A winning strategy for Dots and Boxes

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Abstract

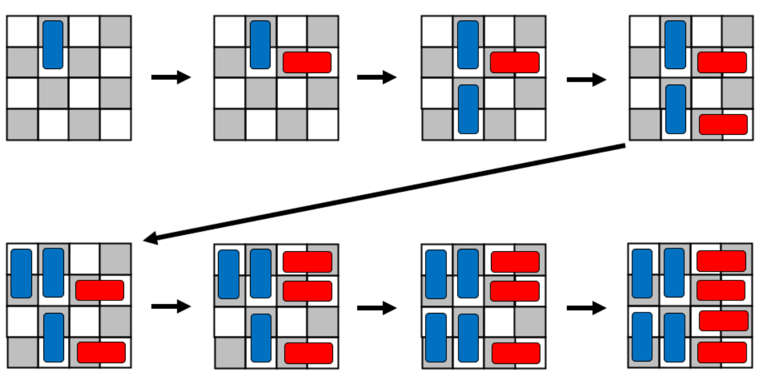
Put a short summary of what your write-up accomplishes here.

1 Introduction

1.1 What is a combinatorial game?

Combinatorial Game Theory is the study of two player games with perfect information. More specifically, a combinatorial game consists of two players alternating moves until one player cannot make a move, and the player who cannot move loses the game. Using the notation introduced by Michael Albert, Richard Nowakowski and David Wolfe in [1], we assume all combinatorial games are played by Louise and Richard, and we denote Louise's moves as *left* moves and Richard's moves as *right* moves.

Example 1. The game DOMINEERING is played on an $n \times m$ chessboard as follows. Louise and Richard take turns placing 2×1 dominoes on the chessboard, with no overlaps allowed, until there are no available moves left on the board. The catch is that Louise can only place her dominoes vertically and Richard can only place his dominoes horizontally. Below is a game of DOMINEERING played on a 3×3 board. Note that in this example, as well as all combinatorial games, Louise's moves are coloured blue and Richard's moves are coloured red.



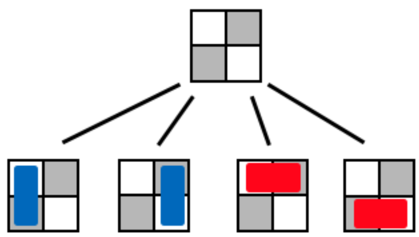
Since Richard is the last player to make a move in this game, he is the winner.

1.2 The formal definition of a game

Combinatorial games can be defined recursively as follows. The base case, called the game ZERO, is the game in which neither Alice nor Bob has a move. Then, letting $\mathcal{G}^L$ and $\mathcal{G}^R$ be sets of games, we can define the game $G = \left\{\mathcal{G}^L \middle| \mathcal{G}^R\right\}$ as the game in which Alice can move to any of the games in $\mathcal{G}^L$ and Bob can move to any of the games in $\mathcal{G}^R$ (note that ZERO $= \{ \mid \}$).

1.3 Game trees

As indicated by its definition, the position of a combinatorial game can be represented by a *game tree*. The root of the tree is the original position from which the game is started. Then a child node is created for each of the positions that either Alice or Bob can reach in one move from the parent node. The leaf nodes are those positions that there are no more moves possible. Game trees are very useful for analyzing a game inductively.



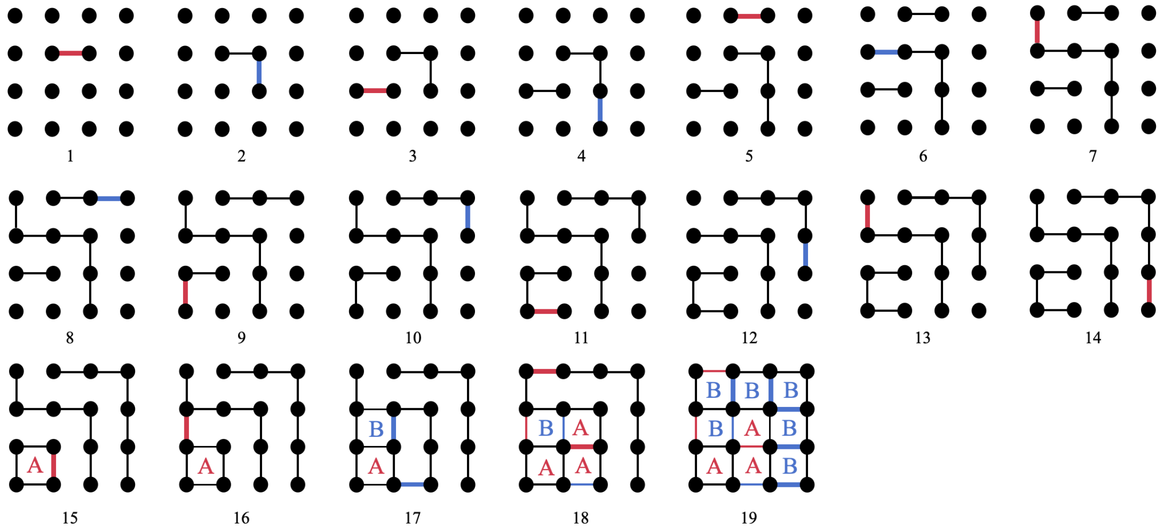
2 Main Result

2.1 Background

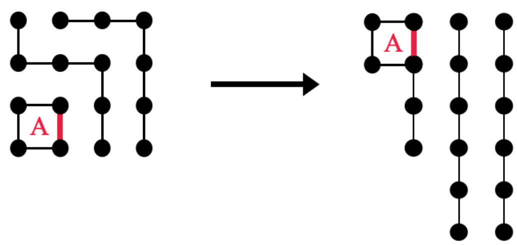
This study focuses on the game *Dots and Boxes* and its winning strategy.

Dots and Boxes is a two player combinatorial game. The game starts with a board with arrays of vertices (dots) and each player take turns to connect any two neighbouring dots vertically or horizontally. The player who completes the fourth side of a unit square (box) scores one point. The game finished when all the boxes on the board are scored and whoever scores the most is the winner. Note that there are two phases of the game: 1. connecting dots 2. collecting boxes. In this study, the strategy is designed for the second phase.

Example 2. The following figure shows an example of the game on a 4×4 board. Let Alice be the first player and Bob be the second player. Their moves at each step are shown in red and blue for Alice and Bob respectively. The study focuses on stage 15 and onward.



To make the analysis more intuitive, one can redraw the game board at stage 15 so that the board looks like *chains* of potential boxes.



At step 17, Bob could have greedily taken all 3 boxes on the chain. However, by rejecting the last 2 boxes, he gains control of the game and collects all boxes on the longest chain, making him the winner. In general, the kind of strategy when a player (Alice) opens a chain for its opponent (Bob), the opponent can sacrifice 2 boxes and gains control of the game, because Alice is forced to open a longer chain for Bob. This kind of strategy is called *doublecrossing*.

This study analyzes the situations when a player might want to doublecross its opponent. Doublecross is a winning strategy for most of the times when there are several *long chains*. A long chain is a chain with at least 3 boxes because 2 boxes can be easily broken into two individual boxes, preventing the player from doublecrossing.

3 Details

3.1 Analysis

To analyze how good or bad doublecrossing is, let's first start with a board with long chains only. Let m be the total number of boxes and n be the number of long chains. Assume both player wants to maximize their scores, then each player will open the shortest chain for their opponent when gaining control. Let k be the number of boxes on the longest chain, this will be the last chain Alice and Bob move on. Without loss of generality, assume Alice opens a chain for Bob. We would like to know if Bob should doublecross or take the whole chain greedily.

If Bob uses the doublecross strategy during the whole game (as he has the control in each turn), then he would score for at least $(n-1)+k$ points. On the other hand, since Alice does not have the control at any point, she can score for at most $2(n-1)$ points. Then Bob would win the game if $k \geq (n-1)$.

If Bob uses the greedy strategy (i.e., taking the whole chain opened by Alice), then he loses the control. Alice may use the dboublecross strategy which follows the analysis in the previous case with the role of the players switched, giving her chance to win. On the other hand, if Alice greedily take the whole chain again, the game back to the point where we started our analysis with the number of chains reduces by 1.

Now if the board has short chains (chains with length ≤ 2), depending on the parity of the number of the short chains, the analysis can be reduced to the first case where there are long chains only. The doublecrossing strategy is not applicable to the short chains, so whoever is forced to take the short chains will have to take the chain greedily. Since no player has the motivations to open a longer chain to the opponent, Alice and Bob will play on the short chain until there are no short chains. Thus the analysis recurse back to the previous case.

3.2 Value of the Game

The value of a game is defined recursively:

$$\begin{aligned}\text{Base case: } 0 &= \{|\} \\ n &= \{n-1|\}\end{aligned}$$

Intuitively, the value n means that Left will have n moves until he wins the game. The negative value, say $-n$, simply implies that if reverse the role of Left and Right, then Left would win the game after n moves.

Let $n_1, ..., n_k$, where $\forall i, n_i \geq 3$, be the number of boxes on each chains. Assume Alice (Right) opens a chain for Bob (Left). Then the value of the game is

$$\sum_{i=1}^{k} n_i - 4(k-1), \text{ if doublecrossing}$$

or

$$n_1 + 4(k-2) - \sum_{i=2}^{k} n_i, \text{ if greedy.}$$

References

[1] Michael H. Albert, Richard J. Nowakowski and David Wolfe. "Lessons In Play. An Introduction To Combinatorial Game Theory." *Taylor & Francis Inc.* 2007.