Building Up to Lorentzian Causality Theory

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Abstract

Lorentzian causality theory is the subfield of mathematical relativity concerned with the study of causal relations between points on a Lorentzian manifold. Lorentzian manifolds are objects we use to model space-time. In this paper, we will focus on discussing the foundations of causality theory (which consists of differential geometry and special relativity) and providing a brief introduction to Lorentzian causality theory.

Contents

1. Introduction

Lorentzian causality theory is the field of mathematical relativity concerned with causal relations between points on a Lorentzian manifold [Min19]. This field began with the publication of Penrose's paper in 1964, "Gravitational Collapse and Space-time Singularities" [Pen64]. In this paper, Penrose predicted the conditions under which a space-time singularity would form using differential geometry. (A space-time singularity is a location in space-time where the gravitational field becomes infinite, independent of the chosen coordinate system). Penrose's paper marks the birth of Lorentzian causality theory since Penrose used several causality arguments there.

In 1966, Stephen Hawking applied Penrose's arguments to the universe as a whole to predict that the universe began as a singularity [Haw66]. This paper sparked the development of mathematical relativity; the pioneers of this field at the time were Penrose, Hawking, Geroch, and Tipler. Initially, theoretical physicists were not extremely interested in these results; the techniques of differential geometry were perceived to be too technical and new. They were more interested in studying black holes, objects that they believed could shed light on the unification of gravity with the other three fundamental forces of nature (electromagnetism, the strong nuclear force, and the weak nuclear interaction). Moreover, they could employ familiar techniques from quantum field theory to study black holes. Therefore, causality theory developed as and continues to be a field primarily studied by mathematicians and mathematical physicists.

Before we can introduce causality theory, we need to first discuss concepts from differential geometry and special relativity. Special relativity will be important for building intuition and differential geometry will be important since it is the main tool used in causality theory.

1.1. Differential Geometry

The following definitions are taken from *A Comprehensive Introduction to Differential Geometry* by Michael Spivak [Spi99] and Wikipedia. The precise definitions of topological spaces, Hausdorff spaces, and second-countable spaces are not extremely important for us; interested readers can refer to the Appendix.

Definition 1 (Manifold)**.** A **manifold** is a topological space M with the following property: if $x \in M$, then there exists an open neighbourhood Ω of x and some $n \in \mathbb{Z}^+$ such that Ω is homeomorphic to $\mathbb{R}^n$.

In general, we require our manifolds to be Hausdorff and second countable. Lorentzian causality theory focuses on differentiable manifolds. Informally, a **differentiable manifold** is one that locally resembles a vector space where calculus operations, such as differentiation, are well-defined[1].

Definition 2 (Tangent Space)**.** Informally, let M be a differentiable manifold. Then, the **tangent space** of a manifold M at a point p, denoted $T_p(M)$, is a vector space containing all the possible ways in which one can tangentially pass through p.

Informally, a surface embedded in $\mathbb{R}^3$ is an example of a manifold, as depicted in Figure 1. The surface M is a manifold of dimension two. The tangent space at a point x is a plane, since all the ways in which one can tangentially pass through x are confined to the plane that is tangent to M at x.

Definition 3 (Convex Cone)**.** Let V be a vector space. Then, a **convex cone** C is a subset of V that is closed under positive linear combinations; that is, $\forall \alpha, \beta$ positive real numbers, $x, y \in C \Rightarrow \alpha x + \beta y \in C$.

[1] For readers with a background in differential geometry, this simply means that all the transition maps between the different charts that "parameterise" a manifold are differentiable.

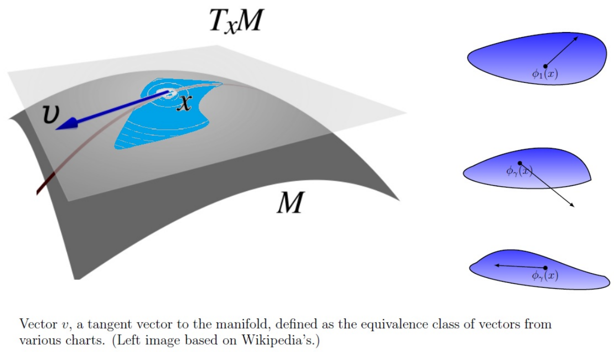

Figure 1: A visual depiction of a manifold M and a tangent space at $x \in M$. Credits: `https://blogs.ams.org/mathgradblog/2017/04/27/manifold-66/`

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1.2. Special Relativity

Special relativity is the branch of physics concerned with the relationship between space and time. The theory of special relativity was proposed by Albert Einstein in 1905 in response to the contradiction between Newtonian mechanics, Maxwell's Theory of electromagnetism, and the notion of a constant speed of light [SF17]. In short, Maxwell's theory of electromagnetism and Newtonian physics are incompatible if one assumes a constant speed of light for all observers. Special relativity is based on two key principles:

(i) The speed of light in a vacuum, c, is the same for all reference frames.
(ii) The laws of physics are invariant under changes in reference frames.

Special relativity only applies to flat space-times (zero curvature); Einstein later developed general relativity to incorporate curved space-time into the theory.

In special relativity, space-time is a four-dimensional manifold (called **Minkowski Space**) consisting of one time-dimension and three spatial dimensions. A point in Minkowski space (t, x, y, z) is called an **event**. The cornerstone of special relativity is the **Lorentz transformation**, which is used to shift reference frames from one frame to another. If we wish to change our reference frame from (t, x, y, z) to (t', x', y', z'), then the Lorentz transformation is given by:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, γ is called the **Lorentz factor**, x, y, and z refer to the coordinates of the current reference frame, t refers to the time measured with respect to the current reference frame, x', y', and z' refer to the coordinates of the reference frame that we wish to shift to, t' refers to the time measured with respect to the reference frame we wish to shift to, c is the speed of light, and v is the velocity of the reference frame we wish to shift to. Observe the transformation's effect will be negligible for reference frames travelling at velocities much lower than the speed of light. Thus, special relativity is only suitable for studying reference frames moving at velocities comparable to the speed of light.

We are interested in quantities that remain invariant under the Lorentz transformations described above. Let $\mathbf{x_i} := (x_i, y_i, z_i)$ and consider two events $(t_1, \mathbf{x_1})$ and $(t_2, \mathbf{x_2})$. In standard Galilean relativity, the spatial separation $(||\mathbf{x_1} - \mathbf{x_2}||)$ and temporal separation $(|t_1 - t_2|)$ between two events are invariant under reference frame shifts. However, this fails to be true in special relativity. A quantity that is invariant under the Lorentz transformation is one that combines spatial and temporal separation. This quantity is called the **invariant interval** and is defined as:

$$\Delta s^2 := c^2t^2 - (x^2 + y^2 + z^2) \tag{1}$$

The notion of an invariant interval gives rise to the **Minkowski space-time cone** and the ideas of space-like, time-like, and light-like separation of events. Consider an observer situated at the origin with a flashlight. When the observer turns the light on, light waves propagate outwards with speed c. We thus know that at any time t, the position of the light wave in $\mathbb{R}^3$ is given by:

$$x^2 + y^2 + z^2 = c^2t^2. \tag{2}$$

We call the set of all points that are eventually reached by the light wave the **Minkowski light cone**. Mathematically, this is the set of all points satisfying Equation 2, which is equivalent to $\Delta s^2 = 0$ [Mal].

Figure 2 depicts a simplified version of the Minkowski light cone. Here, spacetime is compressed into three dimensions: one time-dimension and two spatial-dimensions. The upwards pointing half is called the **future light cone** and the downwards pointing half is called the **past light cone**.

We are now ready to introduce the notions of space-like, time-like, and light-like separation. Let A denote the origin; therefore, Figure 2 is from A's reference frame. Recall that the closure of a set Ω, $\overline{\Omega} = \partial\Omega \cup \mathrm{int}(\Omega)$. If an event B is outside the closure of the cone, then $c^2t^2 - (x^2 + y^2 + z^2) > 0$ and we say that the events A and B are **space-like separated**. Intuitively, events that are space-like separated cannot influence each other, since doing so would require travelling faster the speed of light, which is impossible [Pan13]. If an event B is exactly on the boundary of the cone, then $c^2t^2 - (x^2 + y^2 + z^2) = 0$, and we say that A and B are **light-like separated** if $B \neq A$ and **null-separated** if $B = A$. Finally, if B is in the interior of the cone, then $c^2t^2 - (x^2 + y^2 + z^2) < 0$ and we say that A and B are **time-like separated**. Events that are time-like separated can influence each other. Events that are either time-like or light-like are said to be **causally separated**.

Since these definitions of space-like, time-like, and light-like separation are formulated in terms of the invariant interval, these notions are invariant under the Lorentz transformation, and are therefore consistent from reference frame to reference frame.

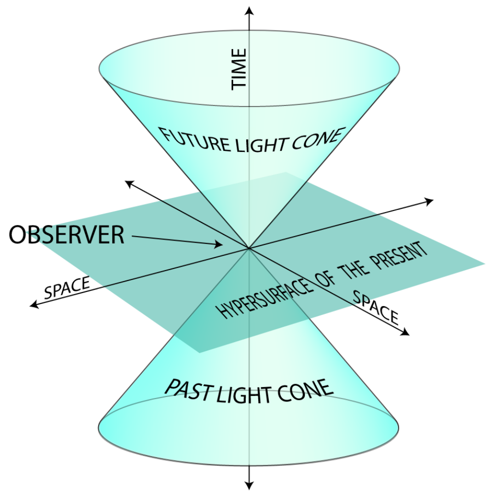

Figure 2: A light-cone in $\mathbb{R}^3$. This light cone is time-oriented since the top component has been labelled as "future" and the bottom component has been labelled as "past." (Wikipedia)

Now that we have built up the necessary background, we will first discuss the first fundamental form and how it can be used to generalize the notion of distance in Section 2. Moreover, as a grounding exercise, we will compute the first fundamental form for the mercator projection of Earth. In Section 3, we will finally introduce some basic results and definitions from Lorentzian causality theory.

2. The First Fundamental Form

Let $S \subseteq \mathbb{R}^3$ be a surface. In this section we are only interested in surfaces that are **regular**. For our purposes, a regular surface S is a surface such that for all points $p \in S$, we can reasonably define a tangent plane through p [Car16].

In $\mathbb{R}^3$, if we wish to describe metric properties such as lengths of curves or areas of regions, we have the standard inner product $\langle \cdot, \cdot \rangle$. Therefore, if we wish to study metric properties on a surface S embedded in $\mathbb{R}^3$, we can just use the ambient space's inner product. However, if we want to study metric properties on S without referring to the space S resides in, we must use the first fundamental form. The first fundamental form captures how exactly a surface S (or more generally, a manifold M) inherits the inner product from its ambient space.

Definition 4 (First Fundamental Form)**.** Let $S \subseteq \mathbb{R}^3$ be a regular surface and let $w \in S$. Then, the quadratic form

$$I_p(w) := \langle w, w \rangle_p = ||w||^2 \geq 0 \tag{3}$$

where $\langle \cdot, \cdot \rangle$ is the inner product of the ambient space, is called the **First Fundamental Form** of S at the point p.

A more enlightening way to express the first fundamental form is in terms of a metric tensor. A **metric tensor** is a generalization of the inner product of $\mathbb{R}^n$ to differentiable manifolds. If we parameterise S by $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$, $(a, b) \mapsto (x, y, z)$, then the first fundamental form can be expressed as:

$$I_p(w) := w^t \begin{bmatrix} E(a,b) & F(a,b) \\ F(a,b) & G(a,b) \end{bmatrix} w$$

The matrix is called the metric tensor of S. Observe that the metric tensor depends on the location of p on S. This is in stark contrast to the standard inner product on $\mathbb{R}^2$, which is expressed as:

$$\langle w, w \rangle := w^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} w$$

and is clearly independent of location.

We will often refer to the **signature of a metric**; this is simply the number of positive, negative, and zero eigenvalues of a metric tensor. For example, if a metric g has a signature $(-, +, +, +)$ then it has three positive eigenvalues and one negative eigenvalue. As we will see later, the Lorentzian metric has this signature. The signature of a metric tells us if the quadratic form is positive definite or not, which then allows us to determine if the metric is Riemannian or not. For example, the signature of the standard Euclidean metric in $\mathbb{R}^4$ is $(+, +, +, +)$; in this case, we say that the Euclidean metric induces a **Riemannian metric**, since it is positive definite. However, we say that Minkowski space-time, which has a metric with a Lorentzian signature, is **pseudo-Riemannian** since a quadratic form with a Lorentzian signature is not positive definite.

2.1. The Mercator Projection

As a grounding exercise, we can try to write the metric for the Mercator projection. The Mercator projection is one way that cartographers can project the spherical Earth onto a two-dimensional map. The transformation is given by:

$$\begin{aligned} x &= R\lambda \\ y &= R \ln\left(\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)\right) \end{aligned}$$

where λ is the longitude and φ is the latitude of the location we are interested in. If we wish to compute the metric tensor for the Mercator projection, we must first compose it with the spherical paramterisation of Earth, since it is spherical. Then, using this composition we must compute the functions E, F, and G.

Let $S := (\theta, \varphi) \mapsto (x, y, z)$ denote the spherical parametrisation of Earth and let $M := (\lambda, \varphi) \mapsto (x, y)$ denote the Mercator projection. In terms of the Mercator projection, finding the first fundamental form will allow us to compute accurate distances between points on a map without having to refer back to $\mathbb{R}^3$. In this case, the first fundamental form will allow our notion of distance to incorporate the effect of Earth's curvature and the Mercator projection's distortion of lengths at the poles. Therefore, we are after the first fundamental form of the composition map, which we will denote by P.

As outlined in *Differential Geometry of Curves and Surfaces* [Car16], we can compute the functions E, F, and G as such:

$$\begin{aligned} E(\lambda, \varphi) &= \langle P_\lambda, P_\lambda \rangle \\ F(\lambda, \varphi) &= \langle P_\lambda, P_\varphi \rangle \\ G(\lambda, \varphi) &= \langle P_\varphi, P_\varphi \rangle \end{aligned}$$

Here, P_λ denotes the gradient of P with respect to λ and P_φ denotes the gradient of P with respect to φ. Carrying out those computations gives us the metric tensor:

$$\begin{bmatrix} R^2 & 0 \\ 0 & R^2 \end{bmatrix}$$

Since the Mercator projection by definition[2] must be conformal to the standard Euclidean metric (meaning that it is a scalar multiple of the identity matrix), we can check the above by observing that we can factor out R^2 to obtain a scalar multiple of the identity matrix.

3. A Brief Introduction to Lorentzian Causality Theory

For the remainder of the paper, I primarily draw from *Lorentzian Causality Theory* [Min19]. Some common definitions are taken from Wikipedia.

3.1. Basic Definitions and the Setup

Lorentzian causality theory studies differentiable manifolds M equipped with a convex sharp cone distribution in M's tangent space; mathematically, this is represented by the map

$$x \mapsto C_x \subseteq T_x(M) \setminus \{0\}$$

In particular, we study the behaviour of trajectories $\dot{x}(t)$ in the cone $C_{x(t)}$. We assume that spacetime is a Lorentzian manifold denoted by (M, g), where g is the metric and M is a manifold. Lorentzian manifolds have metrics with signature $(-, +, +, +)$. We define the **future causal cone** at x as one of the two cones that satisfy

$$C_x := \{y \in T_x(M) \setminus \{0\} \mid g(y, y) \leq 0\}$$

and we define the **Minkowski metric tensor** as:

$$\eta := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For the rest of the paper, we will be working in the following setup: let V be an $(n+1)$-dimensional vector space modelling the tangent space at a given point x on (M, g), and let

[2] One can check Wikipedia for the full derivation of the Mercator projection, but one of the restrictions imposed during the derivation of the Mercator projection that we require it to be a conformal mapping. We will rigorously define conformal mappings in Section 3.3: Cone Distributions and Conformal Invariance.

$g : V \times V \to \mathbb{R}$ be an inner product on V. If g has a Lorentzian signature, then we can find a basis $\{\mathbf{e_0}, ..., \mathbf{e_n}\}$ and coefficients $\{v^0, ..., v^n\}$ such that the inner product can be formulated in terms of the Minkowski metric defined above. That is, for all $v \in V$, we can express v as $v = \sum_{i=1}^{n} v^i \mathbf{e_i}$. The coefficients $\{v^0, ..., v^n\}$ are called the **canonical coordinates** of v. In this setup, the vectors v describe events, and so the definitions from the previous section can be used to determine if v is space-like, time-like, or light-like separated from the reference frame's observer.

The vector space equipped with a metric g, denoted by (V, g), is said to be **time-oriented** if one of the two components of the light-cone emanating from the origin has been labelled as "future." We will assume that if $v^0 > 0$, then v is in the future causal cone; thus, the sign of the first canonical coordinate tells us the time-orientation of vectors that are causally-separated from the origin.

The **observer space** is defined as the set of all future-directed vectors in V such that $g(v, v) = -1$. This generates a hyperboloid in $\mathbb{R}^n$. Since the notions of time-like, space-like, and light-like were derived by searching for invariant quantities under the Lorentz transformation, we define a **Lorentz map** $\Lambda : V \to V$ to be an endomorphism that ensures that g is invariant. Mathematically, this means that $\forall\ v, w \in V$, $g(\Lambda(v), \Lambda(w)) = g(v, w)$.

We are now ready to introduce two propositions that will be important in the proofs later in the article; namely, the reverse triangle inequality and the reverse Cauchy-Schwartz inequality. For these two theorems, we are in the following setup: let (V, g) be a Minkowski space and let $C \subseteq V$ be the future causal cone. Define:

$$F : C \to [0, \infty[$$
$$F(v) := \sqrt{-g(v, v)}$$

Theorem 1 (Reverse Cauchy Schwartz Inequality)**.** Let $v_1, v_2 \in C$. Then:

$$-g(v_1, v_2) \leq F(v_1)F(v_2)$$

Equality holds if and only if v_1 and v_2 are proportional.

The reverse triangle inequality follows from the reverse Cauchy Schwartz inequality.

Theorem 2 (Reverse Triangle Inequality)**.** $\forall\ v_1, v_2 \in C$:

$$F(v_1 + v_2) \geq F(v_1) + F(v_2)$$

Equality holds if and only if v_1 and v_2 are proportional.

The proofs of these theorems can be found in [Min19].

3.2. Time-Orientability of Lorentzian Manifolds

One important property that a Lorentzian manifold may have is time-orientability.

Definition 5 (Time-Orientable)**.** Let M be a Lorentzian manifold. We say that M is **time-orientable** if for all points $x \in M$, we can make a choice for the future cone of $(T_x(M), g_x)$ so that the choice varies continuously with x along M.

The notion of time-orientability is important since we define space-time to be a connected, non-compact, time-oriented smooth Lorentzian manifold. A **connected space** M is a topological space such that we cannot find open sets A, B such that $M = A \sqcup B$ ($\sqcup$ denotes the disjoint union) [Mun17]. A **compact** space M is one where every open cover of M admits a finite sub-cover. We require spacetime to be non-compact since it is unlikely that a compact spacetime can accurately model our universe.

It turns out that if a Lorentzian manifold M is not time-orientable, then it admits a time-orientable covering. We can construct a time-orientable covering of every Lorentzian manifold according to the following algorithm: let $p_0 \in M$ be a reference point, and consider the following set:

$$\mathcal{F} := \{(p, \gamma) \mid p \in M,\ \gamma \text{ continuous curve from } p_0 \text{ to } p\}$$

Here, we are considering all the points p in M that can be connected to the reference point p_0 with a continuous curve γ. Taking the set of all possible p's with their corresponding curve γ gives us $\mathcal{F}$. Now define the following equivalence relation on $\mathcal{F}$: $(p, \gamma) \sim (p', \gamma')$ if:

(i) $p = p'$
(ii) Time-like vectors at p that are continuously moved from p to p_0 along γ, then back to $p = p'$ along γ', do not change their time-directions.[3]

The manifold that is generated by taking all possible equivalence classes based on the rule above is called a **Lorentzian covering** of M. If M is a time-orientable manifold, then the Lorentzian covering of M is actually M itself; if M is not time-orientable, then it is a double-covering of M.

A natural question one can ask is: what are the necessary and sufficient conditions for a manifold to admit a Lorentzian metric? The following theorem answers this question.

Theorem 3. Let M be a smooth manifold. Then, the following are equivalent:

(i) M has a Lorentzian metric.
(ii) M has a continuous line-field.[4]
(iii) M has a non-vanishing continuous vector field.[5]
(iv) M has a space-time structure.[6]
(v) M is non-compact, or M is compact but has zero Euler characteristic.[7]

3.3. Cone Distributions and Conformal Invariance

In this section, we investigate the relation between different metrics on the same manifold, and how this relates to volume forms. Roughly speaking, a **conformal mapping** is a map

[3] We are glossing over what exactly it means to "move vectors along a path", since it requires introducing the notion of connection, which is beyond the scope of this paper."

[4] See here for the rigorous definition of a line-field: `https://en.wikipedia.org/wiki/Line_field`

[5] Informally, a vector field assigns a vector to each point in a space.

[6] manifold is said to have **space-time structure** if we can describe points on M with space- and time-coordinates.

[7] The **Euler characteristic** is a purely topological property that relates the number of vertices, edges, and faces of a geometric object. The famous Gauss-Bonnet theorem [8] from differential geometry relates this quantity to the curvature of a manifold.

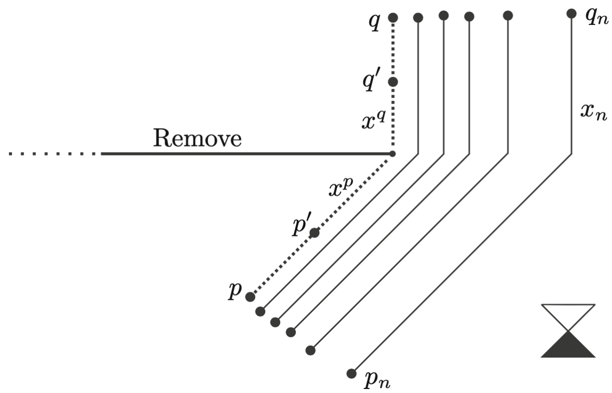

Figure 3: The construction of a sequence of points in J that converges to a point not in J [Min19].

that preserves angles but not necessarily lengths. Before discussing these relations, we first need to state the following useful algebraic result:

Proposition 1. Let V be an $(n+1)$-dimensional vector space, and let g and $\overline{g}$ be two Lorentzian bilinear forms on V. Then, g and $\overline{g}$ induce the same double cone of causal vectors if and only if there exists a constant $\Omega^2 \in]0, \infty[$ such that $\overline{g} = \Omega^2 g$; that is, g and $\overline{g}$ are conformally related.

Proposition 1 can be used to determine when two spacetimes based on the same manifold M, with different metrics g and $\overline{g}$, have the same causal cones. The spacetimes (M, g) and $(M, \overline{g})$ share the same causal cones if and only if the metrics g and $\overline{g}$ are conformally related. Since metrics are a generalisation of the inner product, it is not surprising that a metric g induces a **volume form**, which is given by:

$$\mu(X_0, X_1, ..., X_n) := \sqrt{|\det g(X_i, X_j)|} \tag{4}$$

Volume forms are what induce measures on spaces, which are critical for integration and studying size in general. As a consequence of the previous proposition, spaces-times which are based on the same oriented manifold and with the same causal cones will share the same volume form if and only if they are actually the same spacetime.

3.4. Causality Relations

We now define what it means for a piece-wise C^1 curve to be causal, time-like, or light-like.

Definition 6. Let $x : I \to M$, $t \mapsto x(t)$, $I \subseteq \mathbb{R}$ be a piecewise C^1 curve. Then, x is said to be **causal**, **space-like**, or **light-like** if its tangent vector has the corresponding behaviour at every point on the curve.

Therefore, we can discuss causal relations between points based on the character of the curves that connect them. Namely,

Definition 7 (Causal Relations and Chronological Relations)**.** A **causal relation** is defined as:

$$J := \{(p, q) \in M \mid \exists \text{ a causal curve connecting } p \text{ to } q \text{ or if } p = q\}$$

and a **chronological relation** is defined as:

$$I := \{(p, q) \in M \mid \exists \text{ a time-like curve connecting } p \text{ to } q\ \}$$

Observe that the causal relation J is not necessarily closed. To see why, observe Figure 3. Recall that a set is closed if and only if it contains all of its cluster points. It is possible to construct a sequence in J that converges to a point outside of J; that is, it's possible to construct a sequence of curves connected by causal curves that converges to two points which cannot be connected by a causal curve. This construction is carried out in Figure 3. In this figure, we have a Minkowski $1+1$ space-time where one point is removed. For every pair of points in the sequence, (p_n, q_n), we can connect p_n to q_n with a causal curve. However, due to the removed point, we cannot connect the components of the accumulation point of the sequence $(p_n, q_n)_{n\in\mathbb{N}}$, (p, q), as the curve would not be in the closure of the light-cone.

4. Future Work & Acknowledgements

This DRP project only scratched the surface of causality theory and special relativity; most of the time was spent learning the necessary background differential geometry and physics. So, a natural progression of this paper is to continue reading the primary text that this paper drew from.

I would like to thank Vladmir Sicca Goncalves for the time he spent during this semester (and during the winter holiday!) for mentoring me and answering all my questions. I would also like to thank the DRP committee for organising this program and helping edit our papers. I learned a lot from this fantastic experience.

5. Appendix

This section of the paper presents some definitions from topology that are too technical for the paper. Readers who are interested can refer to [Mun17] for more details.

Definition 8 (Topological Space)**.** A **topological space** is a set X equipped with a collection of sets $\mathcal{O}$ that obeys the following conditions:

(i) $X \in \mathcal{O}$.
(ii) Closed under complements.
(iii) Closed under arbitrary unions.

We call our choice of $\mathcal{O}$ a **topology**.

Definition 9 (Hausdorff)**.** A topological space M is said to be **Hausdorff** if $\forall\ x,\ y \in M$, there exists open sets Ω_1 and Ω_2 such that $x \in \Omega_1$ and $y \in \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$.

Definition 10 (Second-Countable)**.** A topological space M is said to be **second-countable** if there exists a countable collection of open sets of M, $\{\Omega_n\}$, such that for any open set $O \subseteq M$, O can be written as a union of elements Ω in $\{\Omega_n\}$.

For example, $\mathbb{R}$ equipped with the standard topology is a second-countable since we can write every open set as a countable union of open balls with rational radii and rational centres, so in this case our countable "basis" would be the set of open balls with rational radii and rational centres.

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