A Candidate Framework for Structure-Preserving Discretizations

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Motivations:

Conference at the Foundations of Computational Mathematics in Paris (2023) at la Sorbonne.

K. Modin and M. Perrot, *Eulerian and Lagrangian stability in Zeitlin's model of hydrodynamics*, arXiv:2305.08479, (2023).

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Berezin-Toeplitz quantization: $\mathfrak{g} = (C^{\infty}(\mathbb{S}^2, \mathbb{C}), \{\cdot, \cdot\})$ remplaced by $\mathfrak{g}_N = (\mathfrak{gl}(N, \mathbb{C}), [\cdot, \cdot]_N)$ as a discretization of Euler's equations,

$$\dot{\omega} = \{\psi, \omega\}, \quad \Delta \psi = \omega, \quad (\omega, \psi) \in C_0^{\infty}(\mathbb{S}^2, \mathbb{C}) \times C_0^{\infty}(\mathbb{S}^2, \mathbb{C})$$

$$\dot{W}=rac{1}{\hbar}[P,W], \quad \Delta_{\hbar}P=W, \quad (W,P)\in\mathfrak{su}(N) imes\mathfrak{su}(N)$$

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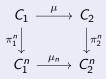
Question: Is the quantization theory a theory of discretization ?

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Axiomatization of the theory of discretization

Definition (Discretization method)

A discretization $\mathfrak{D}(C_1, f)$ of an arrow $(f : C_1 \to C_2)$ is a sequence of arrows $(f_n : C_1^n \to C_2^n)_{n \in \mathbb{N}}$ producing the following diagram



1) $\pi_i^n : C_i \to C_i^n$ surjective, contractive linear maps. 2) $\lim_{n\to\infty} \|\pi_i^n x\|_{C_i^n} = \|x\|_{C_i}$

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Structure preserving discretizations

A discretization $\mathfrak{D}(f)$ is faithful if:

(f is an invertible arrow) \Longrightarrow ($f_n \in \mathfrak{D}(f)$ is invertible for all $n \in \mathbb{N}$).

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1) $(C_1^n, f_n) \in ob(\mathcal{C})$ for all $n \in \mathbb{N}$.

2) the diagram commutes asymptotically:

 $\|f_n\circ\pi_1^n(x)-\pi_2^n\circ f(x)\|\longrightarrow 0$ as $n\to\infty$ and for all $x\in C_1$

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Structure preserving discretizations

A discretization $\mathfrak{D}(f)$ is faithful if:

 $(f \text{ is an invertible arrow}) \Longrightarrow (f_n \in \mathfrak{D}(f) \text{ is invertible for all } n \in \mathbb{N}).$

Definition (Structure preserving discretization)

1) $(C_1^n, f_n) \in ob(\mathcal{C})$ for all $n \in \mathbb{N}$.

2) the diagram commutes asymptotically:

 $\|f_n \circ \pi_1^n(x) - \pi_2^n \circ f(x)\| \longrightarrow 0$ as $n \to \infty$ and for all $x \in C_1$

Definition (Strongly structure preserving discretization)

- 1) $(C_1^n, f_n) \in ob(\mathcal{C})$ for all $n \in \mathbb{N}$.
- 2) the diagram commutes for all $n \in \mathbb{N}$;

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Non-example: Euler's method

Consider the right-shift map on \mathbb{C}^n

$$S: \mathbb{C}^n \to \mathbb{C}^n \quad S(y_1, y_2, \dots, y_n) = (y_n, y_1, y_2, \dots, y_{n-1})$$

and define the family Euler operators by

$$E_n=\frac{n}{2\pi}(S-1)$$

Moreover, we define the maps

 $\pi^n: C^{\infty}(\mathbb{S}^1) \to \operatorname{Hom}(X_n, \mathbb{C}^n) \qquad \pi^n(f) = (f(x_1), f(x_2), \cdots, f(x_n)).$

Consider now the differential operator

$$rac{d}{d heta}: C^\infty(\mathbb{S}^1) o C^\infty(\mathbb{S}^1) \qquad f \mapsto rac{df}{d heta}$$

such that $(C^{\infty}(\mathbb{S}^1), \frac{d}{d\theta})$ is an object of the category of differential algebra

Non-example: Euler's method

We can now look at the Euler discretization method of the pair $(C^{\infty}(\mathbb{S}^1), \frac{d}{d\theta})$ defined by the following diagram:

We readily verify

$$\lim_{n\to\infty}\|\pi^n(f)\|_{\infty}=\|f\|_{\infty}\quad\text{and}\quad\|E_n\circ\pi^n(f)-\pi^n\circ\frac{d}{d\theta}(f)\|\longrightarrow0$$

and thus, the Euler method defines a discretization. However, this is not the case since the pair $(\text{Hom}(X_n, \mathbb{C}^n), E_n)$ does not define a differential algebra since E_n does not satisfy the Leibniz rule.

Convergence

The graph of an arrow f, denoted $\operatorname{Gr}(f)$, is defined as the subset of $C_1 \times C_2$ such that

$$\operatorname{Gr}(f) = \{(x, y) \in C_1 \times C_2 : y = f(x)\}$$

We denote by $p_i : Gr(f) \to C_i$ for i = 1, 2 the obvious coordinate projections.

Definition

Consider a discretization $\mathfrak{D}(f)$ of an arrow f. We say that $\mathfrak{D}(f)$ is *convergent* if to any of the projection maps $\pi_i^n : C_i \to C_i^n$, we can associate an injective contractive linear map $s_i^n : C_i^n \to C_i$ such that

$$\lim_{n \to +\infty} \|x - s_i^n \circ \pi_i^n(x)\| = 0, \quad \text{for all } x \in p_i(\operatorname{Gr}(f)), \text{ and for } i = 1, 2.$$

The maps s_i^n will be called *section* map.

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Example: Finite Element Exterior Calculus

The framework of the finite element exterior calculus is given by the L^2 -de Rham complex, represented by the following complex

$$0 \to \mathrm{H}\Lambda^0(\Omega) \xrightarrow{d} \mathrm{H}\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \mathrm{H}\Lambda^n(\Omega) \to 0$$

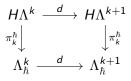
where $\Omega \subseteq \mathbb{R}^n$ is a Lipschitz domain. In order to give a numerical approximation of a PDE, the method builds a finite-dimensional subcomplex

$$0 \to \Lambda^1_{\hbar}(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n_{\hbar}(\Omega) \to 0$$

of the de Rham complex.

Example: Finite Element Exterior Calculus

In order to construct this subcomplex, there exists morphism π^{\hbar} projecting the de Rham complex down to the appropriate subcomplex; so that each map $\pi_k^{\hbar}: \mathrm{H}\Lambda^k \to \Lambda_{\hbar}^k$ is a projection onto the subspace Λ_{\hbar}^k . This morphism defines induces the following commuting diagram

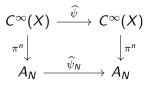


Theorem

The Finite Element Exterior Calculus is a strongly structure preserving discretization.

Example: Diffeomorphism groups

Fix a diffeomorphism ψ , in order to present a structure preserving discretization of the pair $(C^{\infty}(X), \widehat{\psi})$ (it as an object of the category of *-algebra dynamical system). Therefore, following the definition, a *faithful* structure preserving discretization is the following commuting diagram



Theorem

For any diffeomorphism $\psi \in \text{Diff}(X)$, there exists a faithful structure preserving discretization of $\mathfrak{D}(C^{\infty}(X), \widehat{\psi})$. In addition, the discretization does not depend on the choice of ψ .

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Example: Diffeomorphism groups

Theorem

Let $(A_N, \widehat{\psi}_N)$ be a structure preserving discretization of $(C^{\infty}(X), \psi)$.

- i) $\mathfrak{D}(\text{Diff}(X)) = \text{GL}_N(\mathbb{C}).$
- ii) $\mathfrak{D}(\mathrm{Diff}_{\omega}(X)) = \mathrm{SL}_{N}(\mathbb{C}).$
- iii) $\mathfrak{D}(\mathrm{Diff}_0^+(X)) = \mathrm{SO}_N(\mathbb{C}).$
- iv) $\mathfrak{D}(\mathrm{Diff}_{\mathrm{m}}(X)) = \mathrm{GL}^{\mathrm{st}}_{N}(\mathbb{C}).$

The pullback representation

$$\operatorname{Diff}(X) o \operatorname{Der}(\mathcal{C}^\infty(X)) \quad \psi \mapsto \psi^*$$

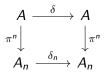
can be thought as a Lie group anti-isomorphism. Differentiating it at the identity $\psi=\mathrm{id}$ gives a linear map

$$\operatorname{Vect}(X) \to \operatorname{Der}(\mathcal{C}^{\infty}(X)) \quad X \mapsto \mathcal{L}_X$$

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Example: Derivations

Consider a derivation $\delta : A \to A$ on a *-algebra. A structure preserving discretization is given by the following commuting diagram:



Theorem

If A_n is isomorphic to a matrix algebra then there exists a self-adjoint element D_n such that

$$d_n(a) = [D_n, a]$$

In addition, if the discretization (A_n, d_n) is structure preserving, then

$$\lim_{n\to\infty}\|\pi^n\circ d(a)-[D_n,\pi^n(a)]\|=0$$

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Example: Derivations

Q: Existence of such discretization ?

Definition

A linear operator D on a Hilbert space H is called *block diagonal*, respectively *quasidiagonal*), if there exists an increasing sequence of finite rank projections, $P_1 < P_2 < P_3 < \cdots$ such $\|[D, P_n]\| = \|DP_n - P_nD\| = 0$, respectively $\rightarrow 0$ for all $n \in \mathbb{N}$ and $P_n \to 1_{\mathcal{H}}$ (in the strong operator topology) as $n \to \infty$.

The surjection maps are given by $\pi^n(a) := P_n a P_n$

$$\pi^{n}\delta(a) = \delta\pi^{n}(a) \Longleftrightarrow P_{n}[D,a]P = [P_{n}DP_{n}, P_{n}aP_{n}]$$

An example of projectors P is given by the spectral projectors of $D = D^*$. The discrete geometry can then be summarized by the data: (P_nAP_n, H_n, P_nDP_n)

Example: The Berezin-Toeplitz Quantization

Let (M, ω) be a compact Kähler manifold.

$$\mathcal{H} = \bigoplus_{m} (\mathcal{H}^{(m)}, \langle \cdot, \cdot \rangle_{m}), \quad \mathcal{H}^{(m)} = \Gamma_{hol}(M, L^{m}), \quad \Pi^{(m)} : \mathcal{H} \to \mathcal{H}^{(m)}$$

For $f \in C^{\infty}(M)$ the Toeplitz operator $T_f^{(m)}$ (of level m) is defined by

$$T_f^{(m)} := \Pi^{(m)} f \Pi^{(m)} : \mathcal{H}^{(m)} o \mathcal{H}^{(m)}$$

Theorem (Bordemann, Meinrenken, Schlichenmaier)

For all functions $f, g \in C^{\infty}(M)$, we have

(a)
$$\lim_{n \to \infty} \|T_f^{(m)}\| = \|f\|_{\infty}$$

(b) $\|im[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}\| = O(m^{-1})$

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Example: The Berezin-Toeplitz Quantization

Theorem

The Berezin-Toeplitz quantization induces a structure preserving discretization of the differential algebra $(C^{\infty}(M), D)$ given by the following commuting diagram

$$\begin{array}{ccc} C^{\infty}(M) & \stackrel{\mathcal{D}}{\longrightarrow} & C^{\infty}(M) \\ T^{(m)} & & & \downarrow_{T^{(m)}} \\ \operatorname{End}(\mathcal{H}_m) & \stackrel{d_m}{\longrightarrow} & \operatorname{End}(\mathcal{H}_m) \end{array}$$
(1)

where the differential d_m is given by the following commutator:

$$A \mapsto d_m(A) = [D_m, A], \qquad D_m := \sum_{k=1}^d \partial_m^k := \sum_{k=1}^d \Pi^{(m)} X^k \Pi^{(m)}$$
 (2)

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Noncommutative Laplace operator

Assume that X^1, \ldots, X^n are isometric embedding (Nash's embedding) coordinates of a compact Kähler M. The Bochner-Laplace operator can be writtenn as:

$$\Delta \varphi = \sum_{k=1}^{n} \{ X^k, \{ X^k, \varphi \} \}$$

After applying the Berezin-Toeplitz projection, we get:

$$T_m(\Delta\varphi) = -m^2[T_m(X^k), [T_m(X^k), T_m(\varphi)]] = \Delta_m \Phi_m$$

Theorem

The discrete Bochner-Laplace operator

$$\widetilde{\Delta}_m: C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M}), \qquad \widetilde{\Delta}_m(f) = \sigma^{(m)} \circ \Delta_m \circ T^{(m)}$$

is a self-adjoint operator on $(\mathcal{H}_m, <\cdot, \cdot >_m)$ with discrete spectrum.

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Convergence and Stability

Theorem

The discrete Bochner-Laplace operator $\hat{\Delta}_m$ converges in the strong-graph limit sense to the Bochner-Laplace operator Δ :

$$\operatorname{sgr} - \operatorname{lim} \widetilde{\Delta}_m = \Delta \tag{3}$$

Theorem (No spurious eigenvalues)

Let A_n ($n \in \mathbb{N}$) and A be (unbounded) self-adjoint operators and assume that $D(A) = D(A_n)$. Assume, furthermore, that there are null-sequences (a_n) and (b_n) from \mathbb{R} for which

$$\|(A-A_n)f\| \le a_n \|f\| + b_n \|Af\| \quad \text{for all } f \in D(A). \tag{4}$$

Then $\sigma(A) = \lim_{n\to\infty} \sigma(A_n)$.

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Proof.

On one hand, using structure preserving we have

$$\begin{split} &\|\Delta(\varphi) - \sigma^{(m)} \circ \Delta_m \circ T^{(m)}(\varphi)\|_{\infty} \\ &\leq \|\Delta(\varphi) - \sigma^{(m)} \circ T^{(m)}(\Delta(\varphi))\|_{\infty} \\ &+ \|\sigma^{(m)} \circ T^{(m)}(\Delta(\varphi)) - \sigma^{(m)} \circ \Delta_m \circ T^{(m)}(\varphi)\|_{\infty} \leq a_m \|\Delta(\varphi)\|_{\infty} \end{split}$$

and on the other hand, using Sobolev's embedding

$$\|\Delta(\varphi) - \sigma^{(m)} \circ \Delta_m \circ T^{(m)}(\varphi)\|_2 \le a_m \|\varphi\|_{H^k}$$

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- Effective algorithm to construct such basis.
- Stability in all generality.

Thank You !

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