value by $\phi(w)$. In which case, we define

$$\langle \phi, (\tau) \otimes P\rangle_o := \langle \phi(\tau), P\rangle_{\mathscr{P}_n}.$$

Lemma 4.2.11. *Fix*

$$\sum_i \gamma_i \otimes \sum_j (\sigma_{ij} \otimes P_{ij}) \in Z_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$$

and $F \in \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)$ such that $\sigma_{ij} \in K\backslash\mathbb{Q}_p$, and for each i, the translated function $F|\gamma_i$ has no residues near σ_{ij} for all j. Then

$$\sum_i \left\langle F|\gamma_i, \sum_j \sigma_{ij} \otimes P_{ij}\right\rangle_B = \sum_i \left\langle F, \sum_j \sigma_{ij} P_{ij}\right\rangle_B.$$

Proof. Unpacking the definitions we have

$$\sum_i \left\langle F|\gamma_i, \sum_j \sigma_{ij} \otimes P_{ij}\right\rangle_B = \sum_i \left\langle F|\gamma_i, \sum_j \alpha_{\sigma_{ij}, P_{ij}, U_{ij}}\right\rangle_B$$

$$= \sum_i \left\langle F, \sum_j \alpha_{\sigma_{ij}, P_{ij}, U_{ij}}|\gamma_i^{-1}\right\rangle_B = \sum_i \left\langle F, \sum_j \alpha_{\gamma_i\sigma_{ij}, P_{ij}|\gamma_i^{-1}, \gamma_i U_{ij}}\right\rangle_B$$
$$= \sum_i \left\langle F, \sum_j \alpha_{\sigma_{ij}, P_{ij}, U'_{ij}}\right\rangle_B.$$

The final equality follows from the condition on the divisor belonging to

$$Z_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n).$$

□

Definition 4.2.12. It follows from Lemma 4.2.11 that the extension of the Breuil pairing to meromorphic functions does not depend on the choice of cocycle class since it vanishes on coboundaries. The value

$$\left\langle \mathfrak{J}(\gamma_\sigma), \frac{(t-\sigma)^{n/2}(t-\overline{\sigma})^{n/2}}{(\sigma-\overline{\sigma})^{n/2}} \log_{\mathscr{L}}(t-\sigma) \right\rangle_B$$

depends only on the cocycle class in $\mathfrak{J} \in H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K))$. Denote this value by $\mathfrak{J}[\tau]$.

Similarly, for $J \in H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_K))$ and a degree 0 divisor $\mathfrak{D} \in H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$. Denote the associated pairing by $J[\mathfrak{D}]_{\mathscr{L}}$. Remark that the dependence on $\mathscr{L}$ is from the inclusion

$$\mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n \longrightarrow \Sigma(n, \mathbb{Q}_p)$$

We verify below that this extended pairing when restricted to the setting of Darmon-Vonk's multiplicative cocycles recovers simply the evaluation of the cocycles at RM-points.

Proposition 4.2.13. *If $J^\times \in H^1(\Gamma, \mathcal{M}^\times)$, and τ an RM-point. Assume $\log_{\mathscr{L}}(J^\times)$ has no residues near τ then*

$$\left(\log_{\mathscr{L}}(J^\times)\right)[\tau] = \log_{\mathscr{L}}\left(J^\times[\tau]\right).$$

Proof. Let U be an open neighborhood of τ in which $\mathrm{dlog}(J^\times)$ has no residues, and let V denote $\mathbb{P}^1(K)\backslash U$. Pick any $\alpha \in \mathcal{H}_K$. Write

$$\alpha_{\tau,U}(t) = \log_{\mathscr{L}}(t-\alpha) - \mathbb{1}_U(t)\log_{\mathscr{L}}(t-\alpha) + \mathbb{1}_V(t)\log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right).$$

Consider the pairing of each of the terms on the right hand above separately. From the properties of the Breuil pairing, we see that

$$\left\langle \log_{\mathscr{L}}(J^{\times})(\gamma_\tau), \log_{\mathscr{L}}(t-\alpha)\right\rangle_B = \log_{\mathscr{L}}(J^{\times}(\gamma_\tau)(\alpha)).$$

Next, observe that $\mathbb{1}_U(t)\log_{\mathscr{L}}(t-\alpha)$ belongs to $C(n,K)$, and therefore

$$\left\langle \log_{\mathscr{L}}(J^{\times})(\gamma_\tau), \mathbb{1}_U(t)\log_{\mathscr{L}}(t-\alpha)\right\rangle_B = \left\langle \mathrm{dlog}_{\mathscr{L}}(J^{\times})(\gamma_\tau)(z), \mathbb{1}_U(t)\log_{\mathscr{L}}(t-\alpha)\right\rangle_M.$$

But the Morita pairing is calculated via integrating against a measure. Note that the measure involved vanishes on U, and the integrand $\mathbb{1}_U(t)\log_{\mathscr{L}}(t-\alpha)$ is supported on U, and therefore this term vanishes.

For the final term, again $\mathbb{1}_V(t)\log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right)$ belongs to $C(n,K)$, and so we are left to compute

$$\left\langle \mathrm{dlog}_{\mathscr{L}}(J^{\times})(\gamma_\tau), \mathbb{1}_V(t)\log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right)\right\rangle_M.$$

Since $\mathrm{dlog}(J^{\times})(\gamma_\tau)$ has no residues in U,

$$\left\langle \mathrm{dlog}_{\mathscr{L}}(J^{\times})(\gamma_\tau), \mathbb{1}_V(t)\log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right)\right\rangle_M = \left\langle \mathrm{dlog}_{\mathscr{L}}(J^{\times})(\gamma_\tau), \log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right)\right\rangle_M.$$

The result follows if (2.4) holds for this extended pairing. That is, if the expression above equals

$$G(\tau) - G(\alpha)$$

for G any primitive of $\mathrm{dlog}_{\mathscr{L}}(J^{\times})(\gamma_\tau)$. We take $G = \log_{\mathscr{L}}(J^{\times}(\gamma_\tau))$, and conclude

$$\begin{aligned}\left\langle \log_{\mathscr{L}}(J^{\times})(\gamma_\tau), \alpha_{\tau,U}\right\rangle_B &= \log_{\mathscr{L}}(J^{\times}(\gamma_\tau)(\alpha)) + 0 + (\log_{\mathscr{L}}(J^{\times}(\gamma_\tau)(\tau)) - \log_{\mathscr{L}}(J^{\times}(\gamma_\tau)(\alpha)))\\ &= \log_{\mathscr{L}}(J^{\times}(\gamma_\tau)(\tau)).\end{aligned}$$

We have to show that for any $f \in \mathcal{O}(2,\mathcal{H}_K)$ with no residues near τ, then

$$\left\langle f, \log_{\mathscr{L}}\left(\frac{t-\tau}{t-\alpha}\right)\right\rangle_M = F(\tau) - F(\alpha) \tag{4.6}$$

for any primitive F of f. Rigid functions of the form $\frac{1}{z-\beta}$, for $\beta \in K$, are dense in $\mathcal{O}(2, \mathcal{H}_K)$. So we only need to verify (4.6) for $f(z) = \frac{1}{z-\beta}$ with $\beta \neq \tau$. The measure associated to f is the Dirac measure at β and so

$$\int_{\mathbb{P}^1(K)} \log_{\mathscr{L}} \left(\frac{t-\tau}{t-\alpha} \right) \, d\mu_f = \log_{\mathscr{L}}(\tau - \beta) - \log_{\mathscr{L}}(\alpha - \beta).$$

□

Proposition 4.2.14. *Let* $J^\times \in H^1(\Gamma, \mathcal{M}^\times)$*, and* $J = \mathrm{dlog}(J^\times)$*. Define a cocycle valued in functions on vertices of* $\mathcal{T}_K$ *by*

$$\tilde{\varphi}(\gamma)(v) = \mathrm{ord}_p(J^\times(\gamma)(z)) \quad \textit{for any } z \in \mathrm{red}_K^{-1}(v).$$

Then $\tilde{\varphi}$ *is well-defined and its image in* $C^1_K(\mathscr{P}_n)$ *under the map given in (4.3) equals* φ_J*, the residue cocycle associated to* J*. Finally,*

$$\langle \tilde{\varphi}, \gamma_\tau \otimes (\tau) \rangle_o = \mathrm{ord}_p(J^\times[\gamma_\tau]).$$

Proof. That $\tilde{\varphi}$ is well-defined can be proven directly just as in de Shalit's Lemma 4.1.4. Alternatively, it follows from Lemma 4.1.4 applied to $\log_{\mathscr{L}_1}(J^\times)$ and $\log_{\mathscr{L}_2}(J^\times)$ and with z_1, z_3 reducing to the same vertex and z_2 reducing to a neighboring vertex. The identity

$$\log_{\mathscr{L}_1} - \log_{\mathscr{L}_2} = (\mathscr{L}_1 - \mathscr{L}_2)\mathrm{ord}_p$$

would then imply that

$$\mathrm{ord}_p(J^\times(\gamma)(z_1)) = \mathrm{ord}_p(J^\times(\gamma)(z_3)).$$

Lemma 4.1.4 also implies that $\tilde{\varphi}$ is a lift of φ_J.

Write $v = \mathrm{red}_K(\tau)$, then by definition

$$\langle \tilde{\varphi}, \gamma_\tau \otimes (\tau) \rangle_o = \tilde{\varphi}(\gamma_\tau)(v) = \mathrm{ord}_p(J^\times(\gamma_\tau)(\tau)).$$

□

4.3 Higher Green's Functions

For this section, we will fix a choice of branch of p-adic logarithm, $\log_{\mathscr{L}}$. Recall that $\mathcal{RM}$ denotes the free $\mathbb{Z}$-module of orbits of RM-points on $\mathcal{H}_p$. For any $\mathcal{D} \in \mathcal{RM}$ there is a uniquely defined cocycle

$$J_{k,\mathcal{D}} \in H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p, K)),$$

whose divisor is $\mathrm{Div}_{k,\mathcal{D}}$ and such that $\mathrm{Obs}_o(J_{k,D})$ is trivial. We also associate to each such divisor $\mathcal{D}$ a homology class

$$y_{k,\mathcal{D}} \in H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n),$$

and by Lemma 4.2.6 a unique unitary lift

$$y^{\natural}_{k,D} \in \mathcal{U} \subset H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n).$$

Definition 4.3.1. For a pair of disjoint RM-divisors $\mathcal{D}_1, \mathcal{D}_2 \in \mathcal{RM}$ define the higher Green's function by

$$G_k^{\mathscr{L}}(\mathcal{D}_1, \mathcal{D}_2) = J_{k,\mathcal{D}_1}[y^{\natural}_{k,\mathcal{D}_2}]_{\mathscr{L}}.$$

Definition 4.3.2. An RM-divisor $\mathcal{D}$ is *principal* if for every $\mathscr{L} \in \mathbb{C}_p$,

$$\mathrm{Obs}_{\mathscr{L}}(J_{k,\mathcal{D}}) = 0. \tag{4.7}$$

Equivalently, by Lemma 4.1.3 it suffices that (4.7) is satisfied for a single $\mathscr{L} \in \mathbb{C}_p$.

Remark 4.3.3. It follows from the definition of the obstruction map that if

$$\mathrm{Obs}_{\mathscr{L}}(J_{k,\mathcal{D}}) = 0$$

for some $\mathscr{L} \in \mathbb{C}_p$, then there exists a cocycle $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}} \in H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K))$ such that

$$d^{n+1}\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}} = J_{k,\mathcal{D}}.$$

Proposition 4.3.4. *If there exists a lift $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1} \in H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K))$ such that*

$$d^{n+1}\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1} = J_{k,\mathcal{D}_1},$$

then

$$G_k^{\mathscr{L}}(\mathcal{D}_1, \mathcal{D}_2) = J_{k,\mathcal{D}_1}[y^{\natural}_{k,\mathcal{D}_2}]_{\mathscr{L}} = \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\left[y_{k,\mathcal{D}_2}\right].$$

Proof. This follows from the commutativity of the diagram in Proposition 4.2.2. □

Lemma 4.3.5. *For an RM-divisor $\mathcal{D}$, if there exists a lift*

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}} \in H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)),$$

then this lift is unique and is in the image of

$$\mathrm{MS}^{\Gamma}(\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)) \longrightarrow H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)).$$

Proof. The uniqueness follows from the exact sequence

$$H^0(\Gamma, \mathscr{P}_n) \to H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)) \xrightarrow{\mathrm{Obs}_{\mathscr{L}}} H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_K))$$

and the fact that $H^0(\Gamma, \mathscr{P}_n) = 0$. The existence of a parabolic lift follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{MS}^{\Gamma}(\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)) \arrow[r] \arrow[d] & \mathrm{MS}^{\Gamma}(\mathcal{O}(k, \mathcal{H}_K)) \arrow[r] \arrow[d] & H^1(\Gamma, \mathrm{MS}(\mathscr{P}_n)) \arrow[d] \\
H^1(\Gamma, \mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K)) \arrow[r] & H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_K)) \arrow[r] & H^2(\Gamma, \mathscr{P}_n)
\end{tikzcd}$$

and Proposition 3.2.8 showing that the rightmost vertical map is an isomorphism. □

5

Computations and Conjectures

5.1 Efficient Computation

In this section, we will describe the computations of values

$$G_k^{\mathscr{L}}(\mathcal{D}_1, \mathcal{D}_2),$$

where $\mathrm{Deg}_{k,\mathcal{D}_1}$ is trivial and $\mathcal{D}_1$ is principal. We follow a similar approach to that of Darmon-Vonk [DV21] Section 3.5 to efficiently compute rigid meromorphic cocycles. The main difference in this higher weight analogue is that we work with an additive version corresponding to the logarithm of their multiplicative cocycles.

Since $\mathcal{D}_1$ is principal, by Lemma 4.3.5, there exists a unique

$$\mathfrak{J}_{k,\mathcal{D}_1}^{\mathscr{L}} \in \mathrm{MS}^{\Gamma}(\mathcal{O}_{\mathscr{L}}(2-k, \mathcal{H}_K))$$

such that

$$d^{n+1}\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1} = J_{k,\mathcal{D}_1}.$$

Our first aim is to compute $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$, which by Proposition 3.1.6 determines the modular symbol. Recall the explicit functions $J^{\mathscr{L}}_{k,\mathcal{D}_1}\{r,s\}$ defined in Lemma 4.1.7. The functions are individual primitives of $J_{k,\mathcal{D}_1}\{r,s\}$, but they do not necessarily form a modular symbol. Since the functions $J^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$ and $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$ are both primitives of $J_{k,\mathcal{D}_1}\{0,\infty\}$, the difference

$$J^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\} - \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$$

is a polynomial of degree at most $k-2$. We will compute $J^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$ and use the two-term and three-term relations ((5.3) and (5.4)) to adjust it by a polynomial to obtain $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$. Finally, the function $J^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$ is determined by its restriction to the standard affinoid of $\mathcal{H}_p$. So we will compute it as a sum of a Mittag–Leffler expansion on the standard affinoid and a finite term involving linear combinations of products of polynomials and logarithms.

By Lemma 4.1.10 the function $J^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$ is expressed as

$$\lim_{j\to\infty}\sum_i \sum_{w\in\Sigma^{\leq j}_{\tau_i}\{0,\infty\}} m_i[(w,\overline{w})\cdot(0,\infty)]\left(\frac{(z-w)^{n/2}(z-\overline{w})^n/2}{(w-\overline{w})^n/2}\log_{\mathscr{L}}(t_w(z))+p_w(z)\right).$$

In theory, one could simply compute this limit, however, the size of $\Sigma^{\leq j}_{\tau_i}\{0,\infty\}$ grows exponentially, and quickly becomes infeasible to compute. Recall that $\Phi_{k,\mathcal{D}}$ is the harmonic cochain associated to $J_{k,\mathcal{D}}$. We will instead define several truncations of $\Phi_{k,\mathcal{D}}$ which satisfy a recursive relation. Define

$$\Phi_{k,\mathcal{D},l}\{r,s\}(e) := \sum_i \sum_{\substack{w\in\Sigma_{\tau_i}\{r,s\}\cap U_e\\ \text{level}(w)=l}} m_i[(w,\overline{w})\cdot(r,s)]\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}},$$

and

$$\Phi^{(a)}_{k,\mathcal{D},l}\{r,s\}(e) := \sum_i \sum_{\substack{w\in\Sigma_{\tau_i}\{r,s\}\cap U_e \\ \text{level}(w)=l \\ w\equiv a \mod p}} m_i[(w,\overline{w})\cdot(r,s)]\frac{(T-w)^{n/2}(T-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}}.$$

Note that

$$\Phi_{k,\mathcal{D}} = \sum_{l=0}^{\infty} \Phi_{k,\mathcal{D},l},$$

and for $l \geq 1$ there is a decomposition

$$\Phi_{k,\mathcal{D},l} = \sum_{a\in\mathbb{P}^1(\mathbb{F}_p)} \Phi^{(a)}_{k,\mathcal{D},l}.$$

Definition 5.1.1. Define the involution ϖ on RM divisors by

$$\varpi : [\tau] \to -[p\tau]$$

Remark 5.1.2. The union of the two Γ-orbits $[\tau]$ and $[p\tau]$ is the $\mathrm{GL}_2^+(\mathbb{Z}[1/p])$-orbit of τ.

Lemma 5.1.3. *If $a \in \mathbb{F}_p$, then*

$$\Phi^{(a)}_{k,-\varpi\mathcal{D},l+1}\{0,\infty\} = \sum_{b\in\mathbb{F}_p}\left(\Phi^{(b)}_{k,\mathcal{D},l} \mid \begin{pmatrix} p & a \\ 0 & 1\end{pmatrix}\right)\{0,\infty\},$$

and

$$\Phi^{(\infty)}_{k,-\varpi\mathcal{D},l+1}\{0,\infty\} = \left(\Phi^{(\infty)}_{k,\mathcal{D},l} \mid \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\right)\{0,\infty\} + \sum_{b\in\mathbb{F}_p^\times}\left(\Phi^{(b)}_{k,\mathcal{D},l} \mid \begin{pmatrix} 1 & 0 \\ 0 & p\end{pmatrix}\right)\{0,\infty\}.$$

Proof. Observe that an RM-point w is congruent to $a \neq \infty$, has level $l+1$ and satisfies

$$w > 0 > \overline{w}$$

if and only if $\frac{w-a}{p}$ is not congruent to $\infty \mod p$, is of level l and satisfies

$$\frac{w-a}{p} > -\frac{a}{p} > \frac{\overline{w}-a}{p}.$$

Similarly, if $w \equiv \infty \mod p$, then w has level l and satisfies $w > 0 > \overline{w}$ if and only if pw is not congruent to $0 \mod p$, has level $l+1$ and satisfies $pw > 0 > p\overline{w}$. □

Define the functions associated with these truncations

$$F_{k,\mathcal{D},l} := \langle \Phi_{k,\mathcal{D},l+1},\ (t-z)^n \log_0(l_z(t)) \rangle \,,$$

and

$$F^{(a)}_{k,\mathcal{D},l} := \left\langle \Phi^{(a)}_{k,\mathcal{D},l+1},\ (t-z)^n \log_0(l_z(t)) \right\rangle .$$

Then

$$J^{\mathscr{L}}_{k,\mathcal{D}} = \sum_{l=0}^{\infty} F_{k,\mathcal{D},l}$$

and for $l \geq 1$ we have

$$F_{k,\mathcal{D},l} = \sum_{a \in \mathbb{P}^1(\mathbb{F}_p)} F^{(a)}_{k,\mathcal{D},l}$$

The level 0 function $F_{k,\mathcal{D},0}\{r,s\}$ can be written as

$$F_{k,\mathcal{D},0}\{r,s\} = \sum_i \sum_{w \in \Sigma_{\tau_i}\{r,s\}} m_i[(w,\overline{w})\cdot(r,s)] \frac{(z-w)^{n/2}(z-\overline{w})^{n/2}}{(w-\overline{w})^{n/2}} \log_{\mathscr{L}}(z-w) + P_0(z)$$

where P_0 is some polynomial of degree at most n. Then Lemma 5.1.3 implies

Proposition 5.1.4.

$$F^{(a)}_{k,-\varpi\mathcal{D},l+1}\{0,\infty\} - \sum_{b\in\mathbb{F}_p} \left(F^{(b)}_{k,\mathcal{D},l} \mid \begin{pmatrix} p & a \\ 0 & 1 \end{pmatrix} \right) \{0,\infty\} \tag{5.1}$$

and

$$F^{(\infty)}_{k,-\varpi\mathcal{D},l+1}\{0,\infty\} - \left(F^{(\infty)}_{k,\mathcal{D},l} \mid \begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\right)\{0,\infty\} - \sum_{b=1}^{p-1}\left(F^{(b)}_{k,\mathcal{D},l} \mid \begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\right)\{0,\infty\} \quad (5.2)$$

are polynomials of degree at most n*.*

Proposition 5.1.4 allows us to compute $J^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\}$ modulo polynomials by recursively computing $F^{(a)}_{k,\mathcal{D},l}\{0,\infty\}$. Note that the elements $F^{(b)}_{k,\mathcal{D},l}\{-\frac{a}{p},\infty\}$ can be computed modulo polynomials by finding a unimodular path between the cusps $-\frac{a}{p}$ to ∞, and then using the $\mathrm{SL}_2(\mathbb{Z})$-invariance of $\{F^{(a)}_{k,\mathcal{D},l}\}_a$.

Once we have computed $J^{\mathscr{L}}_{k,\mathcal{D}_1}$ modulo the addition of a polynomial we can use the two term relation,

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\}|S + \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\} = 0, \quad (5.3)$$

and the three term relation,

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\}|U^2 + \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\}|U + \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}}\{0,\infty\} = 0, \quad (5.4)$$

to find a polynomial which fixes the Γ-invariance. This amounts to solving a set of simultaneous linear equations.

Example 5.1.5. When $k = 4$ and $n = 2$, we have computed a function

$$G(z) = \mathfrak{J}^{\mathscr{L}}_{4,\mathcal{D}}\{0,\infty\}(z) + P(z),$$

for some quadratic polynomial $P = az^2 + bz + c$. Now applying the two-term and three-term relations we obtain

$$G + (G|S) = P + (P|S) = (a+c)(z^2+1),$$

and

$$G + (G|U) + (G|U^2) = P + (P|U) + (P|U^2) = (2a+b+2c)(z^2-z+1).$$

This determines the polynomial P up to multiples of $z^2 - 1$.

Observe that the coboundary in $\mathscr{P}_n$ given by

$$\gamma \mapsto (1 \mid \gamma) - 1$$

sends S to $z^2 - 1$, therefore the ambiguity of a multiple of $z^2 - 1$ is a coboundary and does not affect the evaluation.

Algorithm 5.1.6.

- ***Step 1.*** *By computing reduced binary quadratic forms in* $\mathrm{SL}_2(\mathbb{Z})$*-equivalence classes, compute* $\Phi_{k,\mathcal{D}_1,0}\{0,\infty\}$ *and store* $F_{k,\mathcal{D}_1,0}\{0,\infty\}$ *as a tuple of pairs* (w, P) *of points on* $\mathcal{H}_p$ *and polynomials which record the data of functions that are sums of*

$$P(z) \log_{\mathscr{L}}(z - w),$$

  where P *is some polynomial of degree at most* $k-2$*, and* w *an RM-point.*

- ***Step 2.*** *Similarly to the previous step, compute the* $(p+1)$ *functions*

$$F^{(a)}_{k,\mathcal{D}_1,1}\{0,\infty\}$$

  and store them to sufficient precision as a Mittag–Leffler expansion on the standard affinoid.

- ***Step 3.*** *Use the recursion equations* (5.1) *and* (5.2) *to compute*

$$F^{(a)}_{k,\mathcal{D}_1,l}$$

  modulo addition of a polynomial to a sufficiently high level, and sum the resulting functions.

- ***Step 4.*** *Use the two and three-term relations to fix the polynomial part and obtain*

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0,\infty\}$$

as in Example 5.1.5.

- ***Step 5.*** *The function*

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}(\gamma_\sigma)$$

is equal to

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0, \gamma_\sigma \cdot 0\},$$

and can be computed by finding a unimodular sequence (see [DV21, Proposition 1.4]) of $\{s_i\}_{i=1}^r \in \mathbb{P}^1(\mathbb{Q})$ *such that*

$$0 = s_1,\ s_r = \gamma_\sigma \cdot 0$$

and there exist matrices $\gamma_i \in \mathrm{SL}_2(\mathbb{Z})$ *such that* $\gamma_i \cdot \{0, \infty\} = \{s_i, s_{i+1}\}$*. Applying the* Γ*-invariance of the modular symbol* $\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}$ *gives*

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}(\gamma_\sigma) = \sum_i \mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}\{0, \infty\} \mid {\gamma_i}^{-1}$$

- ***Step 6.*** *The value of*

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}[y_{\mathcal{D}_2}]$$

is given by the sum

$$\sum_{j=0}^{n/2} \frac{(-1)^i\binom{n-i}{m-i}}{i!\binom{n}{m}} \frac{\left(\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}(\gamma_\sigma)\right)^{(j)}(\sigma)}{(\sigma - \overline{\sigma})^{n/2-j}}. \tag{5.5}$$

Note that (5.5) follows from Corollary 2.6.5.

Remark 5.1.7. Note that the expression in (5.5) is equal to formally applying iterations of the raising operator δ_{-i} and then evaluating at σ:

$$\delta_{2-k}^{(k-2)/2}\left(\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}_1}(\gamma_i)\right)(\sigma),$$

where

$$\delta_i = \frac{\mathrm{d}}{\mathrm{d}z} + \frac{i}{z - \overline{z}}.$$

5.2 Results and Algebraicity Conjectures

We have implemented Algorithm 5.1.6 in the computer algebra system SageMath. The program and instructions on using it are available on the author's website. For this section, we will fix the Iwasawa branch of p-adic logarithm; that is $\mathscr{L} = 0$.

If $p = 3$ and $k = 4$, then the space of newforms $S_4(\Gamma_0(3))^{p-new}$ is trivial. This implies that for every RM-divisor $\mathcal{D}$, there is a unique lift of $J_{k,\mathcal{D}}$ to

$$\mathfrak{J}^{\mathscr{L}}_{k,\mathcal{D}} \in \mathrm{MS}^{\Gamma}(\mathcal{O}_{\mathcal{L}}(-2, \mathcal{H}_K)),$$

and also

$$\mathfrak{J}^{o}_{k,\mathcal{D}} \in \mathrm{MS}^{\Gamma}(C^0_{K,har}(\mathscr{P}_n)).$$

Example 5.2.1. Let $\phi = \frac{1+\sqrt{5}}{2}$ and

$$\mathcal{D}_1 = (9[\phi] - [2\phi]) + \varpi\,(9[\phi] - 2[2\phi])\,.$$

Then we can verify that $\mathrm{Deg}_{4,\mathcal{D}_1}$ is trivial, and therefore we can use Algorithm 5.1.6 to compute

$$\mathfrak{J}^{\mathscr{L}}_{4,\mathcal{D}_1}$$

and its values.

We find that up to 145 digits of 3-adic accuracy

$$G^{\mathscr{L}}_4(\mathcal{D}_1, [4\phi]) = \frac{1}{40}\log_{\mathscr{L}}\left(\left(\frac{i+2}{i-2}\right)^{5}\left(\frac{i+5}{i-5}\right)^{-26}\left(\frac{i+4}{i-4}\right)^{-63}\left(\frac{i+6}{i-6}\right)^{-17}\right), \tag{5.6}$$

and

$$\begin{aligned} G^{\mathscr{L}}_4(\mathcal{D}_1, [7\phi]) = \frac{1}{140}\log_{\mathscr{L}}\Bigg(&\left(\frac{\sqrt{-7}+1}{\sqrt{-7}-1}\right)^{-2241}\left(\frac{\sqrt{-7}+4}{\sqrt{-7}-4}\right)^{-167} \\ &\times\left(\frac{\sqrt{-7}+17}{\sqrt{-7}-17}\right)^{8}\left(\frac{\sqrt{-7}+6}{\sqrt{-7}-6}\right)^{16}\left(\frac{\sqrt{-7}+23}{\sqrt{-7}-23}\right)^{-64}\Bigg). \end{aligned} \tag{5.7}$$

Example 5.2.2. Let

$$\mathcal{D}_1 = \left(7[\sqrt{2}] - 4[2\sqrt{2}]\right) + \varpi\left(7[\sqrt{2}] - 4[2\sqrt{2}]\right).$$

Again, $\mathrm{Deg}_{4,\mathcal{D}_1}$ is trivial. Up to 145 digits of 3-adic accuracy we have

$$G_4^{\mathscr{L}}(\mathcal{D}_1, [4\sqrt{2}]) = \frac{1}{64}\log_{\mathscr{L}}\left(\left(\frac{i+2}{i-2}\right)^{-208}\left(\frac{i+5}{i-5}\right)^{-30}\left(\frac{i+3}{i-3}\right)^{-91}\right). \tag{5.8}$$

$$G_4^{\mathscr{L}}(\mathcal{D}_1, [\phi]) = \frac{1}{\sqrt{160}}\log_{\mathscr{L}}\left(\left(\frac{\sqrt{-10}+4}{\sqrt{-10}-4}\right)^{2}\right). \tag{5.9}$$

Conjecture 5.2.3. *For any odd prime p, and weight $k > 2$. Let*

$$\mathcal{D}_1 = \sum_i m_i[\tau_i]$$

be a principal RM-divisor, and σ an RM-point. Denote by d_{1i} the discriminant of τ_i and d_2 the discriminant of σ. Then there exist algebraic numbers α_i such that for every $\mathscr{L} \in \mathbb{C}_p$ we have

$$G_k^{\mathscr{L}}(\mathcal{D}_1, \sigma) = \mathfrak{J}_{k,\mathcal{D}_1}^{\mathscr{L}}[\sigma] = \sum_i |d_{1i}d_2|^{(2-k)/4}\log_{\mathscr{L}}(\alpha_i),$$

and

$$\mathfrak{J}_{k,\mathcal{D}_1}^{o}[\sigma] = \sum_i |d_{1i}d_2|^{(2-k)/4}\mathrm{ord}_p(\alpha_i),$$

where α_i lies in the compositum of Hilbert class fields $H_{\tau_i} \cdot H_\sigma$.

5.3 Prime Factorization

Keeping the same notation as in Conjecture 5.2.3, we give a conjectural prime factorization for the algebraic numbers appearing in Conjecture 5.2.3. In particular, for a prime $q \neq p$ a prime number we will give a conjectural value for

$$|d_{1i}d_2|^{(2-k)/4}\mathrm{ord}_q(\alpha_i).$$

Let $\mathcal{O}$ be an order of discriminant D in a real-quadratic field where both p and q are not split. Let $B/\mathbb{Q}$ be the quaternion algebra ramified only at p and q, and let $\mathcal{R}$ be a choice of maximal order in B. Then there is a bijection

$$\Gamma\backslash\mathcal{H}_p^D \cong \Sigma(\mathcal{O}, \mathcal{R}),$$

where $\Sigma(\mathcal{O}, \mathcal{R})$ is the set of $\mathcal{R}_1^\times$ conjugacy classes of oriented optimal embeddings of $\mathcal{O}$ into $\mathcal{R}$. Note that such a bijection is not canonical. We will fix bijections which are compatible with the action of the class group. See [DV21] Section 3.4 for the definitions and more details.

Definition 5.3.1. For any two distinct optimal embeddings of quadratic orders

$$\varphi_i : \mathcal{O}_i \to \mathcal{R},\ i \in \{1, 2\},$$

we define

$$[\varphi_1, \varphi_2]_q := \max t \geq 1 \mid \varphi_1(\mathcal{O}_1), \varphi_2(\mathcal{O}_2) \text{ have the same image in } \mathcal{R}/q^{t-1}\mathcal{R}.$$

Conjecture 5.3.2. *Keeping the same notation as Conjecture 5.2.3, let φ_{1i} be embeddings of the orders $\mathcal{O}_{\tau_i}$ associated to $[\tau_i]$ and let φ_2 be an embedding of $\mathcal{O}_\sigma$ associated to σ. Then for a prime $q \neq p$ which is not split in either order $\mathcal{O}_{\tau_i}$ or $\mathcal{O}_\sigma$ and does not divide any of d_{1i} or d_2, there is an embedding of $H_{\tau_i} \cdot H_\sigma$ in $\overline{\mathbb{Q}_q}$ such that*

$$|d_{1i}d_2|^{(2-k)/4}\mathrm{ord}_q(\alpha_i) = \sum_{\gamma \in \gamma_2^{\mathbb{Z}}\backslash\mathcal{R}_1^\times/\gamma_1^{\mathbb{Z}}} [\gamma\varphi_{1i}\gamma^{-1}, \varphi_2]_q \cdot [(\gamma\tau_i, \overline{\gamma\tau_i}) \cdot (\sigma, \overline{\sigma})] \cdot \langle P_{\gamma\tau_i}, P_\sigma \rangle_{P_n}.$$

Otherwise, if q is split in one of the two orders then

$$\mathrm{ord}_q(\alpha_i) = 0.$$

Conjecture 5.3.2 is analogous to Gross-Zagier's factorization of singular moduli and Darmon–Vonk's factorization of $J_p(\tau_1, \tau_2)$ ([DV21, Conjecture 3.27]). More precisely, Darmon–Vonk conjectures that for $k = 2$,

$$\mathrm{ord}_q(\alpha_i) = \sum_{\gamma \in \gamma_2^{\mathbb{Z}} \backslash \mathcal{R}_1^\times / \gamma_1^{\mathbb{Z}}} [\gamma \varphi_{1i} \gamma^{-1}, \varphi_2]_q \cdot [(\gamma\tau_i, \overline{\gamma\tau_i}) \cdot (\sigma, \overline{\sigma})].$$

Our conjecture is a weighted version of Darmon–Vonk's factorization conjecture, where the intersection of the fibers P_τ is taken into account.

The factorization of archimedean higher Green's functions also exhibit a similar relation to the factorization formulas of Gross–Zagier. That is, the factorization of algerbaic values taken by archimedean higher Green's functions can also be written as weighted versions of Gross–Zagier's factorization formula for singular moduli (see [Li22, Theorem A]). In this case, the weighting reflects the fact that higher Green's functions are local contributions to the arithmetic height pairing of Heegner cycles, and that Heegner cycles belong to the fiber over their respective Heegner points. Therefore, the local intersection of Heegner cycles is related to weighted intersections of Heegner points.

6

Future Research and Conclusion

6.1 Measures Associated to RM-Points

For simplicity, let K in the section be an unramified quadratic extension of $\mathbb{Q}_p$ (unique up to isomorphism), and we will take $k = 2$. Then in this case, there is an inclusion of combinatorial tree

$$\mathcal{T}_p \longrightarrow \mathcal{T}_K.$$

A fundamental domain for the action of Γ on $\mathcal{T}_K$ consists of the standard vertex v_0, its usual neighbor v_1, the standard edge e_0 linking those two vertices, and two infinite rays beginning at v_0 and v_1. The stabilizers of the vertices and edges on those two infinite rays are conjugates of the non-split Cartan congruence groups $\Gamma^{\epsilon}_{ns}(p^r)$, where r denotes the distance of the vertex or edge from $\mathcal{T}_p$.

Applying Shapiro's lemma in this setup, we obtain an exact sequence

$$H^1(\Gamma, C^1_{K,har}(\mathbb{Z})) \to H^1(\Gamma_0(p), \mathbb{Z}) \oplus \left(\bigoplus_{r=1}^{\infty} H^1(\Gamma^{\epsilon}_{ns}(p^r), \mathbb{Z}) \right) \to \left(\bigoplus_{r=0}^{\infty} H^1(\Gamma^{\epsilon}_{ns}(p^r), \mathbb{Z}) \right)^2.$$

Therefore, every rigid meromorphic cocycle corresponds to a trace-compatible family of modular forms

$$(f_1^{(r)}, f_2^{(r)}) \in M_2(\Gamma^{\epsilon}_{ns}(p^r))^2.$$

It is natural then to consider the Hecke-module structure of the limit

$$\lim_{\leftarrow} M_2(\Gamma^{\epsilon}_{ns}(p^r)),$$

and to understand if there is a spectral interpretation to the condition on such a family extending to a meromorphic cocycle on $\mathcal{H}_p$.

Another aspect of these measures is to apply them to the construction of p-adic L-functions described in [BD01, Section 6], and study the resulting analytic functions.

6.2 Shimura Curves

Let $B/\mathbb{Q}$ be a definite quaternion algebra that is split at p. Let R be an Eichler $\mathbb{Z}[1/p]$-order of B, and for this section let Γ instead be the group of norm 1 units of R. Since B is split at p, we fix an isomorphism

$$B \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$$

of quadratic spaces, through which we will identify Γ with its image in $\mathrm{SL}_2(\mathbb{Q}_p)$. It is known that Γ is a Schottky subgroup.

Theorem 6.2.1 (Ceregnik-Drinfeld). *The quotient $\mathcal{H}_p/\Gamma$ is naturally identified with the rigid analytic space $X(\mathbb{C}_p)$, where X is a Shimura curve of an appropriate level.*

Given a field extension $K/\mathbb{Q}_p$, we can follow the approach in Section 3.6 to produce a map

$$H_0(\Gamma, \mathrm{Div}_0(K \backslash \mathbb{Q}_p) \otimes \mathscr{P}_n) \longrightarrow H^0(\Gamma, \mathcal{M}(k, \mathcal{H}_p, K)),$$

that assigns to each divisor a rigid meromorphic modular form for Γ whose divisors are supported on the Γ-orbit of the divisor. Applying this construction in weight 2 for a degree 0 divisor $(\tau_1) - (\tau_2)$, we obtain the logarithmic derivatives of Manin-Drinfeld's theta functions

$$\Theta(\tau_1, \tau_2; z) = \prod_{\gamma \in \Gamma} \frac{z - \gamma\tau_1}{z - \gamma\tau_2}.$$

It is possible to imitate the constructions of this thesis to study the p-adic periods of these higher weight meromorphic modular forms. Specializing to the setting of the CM-points on Shimura curves, we obtain a higher Green's function which is the p-adic analogue the archimedean. In fact, for a pair of CM-points we obtain an $(n+1) \times (n+1)$ matrix of functions in line with the generalization of the higher Green's function in [BF25]. We expect algebraicity for values of the middle entry of this matrix when one of the divisors is principal.

More specifically, assume for the sake of simplification that Γ has no fixed points on $\mathcal{T}_p$. The given a divisor $\mathfrak{D} \in H_0(\Gamma, \mathrm{Div}_0(\mathrm{K}\backslash\mathbb{Q}_\mathrm{p}) \otimes \mathscr{P}_\mathrm{n})$ which is represented by some $D \in \mathrm{Div}_0(K\backslash\mathbb{Q}_p)$. Then, we can write

$$\Phi_{\mathfrak{D}} = \sum_{b \in \Gamma} \Phi^0(D)|b^{-1}.$$

This sum simply converges to a Γ-invariant harmonic cochains, which is independent of the choice of D. It can therefore be integrated to give rise to a rigid meromorphic modular form of weight k for Γ.

The main difference is that for a CM-point $\tau \in \mathcal{H}_p$, we have more divisors to investigate. Specifically, we can study divisors of the form

$$(\tau) \otimes (T - \tau)^a (T - \tau)^b$$

for any $a + b = n$.

It is also interesting to investigate if these higher Green's functions appear as motivic periods as in [BF25] and if the middle entries can be interpreted as a kind of local contribution to a p-adic height pairing for local systems on Shimura curves.

Further, the image of the composition

$$H_1(\Gamma, \mathscr{P}_n) \to H_0(\Gamma, \mathrm{Div}_0(K \backslash \mathbb{Q}_p)) \to H^1(\Gamma, \mathcal{M}(k))$$

is the space of weight k analytic modular forms. By considering the periods, we obtain the periods of some p-adic abelian variety associated to the space of weight k modular forms that is analogous to an intermediate Jacobian. These are precisely the periods of [dS89] but with an approach that relies on special meromorphic functions that is a more direct generalization of Manin–Drinfeld's uniformization of the Jacobian of a Shimura curve.

6.3 Algebraicity Conjecture

In Gross–Zagier's paper, CM-values of the complex higher Green's function $G_k(\tau, \sigma)$ are realized as Fourier coefficients of modular forms that are obtained as follows: Let F be the third real quadratic field contained in the compositum $\mathbb{Q}(\sigma) \cdot \mathbb{Q}(\tau)$. There is a family of Hilbert Eisenstein series $E_s(z_1, z_2)$ attached to a genus character χ of the narrow class group of F. This Eisenstein series vanishes at 0. Therefore, its derivative at 0 is modular. The Rankin-Cohen operator

$$\mathcal{C}_k : \Phi(z_1, z_2) \mapsto \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i}^2 \frac{\delta^{k-1}\Phi}{\delta z_1^i \delta z_2^{k-i}}(z, z)$$

maps Hilbert modular forms of weight $(1, 1)$ to classical modular forms of weight $2k$. The holomorphic projection of

$$\mathcal{C}_k \left(\frac{dE}{ds}|_{s=0} \right)$$

is a holomorphic modular form whose Fourier coefficients can be related to CM-values of higher Green's functions. In this way, Gross, Kohnen and Zagier proved that averaged values of higher Green's functions are algebraic.

In the p-adic setting, both [Daa25] and Darmon–Vonk have constructed infinitesimal deformations of Hilbert Eisenstein (or theta) series for which the diagonal restriction of the derivative has an ordinary projection that encodes the weight 2 Green's function in its Fourier coefficients. This suggests an approach to proving algebraicity of (averaged) p-adic higher Green's functions by adapting the Rankin-Cohen operator to overconvergent Hilbert modular forms. In future work, I aim to apply this approach and to explore other p-adic deformations of Hilbert theta series which would relate to individual values of the higher Green's functions.

Anti-symmetry of the Green's function

This appendix is the result of joint work with Isabella Negrini and Vinayak Vatsal, and will appear in an upcoming joint paper. We will present here a proof of the anti-symmetry of the Green's functions. This comes as a consequence of the symmetry of a certain pairing on divisors. In fact, we use this symmetry to prove various other reciprocity results between cocycles and divisors.

Recall that K is the biquadratic extension of $\mathbb{Q}_p$. Let $\tau_1, \tau_2, \sigma_1, \sigma_2$ be distinct elements of $K \backslash \mathbb{Q}_p$, and $P(T)$, $Q(T)$ be any polynomials of degree at most n. Define

$$f(z) = d^{n+1} \left(P(z) \log \left(\frac{z - \tau_1}{z - \tau_2} \right) \right).$$

Then f belongs to $\mathcal{O}(k, \mathcal{H}_K, \mathbb{Q}_p)$. Also, let

$$g(z) = Q(z) \log_{\mathscr{L}} \left(\frac{z - \sigma_1}{z - \sigma_2} \right),$$

then g is a locally analytic function on $\mathbb{P}^1(K)$ with a pole of order at most n at ∞, σ_1 and σ_2 and logarithmic singularities at σ_1 and σ_2. Further, the function f has no residues near σ_1 and σ_2.

Proposition .0.1. *Let* $((\tau_1) - (\tau_2)) \otimes P(T)$ *and* $((\sigma_1) - (\sigma_2)) \otimes Q(T)$ *be two divisors with* *disjoint support in* $\mathrm{Div}_0(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n$*. Then the pairing*

$$\left\langle P(z)\log_{\mathscr{L}}\left(\frac{z-\tau_1}{z-\tau_2}\right),\ Q(t)\log_{\mathscr{L}}\left(\frac{t-\sigma_1}{t-\sigma_2}\right)\right\rangle_B$$

is well-defined on divisors of $\mathrm{Div}_0(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n$ *with disjoint support. Further, it is* Γ*-invariant and symmetric.*

Proof. The map

$$(\sigma_i)\otimes Q(T)\mapsto Q(t)\log_{\mathscr{L}}(t-\sigma_i)$$

is a Γ-invariant morphism $\mathrm{Div}(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n\longrightarrow\Sigma_{\mathscr{L}}(-n)$. On the other hand the map

$$((\tau_1)-(\tau_2))\otimes P(T)\mapsto d^{n+1}\left(P(z)\log_{\mathscr{L}}\left(\frac{z-\tau_1}{z-\tau_2}\right)\right)$$

is a morphism of Γ-modules $\mathrm{Div}_0(K\backslash\mathbb{Q}_p)\longrightarrow\mathcal{M}(k,\mathcal{H}_p,K)$. Note that the degree zero assumption is crucial here.

The Γ-invariance then follows from the observation that

$$\left\langle P(z)\log_{\mathscr{L}}\left(\frac{z-\tau_1}{z-\tau_2}\right),\ Q(t)\log_{\mathscr{L}}\left(\frac{t-\sigma_1}{t-\sigma_2}\right)\right\rangle_B \tag{1}$$

$$=\left\langle d^{n+1}\left(P(z)\log_{\mathscr{L}}\left(\frac{z-\tau_1}{z-\tau_2}\right)\right),\ Q(t)\log_{\mathscr{L}}\left(\frac{t-\sigma_1}{t-\sigma_2}\right)\right\rangle_M, \tag{2}$$

and that the Morita pairing is Γ-invariant.

We will prove the symmetry by expanding the pairing in two different ways. First, note that (2) can be computed as the sum of residues at τ_1 and τ_2 of expressions of the form

$$\mathrm{res}_{\tau_i}\left(d^{n+1}\left(P(z)\log_{\mathscr{L}}(z-\tau_i)\right)\cdot Q(z)\log_{\mathscr{L}}(z-\sigma_j)\right).$$

We have

$$d^{n+1}(P(z)\log_{\mathscr{L}}(z-\tau))=\sum_{c=0}^{n}(-1)^c(n-c)!\frac{P^{(c)}(\tau)}{(z-\tau)^{n+1-c}}.$$

and also, for z in a small enough neighborhood of τ_i we can expand

$$\begin{aligned}Q(z)\log_{\mathscr{L}}(z-\sigma_j) &= \sum_{a=0}^{n}\frac{Q^{(a)}(\tau_i)}{a!}(z-\tau_i)^a\cdot\left(\log_{\mathscr{L}}(\sigma_j-\tau_i)+\log_{\mathscr{L}}\left(1-\frac{z-\tau_i}{\sigma_j-\tau_i}\right)\right)\\ &= \sum_{a=0}^{n}\frac{Q^{(a)}(\tau_i)}{a!}(z-\tau_i)^a\cdot\left(\log_{\mathscr{L}}(\tau_i-\sigma_j)+\sum_{b=1}^{\infty}\frac{(-1)^{b-1}(z-\tau_i)^b}{b(\tau_i-\sigma_j)^b}\right)\\ &= \log_{\mathscr{L}}(\tau_i-\sigma_j)\sum_{a=0}^{n}\frac{Q^{(a)}(\tau_i)}{a!}(z-\tau_i)^a+\sum_{s=1}^{n}\left(\sum_{b=1}^{s}\frac{(-1)^{b-1}Q^{(s-b)}(\tau_i)}{b(s-b)!(\tau_i-\sigma_j)^b}\right)(z-\tau_i)^s\\ &\quad+\text{higher order terms}\end{aligned}$$

We obtain the answer for the residue at τ_i is

$$\log_{\mathscr{L}}(\tau_i-\sigma_j)\sum_{a=0}^{n}(-1)^{n-a}P^{(n-a)}(\tau_i)Q^{(a)}(\tau_i) + \sum_{s=1}^{n}\sum_{b=1}^{s}\frac{(-1)^{n-s}(-1)^{b-1}s!Q^{(s-b)}(\tau_i)P^{(n-s)}(\tau_i)}{b(s-b)!(\tau_i-\sigma_j)^b}. \tag{3}$$

The second method is by using the following identity for the Breuil pairing

$$\langle G(z),(t-\sigma_j)^i\log_{\mathscr{L}}(t-\sigma_j)\rangle_B = (-1)^{n-i}i!G^{(n-i)}(\sigma_j), \tag{4}$$

with the expansions

$$Q(t)\log_{\mathscr{L}}(t-\sigma_j)=\sum_{s=0}^{n}\frac{Q^{(n-s)}(\sigma_j)}{(n-s)!}(t-\sigma_j)^{n-s}\log_{\mathscr{L}}(t-\sigma_j),$$

and

$$(P(z)\log_{\mathscr{L}}(z-\tau_i))^{(s)} = P^{(s)}(z)\log_{\mathscr{L}}(z-\tau_i)+\sum_{b=1}^{s}\binom{s}{b}\frac{(-1)^{b-1}(b-1)!P^{(s-b)}(z)}{(z-\tau_i)^b}.$$

Combining the above expressions we obtain

$$\langle P(z)\log_{\mathscr{L}}(z-\tau_i), Q(t)\log_{\mathscr{L}}(t-\sigma_j)\rangle_B = \log_{\mathscr{L}}(\tau_i-\sigma_j)\sum_{s=0}^{n}(-1)^s P^{(s)}(\sigma_j)Q^{(n-s)}(\sigma_j)$$
$$+\sum_{s=0}^{n}\sum_{b=1}^{s}\frac{(-1)^s(-1)^{b-1}s!P^{(s-b)}(\sigma_j)Q^{(n-s)}(\sigma_j)}{b(s-b)!(\sigma_j-\tau_i)^b} \quad (5)$$

By comparing (3) and (5) we see that they are the same expression with the roles of the two divisors swapped. □

Remark .0.2. The pairing above generalizes the cross-ratio; when $k=2$ we find

$$\left\langle \log_{\mathscr{L}}\left(\frac{z-\tau_1}{z-\tau_2}\right), \log_{\mathscr{L}}\left(\frac{t-\sigma_1}{t-\sigma_2}\right)\right\rangle_B = \log_{\mathscr{L}}\frac{(\tau_1-\sigma_1)(\tau_2-\sigma_2)}{(\tau_1-\sigma_2)(\tau_2-\sigma_1)}$$

Recall in Proposition 3.6.8 we defined a map

$$H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p)) \longrightarrow H^1(\Gamma, \mathcal{M}(k, \mathcal{H}_p))$$

written as

$$\mathfrak{D} \mapsto J_{\mathfrak{D}}.$$

Given divisors $\mathfrak{D}_1$ and $\mathfrak{D}_2$ with Γ-orbits of disjoint support, we note that $J_{\mathcal{D}_1}$ has no residues near $\mathfrak{D}_2$ so we define

$$\langle \mathfrak{D}_1, \mathfrak{D}_2\rangle_{\mathrm{Div}} = \langle J_{\mathfrak{D}_1}, \mathfrak{D}_2\rangle_M.$$

Then $\langle\cdot,\cdot\rangle_{\mathrm{Div}}$ defines a symbol on pairs of divisors which are disjoint modulo Γ.

Proposition .0.3. *The symbol $\langle\cdot,\cdot\rangle_{\mathrm{Div}}$ is antisymmetric.*

Proof. We calculate the pairing for

$$\mathfrak{D}_1 = \sum_i \gamma_i \otimes D_{1i}$$

and

$$\mathfrak{D}_2 = \sum_j \alpha_j \otimes D_{2j}.$$

The pairing

$$\begin{aligned}
\langle \mathfrak{D}_1, \mathfrak{D}_2 \rangle_{\mathrm{Div}} &= \sum_{\substack{b\in\Gamma \\ i,j}} [(\eta_0, \alpha_j\eta_0)\cdot(b\eta_0, b\gamma_i\eta_0)]\langle bD_{1i}, D_{2j}\rangle \\
&= \sum_{\substack{b\in\Gamma \\ i,j}} [(\eta_0, \alpha_j\eta_0)\cdot(b\eta_0, b\gamma_i\eta_0)]\langle D_{2j}, bD_{1i}\rangle \\
&= \sum_{\substack{b\in\Gamma \\ i,j}} -[(b\eta_0, b\gamma_i\eta_0)\cdot(\eta_0, \alpha_j\eta_0)]\langle D_{2j}, bD_{1i}\rangle \\
&= -\sum_{\substack{b\in\Gamma \\ i,j}} [(\eta_0, \gamma_i\eta_0)\cdot(b^{-1}\eta_0, b^{-1}\alpha_j\eta_0)]\langle b^{-1}D_{2j}, D_{1i}\rangle \\
&= -\langle D_2, D_1\rangle.
\end{aligned}$$

The first and last equalities follow since $\Phi_{\mathfrak{D}_1}$ can be written as a limit of truncations

$$\sum_{i=0}^{l}\sum_j\sum_{b\in\Gamma^l}(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_j\xi_0)(\Phi^0(D_j)|b^{-1})(e)$$

but then this is exactly the residue associated to the finite divisor

$$\sum_{i=0}^{l}\sum_j\sum_{b\in\Gamma^{(l)}}(\eta_0, \gamma\eta_0)\cdot(b\xi_0, b\gamma_j\xi_0)bD_j$$

and so evaluation at the other divisor just amounts to the pairing of divisors. The second equality is the symmetry proven in Proposition .0.1 □

Lemma .0.4. *Let $\mathcal{D}$ be an RM-divisor, with an associated element $y_{k,\mathcal{D}} \in H_1(\Gamma, \mathrm{Div}(K\backslash\mathbb{Q}_p)\otimes \mathscr{P}_n)$ and unitary lift $y^\natural_{k,\mathcal{D}} \in \mathcal{U} \subset H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p)\otimes\mathscr{P}_n)$. Then,*

$$J_{y_{k,\mathcal{D}^\natural}} = J_{k,\mathcal{D}},$$

i.e.

$$\mathrm{Obs}_o(J_{y_{k,\mathcal{D}^\natural}}) = 0.$$

Proof. It suffices to verify that for every class $\alpha \in H_2(\Gamma, \mathscr{P}_n)$, the natural pairing between $H^2(\Gamma, \mathscr{P}_n)$ and $H_2(\Gamma, \mathscr{P}_n)$ gives

$$\left\langle \mathrm{Obs}_o(J_{y^\natural_{k,\mathcal{D}}}), \alpha \right\rangle = 0.$$

Let $i(\alpha)$ be the image of α in $H_1(\Gamma, \mathrm{Div}_0(K\backslash\mathbb{Q}_p) \otimes \mathscr{P}_n)$, we have

$$\left\langle \mathrm{Obs}_o(J_{y^\natural_{k,\mathcal{D}}}), \alpha \right\rangle = \left\langle J_{y^\natural_{k,\mathcal{D}}}, i(\alpha) \right\rangle.$$

By the anti-symmetry this equals

$$\left\langle -J_{i(\alpha)}, y^\natural_{k,\mathcal{D}} \right\rangle,$$

and $J_{i(\alpha)}$ belongs to the image of $H^1(\Gamma, \mathcal{O}(k, \mathcal{H}_p))$. Since $y^\natural_{k,\mathcal{D}}$ is unitary, this last pairing equals 0. □

Corollary .0.5. *Given any two RM-divisors $\mathcal{D}_1,\ \mathcal{D}_2$, we have*

$$G_k(\mathcal{D}_1, \mathcal{D}_2) = -G_k(\mathcal{D}_2, \mathcal{D}_2).$$

Proof. We follow the definition

$$G_k(\mathcal{D}_1, \mathcal{D}_2) = \left\langle J_{k,\mathcal{D}_1}, y^\natural_{\mathcal{D}_2} \right\rangle = \left\langle J_{y^\natural_{k,\mathcal{D}_2}}, y^\natural_{\mathcal{D}_2} \right\rangle = -\left\langle J_{y^\natural_{k,\mathcal{D}_1}}, y^\natural_{\mathcal{D}_1} \right\rangle = -G_k(\mathcal{D}_2, \mathcal{D}_1).$$

□

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