The Yamabe Problem

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1 Preliminaries

1.1 Differential Geometry

All definitions, theorems, and propositions are taken from [Aub98] and [Heb97].

Definition 1.1 (Manifolds)**.** Let $n \in \mathbb{N}$. A manifold M of dimension n is a Haussdorff topological space where at every point $p \in M$, there exists an open neighbourhood U of p and a homeomorphism $\varphi : U \to \varphi(U)$ such that $\varphi(U) \subset \mathbb{R}^n$ is open.

Definition 1.2 (Charts)**.** Let M be a manifold of dimension n. A local chart (U, φ) on M consists of an open set $U \subset M$ and a homeomorphism $\varphi : U \to \varphi(U)$ such that $\varphi(U) \subset \mathbb{R}^n$ is open. The homeomorphism φ defines a local coordinate system $(x^1, x^2, ..., x^n)$ of U.

Definition 1.3 (Transition Maps)**.** Let M be a manifold, and let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two local charts such that $U_\alpha \cap U_\beta \neq \emptyset$. The transition map $\tau_{\alpha\beta} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is defined by:

$$\tau_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} \tag{1.1}$$

Note that $\tau_{\alpha\beta}$ is a homeomorphism since both φ_α and φ_β are homeomorphisms.

Definition 1.4 (Atlases)**.** Let M be a manifold. An atlas $\mathcal{A}$ is a collection of local charts $\{(U_i, \varphi_i)\}_{i \in I}$ such that:

$$\bigcup_{i \in I} U_i = M \tag{1.2}$$

Let $0 \leq k \leq \infty$ be some integer. We say that $\mathcal{A}$ is C^k if each transition map between any two local charts in $\mathcal{A}$ is C^k.

Definition 1.5 (Differentiable Manifolds)**.** Let M be a manifold, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two C^k atlases on M. The two atlases are equivalent if their union is also a C^k atlas. We say that M is a C^k differentiable manifold when it is paired with an equivalence class of C^k atlases. Unless otherwise stated, we will assume that M is C^∞.

Definition 1.6 (Differentiable Functions)**.** Let M and N be C^k differentiable manifolds of dimensions m and n respectively, let $p \in M$, and let (U, φ) be a local chart of p on M. Let $f : U \to N$ be a map, let (V, ψ) be a local chart of $f(p)$ on N, and let $r \leq k$. The function f is $C^r(M)$ differentiable at p if the function $\psi \circ f \circ \varphi^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is $C^r(\mathbb{R}^m)$ differentiable at $\varphi(p) \in \mathbb{R}^m$.

Definition 1.7 (Partial Derivatives)**.** Let M be a manifold of dimension n, and let $p \in M$. Let $(x^1, x^2, ..., x^n)$ be a local coordinate system at p, and let $(y^1, y^2, ..., y^n)$ be the natural coordinate system of $\mathbb{R}^n$ at $\varphi(p)$. The partial derivatives of a $C^\infty(M)$ function $f : M \to \mathbb{R}$ at p are defined to be:

$$\left.\frac{\partial f}{\partial x^i}\right|_p = \left.\frac{\partial \left(f \circ \varphi^{-1}\right)}{\partial y^i}\right|_{\varphi(p)} \tag{1.3}$$

for each $i = 1, 2, ..., n$.

Definition 1.8 (Tangent Vectors)**.** Let M be a manifold, and let $p \in M$. Let $\mathcal{F}_p$ be the set of all real-valued functions that are differentiable in a neighbourhood of p. A tangent vector at p is a linear map $X_p : \mathcal{F}_p \to \mathbb{R}$ satisfying:

- $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$ for all α, $\beta \in \mathbb{R}$ and f, $g \in \mathcal{F}_p$
- $X_p(f) = 0$ if the gradient of $f \in \mathcal{F}_p$ is zero at p.
- $X_p(fg) = f(p)X_p(g) + g(p)X_p(f)$

Definition 1.9 (Tangent Spaces)**.** Let M be a manifold, and let $p \in M$. The tangent space $T_p(M)$ at p is the set of all tangent vectors to the manifold at p. The tangent bundle $T(M)$ is the disjoint union of all tangent spaces at a point $p \in M$:

$$T(M) = \bigsqcup_{p \in M} T_p(M) \tag{1.4}$$

Definition 1.10 (Cotangent Spaces)**.** Let M be a manifold, and let $p \in M$. The cotangent space $T_p^*(M)$ at p is the set of all linear functionals $\alpha_p : T_p(M) \to \mathbb{R}$. In other words, it is the dual space of the tangent space $T_p(M)$:

$$T_p^*(M) = (T_p(M))^* \tag{1.5}$$

The cotangent bundle $T^*(M)$ is the disjoint union of all cotangent spaces at a point $p \in M$:

$$T^*(M) = \bigsqcup_{p \in M} T_p^*(M) \tag{1.6}$$

Proposition 1.1 (Canonical Basis for $T_p(M)$)**.** Let M be a manifold of dimension n, and let $p \in M$. If $(x^1, x^2, ..., x^n)$ is a local coordinate system at p, then the set of partial derivatives $\{\partial/\partial x^i\big|_p\}_i \subset T_p(M)$ forms a basis of $T_p(M)$. Furthermore, for all tangent vectors $X_p \in T_p(M)$, we may write:

$$X_p = \sum_{i=1}^{n} (X_p)^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{1.7}$$

where $(X_p)^i = X_p(x^i) \in \mathbb{R}$ is the i-th component of X_p in local coordinates.

In Einstein summation notation, we may write:

$$X_p = (X_p)^i \left.\frac{\partial}{\partial x^i}\right|_p \tag{1.8}$$

Proposition 1.2 (Canonical Basis for $T_p^*(M)$)**.** Let M be a manifold of dimension n, and let $p \in M$. If $(x^1, x^2, ..., x^n)$ is a local coordinate system at p, then the set of coordinate differentials $\{(dx^i)_p\}_i \subset T_p^*(M)$ forms a basis of $T_p^*(M)$. The differentials are uniquely defined by:

$$\left(dx^i\right)_p \left(\left. \frac{\partial}{\partial x^j} \right|_p \right) = \delta^i_j \tag{1.9}$$

where δ is the Kronecker delta.

Furthermore, for all linear functionals $\alpha_p \in T_p^*(M)$, we may write in Einstein summation notation:

$$\alpha_p = f_i(p) \left(dx^i\right)_p \tag{1.10}$$

where each $f_i : M \to \mathbb{R}$ is a smooth function.

Definition 1.11 (Projection Maps)**.** Let M be a manifold. The projection map $\Pi : T(M) \to M$ associated with the fiber bundle over M is defined by:

$$\Pi(X_p) = p \tag{1.11}$$

where $X_p \in T_p(M)$ for some $p \in M$.

Definition 1.12 (Vector Fields)**.** Let M be a manifold. A vector field X on M is an assignment of a tangent vector to each point in M. Formally, X is a mapping from M into $T(M)$ such that $\Pi \circ X : M \to M$ is the identity map. We say that a vector field is a section of $T(M)$.

Let (U, φ) be a local chart of M, and let $(x^1, x^2, ..., x^n)$ be a local coordinate system of U. In the canonical basis, we may write:

$$X = X^i \frac{\partial}{\partial x^i} \tag{1.12}$$

We denote by $\Gamma(M)$ the space of smooth vector fields on M.

Definition 1.13 (Differential Forms)**.** Let M be a manifold of dimension n. A 1-form α on M is a mapping from $T(M)$ to $\mathbb{R}$ whose restriction to each tangent space $T_p(M)$ is a linear functional α_p on the tangent space. We say that a 1-form is a section of $T^*(M)$. Furthermore, a k-form ω on M is a section of $\bigwedge^k T^*(M)$ for $k \leq n$.

Let (U, φ) be a local chart of M, and let $(x^1, x^2, ..., x^n)$ be a local coordinate system of U. In the canonical basis, we may write:

$$\omega = \sum_{i_1 < i_2 < ... < i_k}^{n} f_{i_1 i_2 ... i_k} \, dx^{i_1} \wedge dx^{i_2} \wedge ... \wedge dx^{i_k} \tag{1.13}$$

where each $f_{i_1 i_2 ... i_k} : M \to \mathbb{R}$ is a smooth function and each dx^{i_j} is a differential 1-form corresponding to a coordinate differential. The symbol $\wedge$ denotes the antisymmetric exterior product.

We denote by $\bigwedge^k(M)$ the space of smooth k-forms on M.

Definition 1.14 (Lie Bracket)**.** Let M be a manifold, let X, $Y \in \Gamma(M)$ be two vector fields, and let $f : M \to \mathbb{R}$ be a $C^\infty(M)$ function. The Lie bracket of X and Y is also a smooth vector field defined by:

$$[X, Y](f) = X(Y(f)) - Y(X(f)) \tag{1.14}$$

Definition 1.15 (Riemannian Metrics)**.** Let M be a manifold. A Riemannian metric g on M assigns to each point $p \in M$ a positive definite, bilinear, symmetric form $g_p : T_p(M) \times T_p(M) \to \mathbb{R}$. In other words, for all X_p, $Y_p \in T_p(M)$:

$$g_p(X_p, Y_p) = g_p(Y_p, X_p) \qquad\qquad g_p(X_p, X_p) > 0 \text{ for } X_p \neq 0 \tag{1.15}$$

Let U be an open neighbourhood of $p \in M$. For all smooth vectors fields X, Y in $U \subset M$, the following function is a smooth, real-valued function of p:

$$g(X, Y)(p) = g_p(X_p, Y_p) \tag{1.16}$$

Definition 1.16 (Riemannian Manifolds)**.** Let M be a manifold, and let g be a Riemannian metric. The pair (M, g) is a Riemannian manifold. Unless otherwise stated, we will assume that (M, g) is C^∞.

Definition 1.17 (Connections)**.** Let M be a manifold. A connection on M is a map $D : T(M) \times \Gamma(M) \to T(M)$ such that:

- For all $p \in M$, if $X \in T_p(M)$ and $Y \in \Gamma(M)$, then $D(X, Y) \in T_p(M)$.
- For all $p \in M$, the restriction of D to $T_p(M) \times \Gamma(M)$ is bilinear.
- For all $p \in M$, for all $X \in T_p(M)$, and for all $Y \in \Gamma(M)$, if $f : M \to \mathbb{R}$ is differentiable, then:

$$D(X, fY) = X(f)Y(p) + f(p)D(X, Y) \tag{1.17}$$

- For all $X, Y \in \Gamma(M)$, if $X \in C^k(M)$ and $Y \in C^{k+1}(M)$, then $D(X, Y) \in C^k(M)$, where $D(X, Y)$ is a vector field on M defined for all $p \in M$ by:

$$D(X, Y)(p) = D(X(p), Y) \tag{1.18}$$

We often write $D_X Y$ rather than $D(X, Y)$ and call $D_X Y$ the covariant derivative of Y with respect to X for some fixed $Y \in \Gamma(M)$. We may also extend the definition of the covariant derivative to real-valued functions, 1-forms, and general tensors.

First, let $X \in T_p(M)$, and let $f \in \mathcal{F}_p$. The covariant derivative of f with respect to X is given by:

$$D_X f = X(f) \tag{1.19}$$

Now, let $X \in T_p(M)$, and let $\alpha \in \Lambda(M)$. The covariant derivative of α with respect to X is given by the unique 1-form which satisfies the following identity for all $Y \in T_p(M)$:

$$(D_X \alpha)(Y) = D_X(\alpha(Y)) - \alpha(D_X Y) \tag{1.20}$$

Finally, let $X \in T_p(M)$, and let $T \in \bigotimes^r \Gamma(M) \bigotimes^s \Lambda(M)$ be a tensor of rank (r, s). The covariant derivative of T with respect to X extends naturally from the above definitions when combined with the following identity:

$$D_X(V \otimes W) = (D_X V) \otimes W + V \otimes (D_X W) \tag{1.21}$$

where V and W are tensors of arbitrary rank.

Definition 1.18 (Christoffel Symbols)**.** Let M be a manifold of dimension n, and let $p \in M$. Let $(x^1, x^2, ..., x^n)$ be a local coordinate system at p. In this coordinate system, we denote:

$$\nabla_i X = D_{\left(\frac{\partial}{\partial x^i}\right)} X \tag{1.22}$$

for each $i = 1, 2, ..., n$.

Let (U, φ) be the corresponding local chart of p on M. In this local chart, the Christoffel symbols of this connection D are the $C^\infty(M)$ functions $\Gamma^k_{ij} : U \to \mathbb{R}$ defined by:

$$\nabla_i \left(\frac{\partial}{\partial x^j} \right) (q) = \Gamma^k_{ij}(q) \left. \frac{\partial}{\partial x^k} \right|_q \tag{1.23}$$

for all points $q \in U$. This specifies the covariant derivative of the canonical basis vector field $\partial/\partial x^j$ along the basis vector field $\partial/\partial x^i$.

Proposition 1.3 (Basis Representation of the Covariant Derivative)**.** Let M be a manifold of dimension n, and let $p \in M$. Let (U, φ) be the corresponding local chart of p on M, and let $(x^1, x^2, ..., x^n)$ be a local coordinate system on U.

Let $X \in T_p(M)$, let $Y \in \Gamma(U)$, and let $\alpha \in \Lambda(U)$. In the canonical basis, we may write:

$$X = X^i \left. \frac{\partial}{\partial x^i} \right|_p \qquad Y = Y^i \frac{\partial}{\partial x^i} \qquad \alpha = \alpha_i \left(dx^i \right)_p \tag{1.24}$$

Then the k-th component of the covariant derivative of Y with respect to X is given by:

$$(D_X Y)^k = X^i (\nabla_i Y|_p)^k \tag{1.25}$$

$$= X^i \left(\left. \frac{\partial Y^k}{\partial x^i} \right|_p + \Gamma^k_{ij}(p) Y^j(p) \right) \tag{1.26}$$

The k-th component of the covariant derivative of α with respect to X is given by:

$$(D_X \alpha)_k = X^i (\nabla_i \alpha|_p)_k \tag{1.27}$$

$$= X^i \left(\left. \frac{\partial \alpha_k}{\partial x^i} \right|_p - \Gamma^j_{ik}(p) \alpha_j(p) \right) \tag{1.28}$$

And each component of the covariant derivative of an arbitrary tensor T of rank (r, s) with respect to X is given by:

$$(D_X T)^{j_1, j_2, ..., j_r}_{k_1, k_2, ..., k_s} = X^i (\nabla_i T|_p)^{j_1, j_2, ..., j_r}_{k_1, k_2, ..., k_s} \tag{1.29}$$

$$= X^i \Bigg(\left. \frac{\partial T^{j_1, ..., j_r}_{k_1, ..., k_s}}{\partial x^i} \right|_p + \sum_{m=1}^{r} \Gamma^{j_m}_{i, \nu}(p) T(p)^{j_1, ..., j_{m-1}, \nu, j_{m+1}, ..., j_r}_{k_1, ..., k_s}$$

$$- \sum_{m=1}^{r} \Gamma^{\nu}_{i, k_m}(p) T(p)^{j_1, ..., j_r}_{k_1, ..., k_{m-1}, \nu, k_{m+1}, ..., k_s} \Bigg) \tag{1.30}$$

where the index ν is also implicitly summed over.

Proposition 1.4 (Levi-Civita Connection)**.** Let (M, g) be a Riemannian manifold of dimension n, and let (U, φ) be a local chart of M. If $(x^1, x^2, ..., x^n)$ is a local coordinate system of (U, φ), then there exists a unique connection ∇ on M such that $\Gamma^k_{ij} = \Gamma^k_{ji}$ and for which the covariant derivative of the metric tensor is identically zero:

$$\nabla_k g_{ij} = \nabla_i g_{jk} = \nabla_j g_{ik} = 0 \tag{1.31}$$

In the coordinates of (U, φ), the components of the Christoffel symbols of this connection are given by:

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{j\ell}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell}\right) g^{k\ell} \tag{1.32}$$

where $[g_{ij}]$ denotes the matrix representation of the metric g in local coordinates and $[g^{ij}]$ denotes its inverse.

Definition 1.19 (Riemann Curvature Tensor)**.** Let (M, g) be a Riemannian manifold, and let (U, φ) be a local chart of M. The Riemann curvature tensor is a map $R : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ defined by:

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \tag{1.33}$$

In the coordinates of (U, φ), we have that, for $Z \in \Gamma(M)$:

$$R^\ell{}_{kij} Z^k = \nabla_i \nabla_j Z^\ell - \nabla_j \nabla_i Z^\ell \tag{1.34}$$

It can also be shown that the components of the curvature tensor are given by:

$$R^\ell{}_{kij} = \partial_i \Gamma^\ell_{jk} - \partial_j \Gamma^\ell_{ik} + \Gamma^\ell_{im} \Gamma^m_{jk} - \Gamma^\ell_{jm} \Gamma^m_{ik} \tag{1.35}$$

Definition 1.20 (Ricci Tensor and Scalar Curvature)**.** Let (M, g) be a Riemannian manifold, and let (U, φ) be a local chart of M. From the curvature tensor, the Ricci tensor is the only non-zero tensor (up to a sign difference) obtained by contraction. In the coordinates of (U, φ), its components are given by:

$$R_{ij} = R^k{}_{ikj} \tag{1.36}$$

The Ricci tensor is symmetric, and its contraction $R = g^{ij} R_{ij}$ is called the scalar curvature.

Definition 1.21 (Jacobians)**.** Let M be a manifold of dimension n, and let $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) be two charts of an atlas $\mathcal{A}$ such that $U_\alpha \cap U_\beta \neq \emptyset$. Denote by $(x^1, x^2, ..., x^n)$ the coordinates corresponding to $(U_\alpha, \varphi_\alpha)$ and by $(y^1, y^2, ..., y^n)$ the coordinates corresponding to (U_β, φ_β). Then, in $U_\alpha \cap U_\beta$, the components of the Jacobian matrix $J \in GL(\mathbb{R}^n)$ and its inverse Jacobian matrix $J^{-1} \in GL(\mathbb{R}^n)$ are respectively given by:

$$J^j_i = \frac{\partial y^j}{\partial x^i} \qquad \left(J^{-1}\right)^i_j = \frac{\partial x^i}{\partial y^j} \tag{1.37}$$

These are respectively the Jacobians of the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$.

Definition 1.22 (Orientable Manifolds)**.** Let M be a manifold. M is orientable if there exists an atlas such that all of its transition maps have a positive Jacobian determinant. A transition map is called orientation-preserving if the determinant of its Jacobian matrix is positive.

Theorem 1.5. Let M be a manifold of dimension n. M is orientable if and only if there exists an everywhere non-vanishing n-form on M.

Definition 1.23 (Orientations)**.** Let M be a connected, orientable manifold of dimension n, and denote by $\{\omega_i\}_{i\in I}$ the set of everywhere non-vanishing, differentiable n-forms. Consider the equivalence relation: $\omega_i \sim \omega_j$ if there exists a positive function $f_{ij} : M \to \mathbb{R}$ such that $\omega_i = f_{ij}\omega_j$ for each pair ω_i, $\omega_j \in \{\omega_i\}_{i\in I}$. Since an equivalence relation can also be defined using negative functions, there are two possible equivalence classes with opposing signs. Choosing one of them defines an orientation of M, and M is then called oriented. Note that there are only two possible orientations for an orientable, connected manifold.

Definition 1.24 (Partitions of Unity)**.** Let M be a manifold. A partition of unity on M is a set $\{\rho_i\}_{i\in I}$ of continuous functions $\rho_i : M \to [0, 1]$ such that at every point $p \in M$:

- There exists a neighbourhood U of p where all but a finite number of functions in $\{\rho_i\}_{i\in I}$ are zero.
- The sum of all the functions in $\{\rho_i\}_{i\in I}$ evaluated at p is one.

Theorem 1.6. Let M be a compact (and hence paracompact) manifold, and let $(U_i, \varphi_i)_{i\in I}$ be an atlas of M. There exists a partition of unity $\{\rho_i\}_{i\in I}$ indexed over the same set I such that the support of ρ_i is a subset of U_i for each $i \in I$:

$$\mathrm{supp}(\rho_i) \subset U_i \tag{1.38}$$

Such a partition is said to be subordinate to $\{U_i\}_{i\in I}$.

Definition 1.25 (Integration)**.** Let M be an oriented manifold of dimension n, let $(U_i, \varphi_i)_{i\in I}$ be an atlas compatible with the chosen orientation, and let $\{\rho_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. Let $(x^1, x^2, ..., x^n)$ be a local coordinate system of (U_i, φ_i), and let ω be a differentiable n-form with compact support on M such that, on each U_i, we have $\omega = f_i \; dx^1 \wedge dx^2 \wedge ... \wedge dx^n$. Then the integral of ω on M is given by:

$$\int_M \omega = \sum_{i\in I} \int_{\varphi_i(U_i)} (\rho_i f_i) \circ \varphi_i^{-1} \; dx^1 \wedge dx^2 \wedge ... \wedge dx^n \tag{1.39}$$

Note that the integral does not depend on the partition of unity and the sum is finite.

Equivalently, let (M, g) be a Riemannian manifold (not necessarily oriented) of dimension n. The integral of a function $f : M \to \mathbb{R}$ is given by:

$$\int_M f \; dV_g \tag{1.40}$$

where dV_g denotes the natural volume form of (M, g).

In local coordinates, we may write:

$$\int_M f \; dV_g = \int_M f\sqrt{\det(g)} \; dx \tag{1.41}$$

where $dx = dx^1 \wedge dx^2 \wedge ... \wedge dx^n$ denotes the standard Euclidean volume form on $\mathbb{R}^n$.

Note that the integral is independent of the choice of coordinates.

Definition 1.26 (Normal Coordinates)**.** Let (M, g) be a Riemannian manifold. A local coordinate system $\{x^i\}$ associated with the metric $\widetilde{g}$ is a normal coordinate system at a point $p \in M$ if, for all i, j, and k:

$$\widetilde{g}_{ij}(p) = \delta_{ij} \qquad \partial_k \widetilde{g}_{ij}(p) = 0 \tag{1.42}$$

Proposition 1.7. Let (M, g) be a Riemannian manifold. At every point $p \in M$, there exists a normal coordinate system.

1.2 Analysis on Manifolds

All definitions and theorems, and propositions are taken from [Aub98] and [LP87].

Definition 1.27 (Locally Integrable Functions)**.** Let (M, g) be a Riemannian manifold of dimension n. A function $f : M \to \mathbb{R}$ is locally integrable on M if for each point $p \in M$, there exists an open neighbourhood U of p such that:

$$\int_U |f| \; dV_g < \infty \tag{1.43}$$

Definition 1.28 (Weak Derivatives)**.** Let (M, g) be a Riemannian manifold, let $f : M \to \mathbb{R}$ be a locally integrable function, and let D be an arbitrary linear partial differential operator. The function f is weakly differentiable if there exists a locally integrable function $g : M \to \mathbb{R}$ such that, for all $\varphi \in C_c^\infty(M)$:

$$\int_M g\varphi \; dV_g = \int_M f D^* \varphi \; dV_g \tag{1.44}$$

where D^* is the formal adjoint of D obtained by formally integrating by parts. The function g is called the weak derivative of f and is denoted by Df.

Definition 1.29 (Lebesgue Spaces)**.** Let (M, g) be a Riemannian manifold, and let $q \geq 1$. The Lebesgue space $L^q(M)$ is the set of locally integrable functions u on M whose norm $\|u\|_q$ is finite. The q-norm $\| \cdot \|_q$ is given by:

$$\|u\|_q = \left(\int_M |u|^q \; dV_g \right)^{\frac{1}{q}} \tag{1.45}$$

Definition 1.30 (Sobolev Spaces)**.** Let (M, g) be a Riemannian manifold. The Sobolev space $W^{k,q}(M)$ is the set of functions $u \in L^q(M)$ whose weak derivatives up to order k have a finite $L^q(M)$ norm. The Sobolev norm $\| \cdot \|_{k,q}$ is given by:

$$\|u\|_{k,q} = \left(\sum_{i=0}^{k} \int_M |\nabla^i u|^q \; dV_g \right)^{\frac{1}{q}} \tag{1.46}$$

where the covariant derivatives $\nabla^i u$ are taken in a weak sense. Note that $W^{0,q} = L^q$.

Definition 1.31 (C^k Spaces)**.** Let (M, g) be a Riemannian manifold. The C^k space $C^k(M)$ is the set of k-times continuously differentiable functions u on M whose norm $\|u\|_{C^k}$ is finite. The $C^k(M)$ norm $\| \cdot \|_{C^k}$ is given by:

$$\|u\|_{C^k} = \sum_{i=0}^{k} \sup_M |\nabla^i u| \tag{1.47}$$

Definition 1.32 (Hölder Spaces)**.** Let (M, g) be a Riemannian manifold. The Hölder space $C^{k,\alpha}(M)$, where $0 < \alpha \leq 1$, is the set of functions $u \in C^k(M)$ whose norm $\|u\|_{C^{k,\alpha}}$ is finite. The Hölder norm $\|\cdot\|_{C^{k,\alpha}}$ is given by:

$$\|u\|_{C^{k,\alpha}} = \|u\|_{C^k} + \sup_{x \neq y \in M} \frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x - y|^\alpha} \tag{1.48}$$

where the supremum is taken over all points y contained within a normal coordinate neighbourhood of x for any $x \in M$.

Theorem 1.8 (First Sobolev Embedding Theorem)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Let $W^{k,p}(M)$ and $W^{\ell,q}(M)$ be two Sobolev spaces on M with $k > \ell$. Suppose that:

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{\ell}{n} \tag{1.49}$$

Then the embedding $W^{k,p}(M) \subset W^{\ell,q}(M)$ is continuous.

Theorem 1.9 (Rellich-Kondrachov Embedding Theorem)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Let $W^{k,p}(M)$ and $W^{\ell,q}(M)$ be two Sobolev spaces on M with $k > \ell$. Suppose that:

$$\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{\ell}{n} \tag{1.50}$$

Then the embedding $W^{k,p}(M) \subset W^{\ell,q}(M)$ is compact.

Theorem 1.10 (Second Sobolev Embedding Theorem)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Let $W^{k,p}(M)$ be a Sobolev space on M, and let $C^{r,\alpha}(M)$ be a Hölder space on M with $0 < \alpha < 1$. Suppose that:

$$\frac{1}{p} - \frac{k}{n} \leq -\frac{r + \alpha}{n} \tag{1.51}$$

Then the embedding $W^{k,p}(M) \subset C^{r,\alpha}(M)$ is continuous.

Theorem 1.11 (Global Elliptic Regularity Theorem)**.** Let (M, g) be a compact, Riemannian manifold, and let $u : M \to \mathbb{R}$ be a locally integrable, weak solution to Poisson's equation $\Delta u = f$ for some function $f : M \to \mathbb{R}$.

- If $f \in W^{k,q}(M)$, then $u \in W^{k+2,q}(M)$ and, for some $K > 0$:

$$\|u\|_{k+2,q} \leq K \left(\|\Delta u\|_{k,q} + \|u\|_q\right) \tag{1.52}$$

- If $f \in C^{k,\alpha}(M)$, then $u \in C^{k+2,\alpha}(M)$ and, for some $K > 0$:

$$\|u\|_{C^{k+2,\alpha}} \leq K \left(\|\Delta u\|_{C^{k,\alpha}} + \|u\|_{C^{0,\alpha}}\right) \tag{1.53}$$

Theorem 1.12 (Strong Maximum Principle)**.** Let (M, g) be a connected, Riemannian manifold, and let $h : M \to \mathbb{R}$ be a non-negative, smooth function on M. Let $u : M \to \mathbb{R}$ be a $C^2(M)$ function satisfying:

$$(\Delta + h)\, u \geq 0 \tag{1.54}$$

If u attains its minimum $m \leq 0$ on M, then u is constant on M.

Definition 1.33 (Equicontinuity)**.** Let M be a manifold. A subset $\mathbf{F} \subset C^0(M)$ is equicontinuous if for all $x \in M$ and for all $\varepsilon > 0$, there exists a neighbourhood U of x such that, for all $y \in U$ and for all $f \in \mathbf{F}$:

$$|f(y) - f(x)| < \varepsilon \tag{1.55}$$

Definition 1.34 (Pointwise Boundedness)**.** Let M be a manifold. A subset $\mathbf{F} \subset C^0(M)$ is pointwise bounded if for all $x \in M$:

$$\sup_{f \in \mathbf{F}} |f(x)| < \infty \tag{1.56}$$

Theorem 1.13 (Arzelà-Ascoli Theorem)**.** Let M be a manifold. A subset $\mathbf{F} \subset C^0(M)$ is relatively compact in the topology induced by the supremum norm if and only if it is equicontinuous and pointwise bounded.

2 Introduction

The Yamabe Problem. The Yamabe problem is a classic problem in geometric analysis. It was first posed in 1960 by the mathematician Hidehiko Yamabe, who also attempted to provide a solution. Unfortunately, Neil Trudinger discovered an error in his proof in 1968.

The problem remained open until it was finally solved in 1984 with the combined efforts of Yamabe, Trudinger, Aubin, and Schoen. The statement of the problem is as follows:

Let (M, g) be a compact, Riemannian manifold of dimension $n \geq 3$. Is it possible to find a metric $\widetilde{g}$ conformal to g with constant scalar curvature?

Definition 2.1 (Conformal Metrics)**.** Let (M, g) be a Riemannian manifold of dimension n. A metric $\widetilde{g}$ is conformal to g if there exists a $C^\infty(M)$ function $f : M \to \mathbb{R}$ such that:

$$\widetilde{g} = e^{2f} g \tag{2.1}$$

Let R and $\widetilde{R}$ denote the scalar curvatures of g and $\widetilde{g}$ respectively. It can be shown that the scalar curvatures satisfy:

$$\widetilde{R} = e^{-2f} \left(R + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2 \right) \tag{2.2}$$

where ∇f and $\Delta f = -\nabla^\mu \nabla_\mu f$ respectively denote the covariant derivative of f and the Laplace-Beltrami operator with respect to the metric g.

We can simplify this equation by making the substitution:

$$e^{2f} = \varphi^{p-2} \tag{2.3}$$

where $\varphi : M \to \mathbb{R}$ is a positive, smooth function and $p = 2n/(n-2)$ is the critical Sobolev exponent.

Then Equation (2.2) reduces to:

$$\widetilde{R} = \varphi^{1-p} \left(4\frac{n-1}{n-2}\Delta + R \right) \varphi \tag{2.4}$$

Definition 2.2 (Conformal Laplacian)**.** Let (M, g) be a Riemannian manifold of dimension n. The conformal Laplacian of g is given by:

$$4\frac{n-1}{n-2}\Delta + R \tag{2.5}$$

Let $\widetilde{g} = \varphi^{p-2}g$ be a metric conformal to g, and let $\widetilde{\Delta}$ and $\widetilde{R}$ denote the Laplace-Beltrami operator and the scalar curvature of $\widetilde{g}$ respectively. It can be shown that the conformal Laplacians satisfy:

$$\left(4\frac{n-1}{n-2}\widetilde{\Delta} + \widetilde{R}\right)\left(\frac{u}{\varphi}\right) = \varphi^{1-p}\left(4\frac{n-1}{n-2}\Delta + R\right)u \tag{2.6}$$

for all $C^\infty(M)$ functions $u : M \to \mathbb{R}$.

Definition 2.3 (The Yamabe Equation)**.** Let (M, g) be a Riemannian manifold of dimension n, and let $\widetilde{g} = \varphi^{p-2}g$ be a metric conformal to g. Suppose that $\widetilde{R} = \lambda$ for some constant $\lambda \in \mathbb{R}$. Then the Yamabe equation is obtained by rearranging Equation (2.4):

$$\lambda\varphi^{p-1} = \left(4\frac{n-1}{n-2}\Delta + R\right)\varphi \tag{2.7}$$

Solving the Yamabe problem on M is equivalent to finding a positive, smooth function φ satisfying this equation.

Definition 2.4 (The Yamabe Quotient)**.** Let (M, g) be a Riemannian manifold of dimension n, and let $\widetilde{g} = \varphi^{p-2}g$ be a metric conformal to g. The Yamabe quotient $Q(\widetilde{g})$ is the functional whose Euler-Lagrange equation is the Yamabe equation:

$$Q(\widetilde{g}) = \frac{\int_M \widetilde{R}\, dV_{\widetilde{g}}}{\left(\int_M dV_{\widetilde{g}}\right)^{\frac{2}{p}}} \tag{2.8}$$

where $\widetilde{g}$ is allowed to vary over all metrics which are conformal to g.

Equivalently, the Yamabe quotient can be written as:

$$Q_g(\varphi) = \frac{E(\varphi)}{\|\varphi\|_p^2} \tag{2.9}$$

where

$$E(\varphi) = \int_M \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + R\varphi^2\right) dV_g \tag{2.10}$$

Definition 2.5 (The Yamabe Constant)**.** Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. The Yamabe constant $\lambda(M)$ of M is defined to be:

$$\lambda(M) = \inf\left\{Q(\widetilde{g}) \mid \widetilde{g} \text{ conformal to } g\right\} \tag{2.11}$$

$$= \inf\left\{Q_g(\varphi) \mid \text{positive, smooth function } \varphi \text{ on } M\right\} \tag{2.12}$$

Note that the Yamabe constant is an invariant of the conformal class of (M, g).

Claim 2.1. Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, and let $\varphi \in W^{1,2}(M)$ be a smooth, positive function. Then φ is a critical point of Q_g if and only if it satisfies the Yamabe equation with $\lambda = E(\varphi)/\|\varphi\|_p^p$. Furthermore, if φ is a minimizer of Q_g and $\|\varphi\|_p = 1$, then $\lambda = \lambda(M)$.

Proof. Let $\psi \in C^\infty(M)$ be arbitrary. Then:

$$\begin{aligned} 0 &= \frac{d}{dt} Q_g(\varphi + t\psi)\Big|_{t=0} \\ &= \frac{d}{dt}\left(\frac{E(\varphi + t\psi)}{\|\varphi + t\psi\|_p^2}\right)\Big|_{t=0} \\ &= \left(\frac{1}{\|\varphi + t\psi\|_p^2}\frac{d}{dt}(E(\varphi + t\psi)) + E(\varphi + t\psi)\frac{d}{dt}\left(\frac{1}{\|\varphi + t\psi\|_p^2}\right)\right)\Big|_{t=0} \end{aligned}$$

Expanding the expressions and differentiating:

$$\begin{aligned} 0 &= \left(\frac{\int_M \left(8\frac{n-1}{n-2}\langle \nabla(\varphi + t\psi), \nabla\psi\rangle + 2R(\varphi + t\psi)\psi\right) dV_g}{\|\varphi + t\psi\|_p^2}\right)\Bigg|_{t=0} \\ &\qquad - \left(\frac{2E(\varphi + t\psi)\int_M |\varphi + t\psi|^{p-1}\psi \; dV_g}{\|\varphi + t\psi\|_p^2 \|\varphi + t\psi\|_p^p}\right)\Bigg|_{t=0} \\ &= \frac{\int_M 8\frac{n-1}{n-2}\langle\nabla\varphi, \nabla\psi\rangle \; dV_g}{\|\varphi\|_p^2} + \frac{\int_M 2R\varphi\psi \; dV_g}{\|\varphi\|_p^2} - \frac{2E(\varphi)\int_M \varphi^{p-1}\psi \; dV_g}{\|\varphi\|_p^2\|\varphi\|_p^p} \end{aligned}$$

Applying integration by parts to the first term:

$$\begin{aligned} 0 &= \frac{\int_M 8\frac{n-1}{n-2}(\Delta\varphi)\psi \; dV_g}{\|\varphi\|_p^2} + \frac{\int_M 2R\varphi\psi \; dV_g}{\|\varphi\|_p^2} - \frac{2E(\varphi)\int_M \varphi^{p-1}\psi \; dV_g}{\|\varphi\|_p^2\|\varphi\|_p^p} \\ &= \frac{2}{\|\varphi\|_p^2}\int_M \left(4\frac{n-1}{n-2}\Delta\varphi + R\varphi - \frac{E(\varphi)}{\|\varphi\|_p^p}\varphi^{p-1}\right)\psi \; dV_g \end{aligned}$$

Since this holds for any $\psi \in C^\infty(M)$, it must be that:

$$\frac{E(\varphi)}{\|\varphi\|_p^p}\varphi^{p-1} = \left(4\frac{n-1}{n-2}\Delta + R\right)\varphi \tag{2.13}$$

Then, by direct comparison with the Yamabe equation:

$$\lambda = \frac{E(\varphi)}{\|\varphi\|_p^p} \tag{2.14}$$

If $\|\varphi\|_p = 1$, it follows immediately that $\lambda = Q_g(\varphi)$. Additionally, if φ also minimizes Q_g, then $\lambda = \lambda(M)$. □

From the above results, to solve the Yamabe problem, it is sufficient to show that there exists a positive, smooth function φ which minimizes $Q_g(\varphi)$.

In fact, the solution to the Yamabe problem can be summarized by the following three theorems.

Theorem A (Yamabe, Trudinger, Aubin)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Suppose that $\lambda(M) < \lambda(S^n)$, where S^n is the n-sphere equipped with the round metric. Then there exists a minimizer of $Q_g(\varphi)$, and hence a solution to the Yamabe problem on M.

Theorem B (Aubin)**.** Suppose that (M, g) is a compact, Riemannian manifold of dimension $n \geq 6$ and is not locally, conformally flat. Then $\lambda(M) < \lambda(S^n)$.

Theorem C (Schoen)**.** Suppose that (M, g) is a compact, Riemannian manifold of dimension $n = 3$, 4, or 5 or is locally, conformally flat. Then $\lambda(M) < \lambda(S^n)$ unless M is conformal to S^n.

3 Theorem A (Yamabe, Trudinger, Aubin)

Theorem 3.1 (A Sobolev Inequality)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Let $p = 2n/(n-2)$, and denote the n-dimensional Sobolev constant by σ_n. Then, for all $\varepsilon > 0$, there exists a constant C_ε such that, for all $\varphi \in C^\infty(M)$:

$$\|\varphi\|_p^2 \leq (1+\varepsilon)\sigma_n \int_M |\nabla\varphi|^2 \, dV_g + C_\varepsilon \int_M \varphi^2 \, dV_g \tag{3.1}$$

Theorem 3.2 (Yamabe Constant on the n-sphere)**.** Let (S^n, g_0) denote the n-sphere equipped with the standard, round metric. Then the Yamabe constant $\lambda(S^n)$ is given by:

$$\lambda(S^n) = Q(g_0) \tag{3.2}$$

$$= n(n-1)\operatorname{vol}(S^n)^{\frac{2}{n}} \tag{3.3}$$

where $\operatorname{vol}(S^n)$ denotes the volume of the unit n-sphere.

Additionally, the Sobolev constant σ_n mentioned in Theorem 3.1 is given by:

$$\sigma_n = \frac{4}{\lambda(S^n)} \frac{n-1}{n-2} \tag{3.4}$$

Definition 3.1 (The Subcritical Equation)**.** Let (M, g) be a Riemannian manifold of dimension n. Let $2 \leq s \leq p$. The subcritical equation is analogous to the Yamabe equation for some constant $\kappa \in \mathbb{R}$:

$$\kappa\varphi^{s-1} = \left(4\frac{n-1}{n-2}\Delta + R\right)\varphi \tag{3.5}$$

Consider its respective functional, analogous to the Yamabe quotient:

$$Q^s(\varphi) = \frac{E(\varphi)}{\|\varphi\|_s^2} \tag{3.6}$$

where $E(\varphi)$ is as defined in Equation (2.10).

Define an analogous Yamabe constant:

$$\lambda_s = \inf\{Q^s(\varphi) \mid \text{positive, smooth function } \varphi \text{ on } M\} \tag{3.7}$$

Lemma 3.3. Let (M, g) be a compact, Riemannian manifold. Suppose that $s \leq s'$ for any two numbers s, $s' \in [2, p]$. Then the $L^s(M)$ norm is dominated by the $L^{s'}(M)$ norm.

Proof. Let $\alpha = s'/s$ and $\beta = s'/(s' - s)$ be conjugate exponents. Then, for any $u \in L^s(M)$, by Hölder's inequality:

$$\begin{aligned} \|u\|_s^s &= \int_M |u|^s \, dV_g \\ &\leq \left(\int_M dV_g \right)^{\frac{1}{\beta}} \left(\int_M |u|^{s\alpha} \, dV_g \right)^{\frac{1}{\alpha}} \\ &= C \left(\int_M |u|^{s'} \, dV_g \right)^{\frac{s}{s'}} \\ &= C\|u\|_{s'}^s \end{aligned}$$

for some constant C since M is compact.

It follows that:

$$\|u\|_s \leq C\|u\|_{s'} \tag{3.8}$$

for some new constant C.

In particular, note that $C = 1$ if the metric is chosen such that:

$$\int_M dV_g = 1$$

□

Theorem 3.4 (Regularity Theorem)**.** Let (M, g) be a compact, Riemannian manifold, let $\varphi \in W^{1,2}(M)$ be a non-negative weak solution of the subcritical equation with $2 \leq s \leq p$, and let $|\kappa| < K$ for some constant $K \in \mathbb{R}$. Suppose that $\varphi \in L^r(M)$ for some $r > (s-2)n/2$. Then φ is either identically zero or strictly positive and $C^\infty(M)$. Furthermore, $\|\varphi\|_{C^{2,\alpha}} \leq C$ for some constant C which only depends on M, g, K, and $\|\varphi\|_r$. In particular, this holds if $r = s < p$ or if $r > s = p$.

Proof. We first prove that $\|\varphi\|_{C^{2,\alpha}}$ is bounded by some constant C. Suppose that $\varphi \in L^r(M)$ satisfies the subcritical equation. Then $\kappa\varphi^{s-1} - R\varphi$ and hence $4(n-1)/(n-2)\Delta\varphi$ are both functions in $L^q(M)$ where $q = r/(s-1)$.

Since $\Delta\varphi \in L^q(M)$, it follows by the global elliptic regularity theorem (Theorem 1.11) that φ is also in $W^{2,q}(M)$. By the first Sobolev embedding theorem (Theorem 1.8), we also have that $\varphi \in L^{r'}(M)$ where $r' = nr/(ns - n - 2r)$.

Recall that $r > (s-2)n/2$ by hypothesis. It follows directly that $r' > r$. Then, by repeating the above argument with r', we can iteratively show that $\varphi \in W^{2,q}$ for all $q > 1$.

By the second Sobolev embedding theorem (Theorem 1.10), we also have that $\varphi \in C^{0,\alpha}(M)$ for some $0 < \alpha < 1$. It can be shown that $\varphi^{s-1} \in C^{0,\alpha}(M)$ as well.

Since both φ and φ^{s-1} are in $C^{0,\alpha}(M)$, the subcritical equation implies that $\Delta\varphi$ is in $C^{0,\alpha}(M)$. By the global elliptic regularity theorem (Theorem 1.11), we have that φ is also in $C^{2,\alpha}(M)$.

Each of the above applications of the Sobolev embedding and the global elliptic regularity theorems gives a bound on their corresponding norms, which in turn bounds $\|\varphi\|_{C^{2,\alpha}}$ by some constant C.

We now prove that φ is either identically zero or strictly positive and $C^\infty(M)$. By rearranging the subcritical equation, we obtain:

$$\left(\Delta + \frac{n-2}{4(n-1)}\left(R - \kappa\varphi^{s-2}\right)\right)\varphi = 0 \tag{3.9}$$

Since the scalar curvature R is bounded on compact manifolds, it follows that:

$$(\Delta + m)\,\varphi \geq 0 \tag{3.10}$$

for some constant $m \geq 0$ where

$$m \geq \frac{n-2}{4(n-1)} \sup_M \left(R - \kappa\varphi^{s-2}\right) \tag{3.11}$$

Since φ is non-negative by hypothesis, if $\varphi = 0$ somewhere on M, then φ attains its minimum and is identically zero by the strong maximum principle (Theorem 1.12). Therefore, φ is either strictly positive or identically zero on M.

Suppose now that φ is strictly positive on M. Since $\varphi \in C^{2,\alpha}$ and is nowhere zero, it can be shown that $\varphi^{s-1} \in C^{2,\alpha}$ as well. Then, by applying global elliptic regularity iteratively to the subcritical equation, we may conclude that $\varphi \in C^\infty(M)$. □

Proposition 3.5 (Yamabe)**.** Let $s \in \mathbb{N}$ such that $2 \leq s < p$. Then there exists a smooth, positive solution φ_s to the subcritical equation which minimizes Q^s and for which $\kappa = \lambda_s$ and $\|\varphi_s\|_s = 1$.

Proof. Let $\{u_i\} \subset C^\infty(M)$ be a minimizing sequence for Q^s with $\|u_i\|_s = 1$.

Observe that $Q^s(|u_i|) = Q^s(u_i)$. We may therefore assume without loss of generality that $u_i \geq 0$. Then:

$$\begin{aligned}
\|u_i\|_{1,2}^2 &= \int_M |\nabla u_i|^2 \; dV_g + \int_M |u_i|^2 \; dV_g \\
&= \frac{E(u_i)(n-2)}{4(n-1)} - \frac{n-2}{4(n-1)} \int_M R u_i^2 \; dV_g + \int_M u_i^2 \; dV_g \\
&= \frac{Q^s(u_i)\|u_i\|_s^2(n-2)}{4(n-1)} + \int_M \left(1 - \frac{R(n-2)}{4(n-1)}\right) u_i^2 \; dV_g \\
&= \frac{Q^s(u_i)(n-2)}{4(n-1)} + \int_M \left(1 - \frac{R(n-2)}{4(n-1)}\right) u_i^2 \; dV_g
\end{aligned}$$

Since the first term is bounded, we need only consider the second term. Let $\alpha = p/2$ and $\beta = p/(p-2)$ be conjugate exponents. By Hölder's inequality:

$$\begin{aligned}
\int_M \left(1 - \frac{R(n-2)}{4(n-1)}\right) u_i^2 \; dV_g &\leq \left(\int_M \left|1 - \frac{R(n-2)}{4(n-1)}\right|^\beta \; dV_g\right)^{\frac{1}{\beta}} \left(\int_M |u_i|^{2\alpha} \; dV_g\right)^{\frac{1}{\alpha}} \\
&\leq \left(\int_M \left|1 - \frac{R(n-2)}{4(n-1)}\right|^\beta \; dV_g\right)^{\frac{1}{\beta}} \left(\int_M |u_i|^p \; dV_g\right)^{\frac{2}{p}}
\end{aligned}$$

Since M is compact, the scalar curvature is bounded. Then the first integral is bounded by some constant C:

$$\begin{aligned}\int_M \left(1 - \frac{R(n-2)}{4(n-1)}\right) u_i^2 \, dV_g &\leq C \left(\int_M |u_i|^p \, dV_g\right)^{\frac{2}{p}} \\ &= C\|u_i\|_p^2\end{aligned}$$

Then $\{u_i\}$ is bounded in $W^{1,2}(M)$. By the Rellich-Kondrachov theorem (Theorem 1.9), the inclusion map $W^{1,2}(M) \subset L^s(M)$ is compact. It follows that there exists a subsequence of $\{u_i\}$ which converges weakly in $W^{1,2}(M)$ and strongly in $L^s(M)$ to some function φ_s with $\|\varphi_s\|_s = 1$.

Since $2 \leq s$, the $L^2(M)$ norm is dominated by the $L^s(M)$ norm (Lemma 3.3). It follows that the same subsequence of $\{u_i\}$, which converges strongly in $L^s(M)$, must also converge strongly in $L^2(M)$:

$$\lim_{i\to\infty} \int_M Ru_i^2 \, dV_g = \int_M R\varphi_s^2 \, dV_g$$

Weak convergence in $W^{1,2}(M)$ implies that:

$$\begin{aligned}\int_M |\nabla\varphi_s|^2 \, dV_g &= \int_M \langle\nabla\varphi_s, \nabla\varphi_s\rangle \, dV_g \\ &= \lim_{i\to\infty} \int_M \langle\nabla u_i, \nabla\varphi_s\rangle \, dV_g\end{aligned}$$

By the Cauchy-Schwarz inequality:

$$\int_M |\nabla\varphi_s|^2 \, dV_g \leq \limsup_{i\to\infty} \left(\int_M |\nabla u_i|^2 \, dV_g\right)^{\frac{1}{2}} \left(\int_M |\nabla\varphi_s|^2 \, dV_g\right)^{\frac{1}{2}}$$

It follows that:

$$\int_M |\nabla\varphi_s|^2 \, dV_g \leq \limsup_{i\to\infty} \int_M |\nabla u_i|^2 \, dV_g$$

Observe then that:

$$\begin{aligned}Q^s(\varphi_s) &= \frac{E(\varphi_s)}{\|\varphi_s\|_s^2} \\ &= \frac{\int_M \left(4\frac{n-1}{n-2}|\nabla\varphi_s|^2 + R\varphi_s^2\right) dV_g}{\|\varphi_s\|_s^2} \\ &\leq \limsup_{i\to\infty} \frac{\int_M \left(4\frac{n-1}{n-2}|\nabla u_i|^2 + Ru_i\right) dV_g}{\|u_i\|_s^2} \\ &= \limsup_{i\to\infty} \frac{E(u_i)}{\|u_i\|_s^2} \\ &= \limsup_{i\to\infty} Q^s(u_i)\end{aligned}$$

Since $\{u_i\}$ is defined to be a minimizing sequence for Q^s, the limit superior is equal to the limit:

$$\begin{aligned} Q^s(\varphi_s) &\le \lim_{i\to\infty} Q^s(u_i) \\ &= \lambda_s \end{aligned}$$

However, since λ_s is defined to be the infimum of Q^s, it must be that $Q^s(\varphi_s) = \lambda_s$. We have therefore found a minimizing function, and hence a solution to the subcritical equation.

Since $\varphi_s \in L^s(M)$ and is not identically zero, it follows from the regularity theorem (Theorem 3.4) that φ_s is positive and $C^\infty(M)$. □

Lemma 3.6. Let (M, g) be a compact, Riemannian manifold of dimension n. Without loss of generality, we may scale the metric g such that:

$$\int_M dV_g = 1$$

Then $|\lambda_s|$ is non-increasing as a function of $s \in [2, p]$.

Proof. Let $u : M \to \mathbb{R}$ be an arbitrary non-zero, smooth function on M.

Observe that, for any two s, $s' \in [2, p]$:

$$\|u\|_{s'}^2 Q^{s'}(u) = E(\varphi) = \|u\|_s^2 Q^s(u) \tag{3.12}$$

If $s \le s'$, it follows from Lemma 3.3 that $\|u\|_s \le \|u\|_{s'}$. Then, by observing Equation (3.12):

$$|Q^{s'}(u)| \le |Q^s(u)| \tag{3.13}$$

Since this is true for all $u \in C^\infty(M)$, it follows that:

$$|\lambda_{s'}| \le |\lambda_s| \tag{3.14}$$

Remark that, if $\lambda_s < 0$ for some $s \in [2, p]$, there exists a $C^\infty(M)$ function $u : M \to \mathbb{R}$ such that $Q^s(u) < 0$. By Equation (3.12), it follows that $Q^{s'}(u) < 0$ for any $s' \in [2, p]$ as well. Since $\lambda_{s'}$ is the infimum of $Q^{s'}$, it must be that $\lambda_{s'} < 0$. Therefore, $\lambda_s < 0$ for all s.

Remark also that, if $\lambda_p \ge 0$, then $Q^p(u) \ge 0$ for any $C^\infty(M)$ function $u : M \to \mathbb{R}$ because λ_p is the infimum of Q^p. By Equation (3.12), it follows that $Q^s(u) \ge 0$ for any $s \in [2, p]$ as well. Furthermore, by Equation (3.13), we have that $Q^p(u) \le Q^s(u)$. Since this is true for all $u \in C^\infty(M)$, it must be that $\lambda_p \le \lambda_s$. Therefore, $\lambda_s \ge 0$ for all s. □

Lemma 3.7. Let (M, g) be a compact, Riemannian manifold of dimension n. Without loss of generality, we may scale the metric g such that:

$$\int_M dV_g = 1$$

Suppose that $\lambda(M) \ge 0$. Then λ_s continuous from the left as a function of $s \in [2, p]$.

Proof. Observe that $\lambda(M) = \lambda_p$ by definition. From the remark in Lemma 3.6, it must be that $\lambda_s \geq 0$ for any $s \in [2, p]$.

Choose some $s \in [2, p]$, and let $\varepsilon > 0$. By the definition of infimum, there exists a non-zero, smooth function $u : M \to \mathbb{R}$ on M such that:

$$Q^s(u) < \lambda_s + \varepsilon \tag{3.15}$$

Recall from Equation (3.13) of Lemma 3.6 that, for $s' \leq s$:

$$|Q^s(u)| \leq |Q^{s'}(u)|$$

Since λ_s and $\lambda_{s'}$ are non-negative, and since $\|u\|_s$ is a continuous function of s, the absolute values are unnecessary, and there exists an s' sufficiently close to s such that:

$$0 \leq Q^{s'}(u) - Q^s(u) < \varepsilon$$

Combined with Equation (3.15), we find that:

$$Q^{s'}(u) < \lambda_s + 2\varepsilon$$

However, by the definition of infimum, we also have that $\lambda_{s'} \leq Q^{s'}(u)$. It follows then that, for $s' \leq s$:

$$\lambda_{s'} < \lambda_s + 2\varepsilon \tag{3.16}$$

Since λ_s is non-increasing by Lemma 3.6, we conclude that λ_s is continuous from the left. □

Proposition 3.8 (Trudinger, Aubin)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Without loss of generality, let g be a metric such that:

$$\int_M dV_g = 1$$

Suppose that $\lambda(M) < \lambda(S^n)$, and let $\{\varphi_s\}$ be the sequence of smooth, positive solutions to the subcritical equation as defined in Proposition 3.5. Then there exists constants $s_0 < p$, $r > p$, and $C > 0$ such that $\|\varphi_s\|_r \leq C$ for all $s \geq s_0$.

Proof. Suppose $\lambda(M) < \lambda(S^n)$. Let $\{\varphi_s\}$ be a sequence of smooth, positive solutions to the subcritical equation as defined in Proposition 3.5.

Let $\delta > 0$, and consider the subcritical equation multiplied by $\varphi_s^{1+2\delta}$:

$$\lambda_s \varphi^{s+2\delta} = 4\frac{n-1}{n-2}\left(\Delta\varphi_s\right)\varphi_s^{1+2\delta} + R\varphi_s^{2+2\delta} \tag{3.17}$$

Integrating, and applying integration by parts to the first term on the right-hand side:

$$\lambda_s \int_M \varphi^{s+2\delta}\, dV_g = 4\frac{n-1}{n-2}\int_M \langle\nabla\varphi_s, (1+2\delta)\varphi_s^{2\delta}\nabla\varphi_s\rangle\, dV_g + \int_M R\varphi_s^{2+2\delta}\, dV_g$$

Substituting $u = \varphi_s^{1+\delta}$:

$$\lambda_s \int_M \varphi_s^{s-2} u^2 \, dV_g = 4\frac{n-1}{n-2}\frac{1+2\delta}{(1+\delta)^2}\int_M \langle \nabla u, \nabla u\rangle \, dV_g + \int_M Ru^2 \, dV_g$$
$$= 4\frac{n-1}{n-2}\frac{1+2\delta}{(1+\delta)^2}\int_M |\nabla u|^2 \, dV_g + \int_M Ru^2 \, dV_g$$

Rearranging:

$$4\frac{n-1}{n-2}\int_M |\nabla u|^2 \, dV_g = \lambda_s \frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g - \frac{(1+\delta)^2}{1+2\delta}\int_M Ru^2 \, dV_g$$
$$\leq \lambda_s \frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + \frac{(1+\delta)^2}{1+2\delta}\int_M |R|u^2 \, dV_g$$

Since M is compact, the scalar curvature is bounded by some constant C_1:

$$4\frac{n-1}{n-2}\int_M |\nabla u|^2 \, dV_g \leq \lambda_s \frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + C_1\frac{(1+\delta)^2}{1+2\delta}\int_M u^2 \, dV_g$$
$$= \lambda_s \frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + C_1\frac{(1+\delta)^2}{1+2\delta}\|u\|_2^2$$
$$= \lambda_s \frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + C\|u\|_2^2 \tag{3.18}$$

for some constant C.

Since $u \in C^\infty(M)$, by the Sobolev inequality in Theorem 3.1, there exists a constant C_ε such that:

$$\|u\|_p^2 \leq (1+\varepsilon)\frac{4}{\lambda(S^n)}\frac{n-1}{n-2}\int_M |\nabla u|^2 \, dV_g + C_\varepsilon \int_M u^2 \, dV_g$$
$$\leq (1+\varepsilon)\frac{4}{\lambda(S^n)}\frac{n-1}{n-2}\int_M |\nabla u|^2 \, dV_g + C_\varepsilon \|u\|_2^2 \tag{3.19}$$

Substituting the integral in Equation (3.19) with the inequality derived in Equation (3.18):

$$\|u\|_p^2 \leq (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + C\|u\|_2^2 + C_\varepsilon\|u\|_2^2$$
$$= (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\int_M \varphi_s^{s-2}u^2 \, dV_g + C'_\varepsilon\|u\|_2^2$$

where $C'_\varepsilon = C + C_\varepsilon$ is some constant.

Let $\alpha = n/2$ and $\beta = p/2$ be conjugate exponents. By Hölder's inequality:

$$\|u\|_p^2 \leq (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\left(\int_M |\varphi_s|^{\alpha(s-2)} \, dV_g\right)^{\frac{1}{\alpha}}\left(\int_M |u|^{2\beta} \, dV_g\right)^{\frac{1}{\beta}} + C'_\varepsilon\|u\|_2^2$$
$$= (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\left(\int_M |\varphi_s|^{\frac{(s-2)n}{2}} \, dV_g\right)^{\frac{2}{n}}\left(\int_M |u|^{p} \, dV_g\right)^{\frac{2}{p}} + C'_\varepsilon\|u\|_2^2$$
$$= (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\|\varphi_s\|_{(s-2)n/2}^{s-2}\|u\|_p^2 + C'_\varepsilon\|u\|_2^2$$

Since $(s-2)n/2 < s$, it follows from Lemma 3.3 that $\|\varphi_s\|_{(s-2)n/2} \leq \|\varphi_s\|_s = 1$. Then:

$$\|u\|_p^2 \leq (1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta}\|u\|_p^2 + C'_\varepsilon\|u\|_2^2 \tag{3.20}$$

For simplicity, suppose that $\lambda(M) \geq 0$; the same result holds if $\lambda(M) < 0$ with minor modifications. By hypothesis, $\lambda(M) < \lambda(S^n)$. Then, for some fixed $s_0 < p$:

$$\frac{\lambda_{s_0}}{\lambda(S^n)} < 1$$

By Lemma 3.6 and the remark within, we have that $\lambda_s \leq \lambda_{s_0}$ for all $s \geq s_0$. It follows that:

$$\frac{\lambda_s}{\lambda(S^n)} < 1$$

Additionally, we can always choose sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that:

$$(1+\varepsilon)\frac{\lambda_s}{\lambda(S^n)}\frac{(1+\delta)^2}{1+2\delta} \leq C' \tag{3.21}$$

for some constant $C' < 1$.

Then, applying this inequality to Equation (3.20):

$$\|u\|_p^2 \leq C'\|u\|_p^2 + C'_\varepsilon\|u\|_2^2$$

By combining both p-norms, it follows immediately that:

$$\|u\|_p^2 \leq C\|u\|_2^2 \tag{3.22}$$

for some new constant C.

Since $u = \varphi_s^{1+\delta}$, it can be shown that:

$$\|\varphi_s\|_{p(1+\delta)}^{2(1+\delta)} \leq C\|\varphi_s\|_{2(1+\delta)}^{2(1+\delta)}$$

Let $\alpha = s/(s-2(1+\delta))$ and $\beta = s/(2(1+\delta))$ be conjugate exponents. By Hölder's inequality and using the fact that $\|\varphi_s\|_s = 1$:

$$\begin{aligned}
\|\varphi_s\|_{p(1+\delta)}^{2(1+\delta)} &\leq C\int_M |\varphi_s|^{2(1+\delta)}\, dV_g \\
&\leq C\left(\int_M dV_g\right)^{\frac{1}{\alpha}}\left(\int_M |\varphi_s|^{2\beta(1+\delta)}\, dV_g\right)^{\frac{1}{\beta}} \\
&= C\left(\int_M |\varphi_s|^s\, dV_g\right)^{\frac{2(1+\delta)}{s}} \\
&= C\|\varphi_s\|_s^{2(1+\delta)} \\
&= C
\end{aligned}$$

It follows that, for all $s \geq s_0$:

$$\|\varphi_s\|_{p(1+\delta)} \leq C \tag{3.23}$$

for some new constant C.

Since $\delta > 0$ is arbitrary, $\|\varphi_s\|_r$ is bounded independently of s for all $r > p$. □

We now prove the first main theorem.

Theorem A (Yamabe, Trudinger, Aubin)**.** Let (M, g) be a compact, Riemannian manifold of dimension n. Suppose that $\lambda(M) < \lambda(S^n)$, where S^n is the n-sphere equipped with the standard, round metric. Then there exists a minimizer of Q_g and hence a solution to the Yamabe problem on M.

Proof. Suppose that $\lambda(M) < \lambda(S^n)$. Let $\{\varphi_s\}$ be the sequence of smooth, positive solutions to the subcritical equation as defined in Proposition 3.5.

By Proposition 3.8, the functions $\{\varphi_s\}$ are uniformly bounded in $L^r(M)$ for some $r > p$. The regularity theorem (Theorem 3.4) then implies that they are also uniformly bounded in $C^{2,\alpha}(M)$. Then there exists a $K \in \mathbb{R}$ for all s such that:

$$\|\varphi_s\|_{C^{2,\alpha}} = \|\varphi_s\|_{C^2} + \sup_{x \neq y \in M} \frac{|\nabla^2 \varphi_s(x) - \nabla^2 \varphi_s(y)|}{|x-y|^\alpha} \tag{3.24}$$

$$= \sup_M |\varphi_s| + \sup_M |\nabla \varphi_s| + \sup_M |\nabla^2 \varphi_s| + \sup_{x \neq y \in M} \frac{|\nabla^2 \varphi_s(x) - \nabla^2 \varphi_s(y)|}{|x-y|^\alpha} \leq K \tag{3.25}$$

It follows that the sequences $\{\varphi_s\}$, $\{\nabla \varphi_s\}$, and $\{\nabla^2 \varphi_s\}$ are each individually in $C^0(M)$. Note that they are bounded because M is compact. Additionally, it follows immediately from the $C^{2,\alpha}$ Hölder norm that the sequence $\{\nabla^2 \varphi_s\}$ is equicontinuous. It can also be shown using the mean value theorem that the other two sequences are likewise equicontinuous.

The Arzelà-Ascoli theorem (Theorem 1.13) then implies that the sequences $\{\varphi_s\}$, $\{\nabla \varphi_s\}$, and $\{\nabla^2 \varphi_s\}$ are relatively compact. Hence, as $s \to p$, there exists subsequences $\{\varphi_{s_k}\}$, $\{\nabla \varphi_{s_k}\}$, and $\{\nabla^2 \varphi_{s_k}\}$ which converges respectively to functions φ, $\nabla \varphi$, and $\nabla^2 \varphi$ in $C^0(M)$. Then the sequence $\{\varphi_s\}$ converges in the C^2 norm to a function $\varphi \in C^2(M)$.

We denote by λ the limit of λ_s as $s \to p$. Note that the limit function φ must then satisfy:

$$\left(4\frac{n-1}{n-2}\Delta + R\right)\varphi = \lambda \varphi^{p-1} \qquad Q_g(\varphi) = \lambda \tag{3.26}$$

If $\lambda(M) \geq 0$, it follows by Lemma 3.6 and Lemma 3.7 that $\lambda = \lambda(M)$.

If $\lambda(M) < 0$, it follows by Lemma 3.6 that λ_s is negative and is increasing as $s \to p$. Since $\lambda(M) = \lambda_p$, this implies that $\lambda \leq \lambda(M)$. However, since $\lambda(M)$ is by definition the infimum of Q_g, it must be that $\lambda = \lambda(M)$.

Either way, we have found a limit function φ such that $Q_g(\varphi) = \lambda(M)$. By applying the regularity theorem (Theorem 3.4), we find that φ is in $C^\infty(M)$ and is strictly positive because $\|\varphi\|_p \geq \lim_{s \to p} \|\varphi_s\|_s = 1$. □

4 Theorem B (Aubin)

Definition 4.1 (Locally Conformally Flat Manifolds)**.** Let (M, g) be a Riemannian manifold of dimension n. Then M is locally conformally flat if for each point $p \in M$, there exists a neighbourhood $U \subset M$ of p and a conformal metric $\widetilde{g}$ such that $(U, \widetilde{g})$ is flat (i.e. the Riemann curvature tensor $\widetilde{R}^i{}_{jkl}$ vanishes on U).

Definition 4.2 (The Weyl Tensor)**.** Let (M, g) be a manifold of dimension n. The components of the Riemann tensor that are not accounted for by the Ricci tensor, are encapsulated in the Weyl tensor. In a local chart, its components are given by:

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}\left(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}\right) + \frac{R}{(n-1)(n-2)}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right) \quad (4.1)$$

Note that the Weyl tensor is trace free by construction. It can also be shown that the Weyl tensor is conformally invariant:

$$\widetilde{W}^i{}_{jkl} = W^i{}_{jkl} \quad (4.2)$$

Theorem 4.1 (Weyl–Schouten Theorem)**.** Let (M, g) be a Riemannian manifold of dimension $n > 3$. Then M is locally conformally flat if and only if the Weyl tensor vanishes identically.

Proposition 4.2 (Conformal Normal Coordinates [Cao93] [Gün91])**.** Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, and let p be a point in M. Then there exists a conformal metric $\widetilde{g}$ on M such that, for all points in a neighbourhood U of p:

$$\det(\widetilde{g}) = 1 \quad (4.3)$$

The local coordinate system $\{x^i\}$ at $p \in M$ associated with this metric $\widetilde{g}$ is called a conformal normal coordinate system. In these coordinates, if $n \geq 5$, the scalar curvature of $\widetilde{g}$ satisfies $\widetilde{R} = O(|x|^2)$ and $\Delta\widetilde{R}(p) = \frac{1}{6}|\widetilde{W}(p)|^2$. It also follows that the volume form $dV_{\widetilde{g}}$ on U is equal to the Euclidean volume form dx.

Theorem 4.3 (Sharp Sobolev Inequality on $\mathbb{R}^n$)**.** Let $n \in \mathbb{N}$. On $\mathbb{R}^n$, the sharp Sobolev inequality is given by:

$$\|\varphi\|_p^2 \leq 4\frac{n-1}{n-2}\lambda(S^n)^{-1}\|\nabla\varphi\|_2^2 \quad (4.4)$$

for all $\varphi \in W^{1,p}(\mathbb{R}^n)$.

Additionally, let $u_\alpha : \mathbb{R}^n \to \mathbb{R}$ with $\alpha > 0$ be the Sobolev extremal functions on $\mathbb{R}^n$ given by:

$$u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{\frac{2-n}{2}} \quad (4.5)$$

On $\mathbb{R}^n$, the functions u_α satisfy:

$$\|u_\alpha\|_p^2 = 4\frac{n-1}{n-2}\lambda(S^n)^{-1}\|\nabla u_\alpha\|_2^2 \quad (4.6)$$

Lemma 4.4 (Aubin)**.** Let (M, g) be a compact, Riemannian manifold of dimension $n \geq 3$, then $\lambda(M) \leq \lambda(S^n)$.

Proof. We begin with $(\mathbb{R}^n, \delta)$, where δ is the usual Euclidean metric. Let $\{x^i\}$ be the Euclidean coordinate system on $\mathbb{R}^n$. Let $B_\varepsilon \subset \mathbb{R}^n$ denote the ball of radius $\varepsilon > 0$ centered around the origin of $\mathbb{R}^n$.

Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a smooth, radial, cutoff function supported in $B_{2\varepsilon}$ with the following properties:

$$0 \leq \eta \leq 1 \text{ in } B_{2\varepsilon} \qquad\qquad \eta \equiv 1 \text{ in } B_\varepsilon$$

Consider the smooth, compactly supported function $\varphi = \eta u_\alpha$ where u_α with $\alpha > 0$ are the Sobolev extremal functions given by:

$$u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{\frac{2-n}{2}} \tag{4.7}$$

Since each u_α is only a function of $|x|$, let $r = |x|$. Then, computing its derivative:

$$\frac{\partial u_\alpha}{\partial r} = \frac{(2-n)r}{\alpha}\left(\frac{r^2 + \alpha^2}{\alpha}\right)^{-\frac{n}{2}} \tag{4.8}$$

We make the following observations:

$$u_\alpha \leq \alpha^{\frac{n-2}{2}} r^{2-n} \qquad\qquad |\partial_r u_\alpha| \leq (n-2)\alpha^{\frac{n-2}{2}} r^{1-n} \tag{4.9}$$

Observe that:

$$\begin{aligned}
4\frac{n-1}{n-2}\|\nabla\varphi\|_2^2 &= 4\frac{n-1}{n-2}\int_{\mathbb{R}^n} |\nabla\varphi|^2 \, dx \\
&= 4\frac{n-1}{n-2}\int_{\mathbb{R}^n} |\eta\nabla u_\alpha + u_\alpha\nabla\eta|^2 \, dx \\
&= 4\frac{n-1}{n-2}\int_{\mathbb{R}^n} \left(\eta^2|\nabla u_\alpha|^2 + 2\eta u_\alpha\langle\nabla\eta, \nabla u_\alpha\rangle + u_\alpha^2|\nabla\eta|^2\right) dx \\
&\leq 4\frac{n-1}{n-2}\int_{\mathbb{R}^n} \left(|\nabla u_\alpha|^2 + 2\eta u_\alpha|\nabla\eta||\nabla u_\alpha| + u_\alpha^2|\nabla\eta|^2\right) dx
\end{aligned}$$

Consider the third integral term. Since η is bounded and supported in $B_{2\varepsilon}$ and constant in B_ε:

$$\begin{aligned}
\int_{\mathbb{R}^n} u_\alpha^2|\nabla\eta|^2 \, dx &= \int_{B_{2\varepsilon}} u_\alpha^2|\nabla\eta|^2 \, dx \\
&= \int_{B_{2\varepsilon}-B_\varepsilon} u_\alpha^2|\nabla\eta|^2 \, dx + \int_{B_\varepsilon} u_\alpha^2|\nabla\eta|^2 \, dx \\
&\leq \int_{B_{2\varepsilon}-B_\varepsilon} C_1 u_\alpha^2 \, dx \\
&\leq \int_{B_{2\varepsilon}-B_\varepsilon} C_1\alpha^{n-2} r^{4-2n} \, dx \\
&= C\alpha^{n-2}
\end{aligned}$$

for some constants C_1 and C.

Consider now the second integral term:

$$\begin{aligned}
\int_{\mathbb{R}^n} 2\eta u_\alpha |\nabla\eta||\nabla u_\alpha| \; dx &\leq \int_{B_{2\varepsilon}} 2u_\alpha|\nabla\eta||\nabla u_\alpha| \; dx \\
&= \int_{B_{2\varepsilon}-B_\varepsilon} 2u_\alpha|\nabla\eta||\nabla u_\alpha| \; dx + \int_{B_\varepsilon} 2u_\alpha|\nabla\eta||\nabla u_\alpha| \; dx \\
&\leq \int_{B_{2\varepsilon}-B_\varepsilon} C_2 u_\alpha|\nabla u_\alpha| \; dx \\
&= \int_{B_{2\varepsilon}-B_\varepsilon} C_2 u_\alpha|\partial_r u_\alpha| \; dx \\
&\leq \int_{B_{2\varepsilon}-B_\varepsilon} C_2(n-2)\alpha^{n-2}r^{3-2n} \; dx \\
&= C\alpha^{n-2}
\end{aligned}$$

for some constants C_2 and C.

Observe that, for a fixed $\varepsilon > 0$, these two integral terms are $O(\alpha^{n-2})$ and vanish as $\alpha \to 0$.

Consider finally the first term. Applying Equation (4.6) from Theorem 4.3:

$$\begin{aligned}
4\frac{n-1}{n-2}\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \; dx &= 4\frac{n-1}{n-2}\|\nabla u_\alpha\|_2^2 \\
&= \lambda(S^n)\|u_\alpha\|_p^2 \\
&= \lambda(S^n)\left(\int_{\mathbb{R}^n} |u_\alpha|^p \; dx\right)^{\frac{2}{p}} \\
&= \lambda(S^n)\left(\int_{B_\varepsilon} |u_\alpha|^p \; dx + \int_{\mathbb{R}^n-B_\varepsilon} |u_\alpha|^p \; dx\right)^{\frac{2}{p}} \\
&\leq \lambda(S^n)\left(\int_{B_{2\varepsilon}} |\eta u_\alpha|^p \; dx + \int_{\mathbb{R}^n-B_\varepsilon} \alpha^{p\left(\frac{n-2}{2}\right)} r^{p(2-n)} \; dx\right)^{\frac{2}{p}} \\
&= \lambda(S^n)\left(\int_{B_{2\varepsilon}} |\varphi|^p \; dx + \int_{\mathbb{R}^n-B_\varepsilon} \alpha^{\left(\frac{2n}{n-2}\right)\left(\frac{n-2}{2}\right)} r^{\frac{2n(2-n)}{n-2}} \; dx\right)^{\frac{2}{p}} \\
&= \lambda(S^n)\left(\int_{B_{2\varepsilon}} |\varphi|^p \; dx + \int_{\mathbb{R}^n-B_\varepsilon} \alpha^n r^{-2n} \; dx\right)^{\frac{2}{p}} \\
&= \lambda(S^n)\left(\int_{B_{2\varepsilon}} |\varphi|^p \; dx\right)^{\frac{2}{p}} + O(\alpha^n) \\
&= \lambda(S^n)\|\varphi\|_p^2 + O(\alpha^n)
\end{aligned}$$

Combining the above results:

$$4\frac{n-1}{n-2}\|\nabla\varphi\|_2^2 \leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} \tag{4.10}$$

for some new constant C.

We now consider a compact, Riemannian manifold (M, g) of dimension $n \geq 3$. Let $p \in M$ be a point, and let $\{x^i\}$ be a set of conformal normal coordinates associated with the conformal metric $\widetilde{g}$ in some neighbourhood $B_{2\varepsilon}$ of p.

In these coordinates, there exists a sufficiently small $\varepsilon > 0$ such that $dV_{\widetilde{g}} = dx$ in $B_{2\varepsilon} \subset M$. Observe then that:

$$\begin{aligned}
E(\varphi) &= \int_{B_{2\varepsilon}} \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + \widetilde{R}\varphi^2\right) dV_{\widetilde{g}} \\
&= \int_{B_{2\varepsilon}} \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + \widetilde{R}\varphi^2\right) \, dx \\
&= 4\frac{n-1}{n-2}\|\nabla\varphi\|_2^2 + \int_{B_{2\varepsilon}} \widetilde{R}\varphi^2 \, dx
\end{aligned}$$

Combined with Equation (4.10), we find that:

$$E(\varphi) \leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_{B_{2\varepsilon}} \widetilde{R}u_\alpha^2 \, dx$$

Since M is compact, the scalar curvature is bounded by some constant C_1. Denote by $d\omega$ the standard volume form on the unit $(n-1)$-sphere, and denote by $d\omega_r = r^{n-1}d\omega$ the standard volume form on the $(n-1)$-sphere of radius r in $\mathbb{R}^n$. Then:

$$\begin{aligned}
E(\varphi) &\leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + C_1\int_{B_{2\varepsilon}} u_\alpha^2 \, dx \\
&= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + C_1\int_0^{2\varepsilon}\int_{\partial B_r} u_\alpha^2 r^{n-1} \, d\omega dr \\
&= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + C\int_0^{2\varepsilon} u_\alpha^2 r^{n-1} \, dr
\end{aligned}$$

for some new constant C.

By Lemma 4.5 (see below), the above integral is bounded by a constant multiple of α. Then, choosing first an ε and then a sufficiently small α, we find that:

$$\begin{aligned}
Q_g(\varphi) &= \frac{E(\varphi)}{\|\varphi\|_p^2} \\
&\leq \lambda(S^n) + \frac{C}{\|\varphi\|_p^2}\alpha^{n-2} + \frac{C}{\|\varphi\|_p^2}\int_0^{2\varepsilon} u_\alpha^2 r^{n-1} \, dr \\
&\leq \lambda(S^n) + K\alpha
\end{aligned}$$

for some constant K.

It follows directly that $\lambda(M) \leq \lambda(S^n)$. □

We now prove the second main theorem.

Theorem B (Aubin)**.** Suppose that (M, g) is a compact, Riemannian manifold of dimension $n \geq 6$ and is not locally, conformally flat. Then $\lambda(M) < \lambda(S^n)$.

Proof. Let (M, g) be a compact, Riemannian manifold of dimension $n \geq 6$. If M is not locally conformally flat, then there exists a point $p \in M$ such that:

$$|W(p)| > 0 \tag{4.11}$$

where W is the Weyl tensor.

Let $\{x^i\}$ be a set of conformal normal coordinates associated with the conformal metric $\widetilde{g}$ in $B_{2\varepsilon}(p) \subset M$, and consider the test function $\varphi = \eta u_\alpha$ as defined in Lemma 4.4. By compact support:

$$\begin{aligned} E(\varphi) &= \int_M \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + \widetilde{R}\varphi^2\right) dV_{\widetilde{g}} \\ &= \int_{B_{2\varepsilon}} \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + \widetilde{R}\varphi^2\right) dV_{\widetilde{g}} \end{aligned}$$

In conformal normal coordinates, there exists a sufficiently small $\varepsilon > 0$ such that $dV_{\widetilde{g}} = dx$ in $B_{2\varepsilon}(p)$:

$$\begin{aligned} E(\varphi) &= \int_{B_{2\varepsilon}} \left(4\frac{n-1}{n-2}|\nabla\varphi|^2 + \widetilde{R}\varphi^2\right) dx \\ &= 4\frac{n-1}{n-2}\|\nabla\varphi\|_2^2 + \int_{B_{2\varepsilon}} \widetilde{R}\varphi^2 dx \end{aligned}$$

Using some results and observations from Lemma 4.4:

$$\begin{aligned} E(\varphi) &\leq \lambda(S^n)\|\varphi\|_p^2 + C_1\alpha^{n-2} + \int_{B_{2\varepsilon}} \widetilde{R}\varphi^2 dx \\ &= \lambda(S^n)\|\varphi\|_p^2 + C_1\alpha^{n-2} + \int_{B_\varepsilon} \widetilde{R}\varphi^2 \ dx + \int_{B_{2\varepsilon}-B_\varepsilon} \widetilde{R}\varphi^2 \ dx \\ &\leq \lambda(S^n)\|\varphi\|_p^2 + C_1\alpha^{n-2} + \int_{B_\varepsilon} \widetilde{R}\varphi^2 \ dx + \int_{B_{2\varepsilon}-B_\varepsilon} \widetilde{R}\alpha^{n-2}r^{4-2n} \ dx \\ &\leq \lambda(S^n)\|\varphi\|_p^2 + C_1\alpha^{n-2} + \int_{B_\varepsilon} \widetilde{R}\varphi^2 \ dx + C_2\alpha^{n-2} \\ &= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_{B_\varepsilon} \widetilde{R}\varphi^2 \ dx \end{aligned}$$

for some constants C_1, C_2, and C.

Recall also that, in conformal normal coordinates:

$$\widetilde{R} = O(|x|^2) \qquad\qquad \Delta\widetilde{R}(p) = \frac{1}{6}|\widetilde{W}(p)|^2 \tag{4.12}$$

Taylor expanding up to leading order, in Einstein summation notation:

$$\begin{aligned} E(\varphi) &\leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_{B_\varepsilon} \left(\frac{1}{2}\nabla^2_{ij}\widetilde{R}(p)x^ix^j + O(|x|^3)\right) u_\alpha^2 \ dx \\ &= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_0^\varepsilon \int_{\partial B_r} \left(\frac{1}{2}\nabla^2_{ij}\widetilde{R}(p)x^ix^j + O(|x|^3)\right) u_\alpha^2 r^{n-1} \ d\omega dr \end{aligned}$$

Observe that the terms with $i \neq j$ vanish because the integrand is odd and the domain of integration is symmetric:

$$E(\varphi) \leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_0^\varepsilon \int_{\partial B_r} \left(\frac{1}{2}\nabla_{ii}^2 \widetilde{R}(p)x^i x^i + O(|x|^3)\right) u_\alpha^2 r^{n-1} \, d\omega dr$$

Recall that the Laplacian Δ is defined as the negative of the trace of the second covariant derivative:

$$\begin{aligned}
E(\varphi) &\leq \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_0^\varepsilon \int_{\partial B_r} \left(-\frac{1}{2}\Delta \widetilde{R}(p)x^i x^i + O(|x|^3)\right) u_\alpha^2 r^{n-1} \, d\omega dr \\
&= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + \int_0^\varepsilon \int_{\partial B_r} \left(-\frac{1}{12}|\widetilde{W}(p)|^2 x^i x^i + O(|x|^3)\right) u_\alpha^2 r^{n-1} \, d\omega dr \\
&= \lambda(S^n)\|\varphi\|_p^2 + C\alpha^{n-2} + C\int_0^\varepsilon \left(-|\widetilde{W}(p)|^2 r^2 + O(r^3)\right) u_\alpha^2 r^{n-1} \, dr
\end{aligned}$$

for some new, positive constant C.

Since the Weyl tensor is conformally invariant, $\widetilde{W}(p) = W(p)$. Then, by Lemma 4.5 (see below):

$$E(\varphi) \leq \begin{cases} \lambda(S^n)\|\varphi\|_p^2 - C|W(p)|^2\alpha^4 + O(\alpha^{n-2}), & \text{if } n > 6 \\ \lambda(S^n)\|\varphi\|_p^2 - C|W(p)|^2\alpha^4 \log(1/\alpha) + O(\alpha^{n-2}), & \text{if } n = 6 \end{cases} \tag{4.13}$$

for some new, positive constant C.

It follows that $Q_g(\varphi) < \lambda(S^n)$ for a sufficiently small α. Since $\lambda(M)$ is the infimum of $Q_g(\varphi)$, it must be that $\lambda(M) < \lambda(S^n)$. ☐

Lemma 4.5. We define:

$$I(\alpha) = \int_0^\varepsilon r^k u_\alpha^2 r^{n-1} \, dr \tag{4.14}$$

where $u_\alpha(r)$ is as defined in Equation (4.7).

Suppose $k > -n$. Then, as $\alpha \to 0$, $I(\alpha)$ is bounded above and below. Additionally:

- If $n > k+4$, $I(\alpha)$ is bounded by positive multiples of α^{k+2}.
- If $n = k+4$, $I(\alpha)$ is bounded by positive multiples of $\alpha^{k+2}\log(1/\alpha)$.
- If $n < k+4$, $I(\alpha)$ is bounded by positive multiples of α^{n-2}.

Proof. We substitute $\sigma = r/\alpha$:

$$\begin{aligned}
I(\alpha) &= \int_0^\varepsilon r^k u_\alpha^2 r^{n-1} \, dr \\
&= \int_0^\varepsilon r^{k+n-1} \left(\frac{r^2+\alpha^2}{\alpha}\right)^{2-n} \, dr \\
&= \alpha \int_0^{\frac{\varepsilon}{\alpha}} \alpha^{k+n-1}\sigma^{k+n-1} \left(\frac{\alpha^2\sigma^2+\alpha^2}{\alpha}\right)^{2-n} \, d\sigma \\
&= \alpha^{k+2} \int_0^{\frac{\varepsilon}{\alpha}} \sigma^{k+n-1}\left(\sigma^2+1\right)^{2-n} \, d\sigma \\
&= \alpha^{k+2} \left(\int_0^1 \sigma^{k+n-1}\left(\sigma^2+1\right)^{2-n} \, d\sigma + \int_1^{\frac{\varepsilon}{\alpha}} \sigma^{k+n-1}\left(\sigma^2+1\right)^{2-n} \, d\sigma\right)
\end{aligned}$$

Since the bounds of the first integral are $0 \leq \sigma \leq 1$, the first integral is bounded by some constant C_1:

$$
\begin{aligned}
I(\alpha) &= \alpha^{k+2}\left(C_1 + \int_1^{\frac{\varepsilon}{\alpha}} \sigma^{k+n-1}\left(\sigma^2+1\right)^{2-n}\ d\sigma\right) \\
&\leq \alpha^{k+2}\left(C_1 + \int_1^{\frac{\varepsilon}{\alpha}} \sigma^{k+n-1}\left(2\sigma^2\right)^{2-n}\ d\sigma\right) \\
&= \alpha^{k+2}\left(C_1 + C_2\int_1^{\frac{\varepsilon}{\alpha}} \sigma^{k+3-n}\ d\sigma\right) \\
&= C\alpha^{k+2}\left(1 + \int_1^{\frac{\varepsilon}{\alpha}} \sigma^{k+3-n}\ d\sigma\right)
\end{aligned}
$$

The result for each case follows by direct computation. □

5 Positive Mass Theorem

5.1 Geometric Preliminaries

All definitions, theorems, and propositions are taken from [Aub98], [Heb97], and [LP87].

Definition 5.1 (Pushforward)**.** Let M and N be manifolds of dimension n, and let $\varphi : M \to N$ be a diffeomorphism. Let $p \in M$, and let $X \in T_p(M)$. The pushforward of X to N is the unique tangent vector $\varphi_* X \in T_{\varphi(p)}(N)$ satisfying:

$$(\varphi_*(X))(f) = X(f \circ \varphi) \tag{5.1}$$

for all smooth functions $f : N \to \mathbb{R}$

Definition 5.2 (Pullback)**.** Let M and N be manifolds of dimension n, let $\varphi : M \to N$ be a diffeomorphism, and let S be a tensor of rank $(0, s)$ on N. The pullback of S is the unique tensor of rank $(0, s)$ on M satisfying

$$(\varphi^* S)_p(X_1, X_2, ..., X_s) = S_{\varphi(p)}(\varphi_* X_1, \varphi_* X_2, ..., \varphi_* X_s) \tag{5.2}$$

for all $p \in M$ and $X_i \in T_p(M)$ for each $i = 1, 2, ..., s$.

Definition 5.3 (Isometric Manifolds)**.** Let (M, g) and (M', g') be two Riemannian manifolds of dimension n. Let $\varphi : M \to M'$ be a diffeomorphism such that $g' = \varphi^* g$. Then (M, g) and (M', g') are isometric manifolds and φ is called an isometry.

Definition 5.4 (Green's Functions)**.** Let M be a compact, Riemannian manifold, and let $x_0 \in M$. The Green's function $\Gamma : M \times M \to \mathbb{R}$ of a linear, differential operator L is the unique, smooth function satisfying, in the sense of distributions:

$$L\ \Gamma(x, x_0) = \delta_{x_0}(x) \tag{5.3}$$

where δ_{x_0} is the Dirac measure at x_0.

We often write $\Gamma_{x_0}(x)$ rather than $\Gamma(x, x_0)$ and call Γ_{x_0} the Green's function at x_0 for some fixed $x_0 \in M$.

Theorem 5.1 (Existence of the Green's Function)**.** Let (M, g) be a compact, Riemannian manifold of dimension $n \geq 3$, and let $h : M \to \mathbb{R}$ be a strictly positive, smooth function on M. Then, at each point $x_0 \in M$, the Green's function Γ_{x_0} for the operator $\Delta + h$ exists.

Lemma 5.2. Let (M, g) be a compact, Riemannian metric of dimension $n \geq 3$, and let $\widetilde{g} = \varphi^{p-2} g$ be a metric conformal to g. The volume form $dV_{\widetilde{g}}$ of $\widetilde{g}$ is then given by:

$$dV_{\widetilde{g}} = \varphi_s^p \, dV_g \tag{5.4}$$

Proof. Let $\{x^i\}$ be a local coordinate system on M. The volume form $dV_{\widetilde{g}}$ of $\widetilde{g}$ is then given by:

$$\begin{aligned} dV_{\widetilde{g}} &= (\det(\widetilde{g}))^{\frac{1}{2}} \; dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\ &= \left(\det\left(\varphi_s^{p-2} g\right)\right)^{\frac{1}{2}} \; dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\ &= \left(\varphi_s^{n(p-2)} \det(g)\right)^{\frac{1}{2}} \; dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\ &= \varphi_s^p \, (\det(g))^{\frac{1}{2}} \; dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n \\ &= \varphi_s^p \, dV_g \end{aligned}$$

where we used the fact that $p = 2n/(n-2)$. □

Proposition 5.3. Let (M, g) be a compact, Riemannian metric of dimension $n \geq 3$, let R denote the scalar curvature of g, and suppose that $\lambda(M) > 0$. Then, at each point $x_0 \in M$, the Green's function Γ_{x_0} for the operator $4(n-1)/(n-2)\Delta + R$ exists and is strictly positive.

Proof. Recall the subcritical equation:

$$\kappa \varphi^{s-1} = \left(4\frac{n-1}{n-2}\Delta + R\right)\varphi \tag{5.5}$$

Let $\varphi_s : M \to \mathbb{R}$ be the smooth, positive solution to the subcritical equation for some fixed $s \in [2, p)$ as defined in Proposition 3.5, and define a new, conformal metric:

$$\widetilde{g} = \varphi_s^{p-2} g \tag{5.6}$$

By Equation (2.4), the scalar curvature $\widetilde{R}$ of $\widetilde{g}$ is given by:

$$\begin{aligned} \widetilde{R} &= \varphi_s^{1-p}\left(4\frac{n-1}{n-2}\Delta + R\right)\varphi_s \\ &= \varphi_s^{1-p} \lambda_s \varphi_s^{s-1} \\ &= \lambda_s \varphi_s^{s-p} \end{aligned}$$

Since $\lambda(M) > 0$ by hypothesis, it follows from the remark in Lemma 3.6 that $\lambda_s > 0$ as well. Therefore $\widetilde{R}$ is strictly positive. Since $\widetilde{R}$ is also a smooth function, by Theorem 5.1, the Green's function Γ'_{x_0} for the operator $4(n-1)/(n-2)\widetilde{\Delta} + \widetilde{R}$ exists at each point $x_0 \in M$. It then follows by definition that, for all $C_c^\infty(M)$ functions $f : M \to \mathbb{R}$:

$$\int_M \Gamma'_{x_0}(x)\left(4\frac{n-1}{n-2}\widetilde{\Delta} + \widetilde{R}\right)\left(\frac{f(x)}{\varphi_s(x)}\right) dV_{\widetilde{g}}(x) = \frac{f(x_0)}{\varphi_s(x_0)} \tag{5.7}$$

If $\Gamma'_{x_0} \leq 0$ at its minimum, then Γ'_{x_0} would be constant by the strong maximum principle (Theorem 1.12). This is impossible. Therefore, Γ'_{x_0} is strictly positive.

Now, consider the function:

$$\Gamma_{x_0}(x) = \varphi_s(x_0)\varphi_s(x)\Gamma'_{x_0}(x) \tag{5.8}$$

Clearly, Γ_{x_0} is strictly positive because both φ_s and Γ'_{x_0} are positive. Then, rewriting Equation (5.7):

$$\frac{f(x_0)}{\varphi_s(x_0)} = \int_M \left(\frac{\Gamma_{x_0}(x)}{\varphi_s(x_0)\varphi_s(x)}\right)\left(4\frac{n-1}{n-2}\widetilde{\Delta} + \widetilde{R}\right)\left(\frac{f(x)}{\varphi_s(x)}\right) dV_{\widetilde{g}}(x)$$

By the conformal transformations of the volume form (Lemma 5.2) and the conformal Laplacian (Equation (2.6)):

$$\begin{aligned}
\frac{f(x_0)}{\varphi_s(x_0)} &= \int_M \left(\frac{\Gamma_{x_0}(x)}{\varphi_s(x_0)\varphi_s(x)}\right)\left(4\frac{n-1}{n-2}\widetilde{\Delta} + \widetilde{R}\right)\left(\frac{f(x)}{\varphi_s(x)}\right) \left(\varphi_s^p \, dV_g\right)(x) \\
&= \int_M \left(\frac{\Gamma_{x_0}(x)}{\varphi_s(x_0)\varphi_s(x)}\right)\left(\varphi_s^{1-p}(x)\left(4\frac{n-1}{n-2}\Delta + R\right) f(x)\right) \varphi_s^p(x) \, dV_g(x) \\
&= \frac{1}{\varphi_s(x_0)} \int_M \Gamma_{x_0}(x)\left(4\frac{n-1}{n-2}\Delta + R\right) f(x) \, dV_g(x)
\end{aligned}$$

It follows immediately that, for all functions $f \in C_c^\infty(M)$:

$$\int_M \Gamma_{x_0}(x)\left(4\frac{n-1}{n-2}\Delta + R\right) f(x) \, dV_g(x) = f(x_0) \tag{5.9}$$

This is equivalent to:

$$\left(4\frac{n-1}{n-2}\Delta + R\right)\Gamma_{x_0} = \delta_{x_0} \tag{5.10}$$

Therefore, at each point $x_0 \in M$, the Green's function for the operator $4(n-1)/(n-2)\Delta + R$ exists and is given by Γ_{x_0}. □

Definition 5.5 (Generalized Stereographic Projection)**.** Let (M, g) be a compact, Riemannian manifold of dimension n, let $\lambda(M) > 0$, and fix some point $x_0 \in M$. Let Γ_{x_0} be the Green's function for the operator $4(n-1)/(n-2)\Delta + R$, and define the metric $\hat{g}$ on $\hat{M} = M \backslash \{x_0\}$ by:

$$\hat{g} = G^{p-2} g \tag{5.11}$$

where

$$G = 4(n-1)\operatorname{vol}(S^{n-1})\Gamma_{x_0} \tag{5.12}$$

The Riemannian manifold $(\hat{M}, \hat{g})$ together with the natural map $\sigma : M\backslash\{x_0\} \to \hat{M}$ is called the stereographic projection of M from x_0.

Proposition 5.4. Let (M, g) be a compact, Riemannian manifold of dimension n, and let $(\hat{M}, \hat{g})$ be the image manifold of a stereographic projection of M from some point $x_0 \in M$. Denote by $\hat{R}$ the scalar curvature of $(\hat{M}, \hat{g})$. Then $\hat{R} = 0$ on $\hat{M}$.

Proof. Recall that $\hat{M} = M \backslash \{x_0\}$. Since Γ_{x_0} is positive, it follows that $G = 4(n-1)\operatorname{vol}(S^{n-1})\Gamma_{x_0}$ is likewise positive. Then $\hat{g} = G^{p-2}g$ is a metric conformal to g on $\hat{M}$.

By Equation (2.4), the scalar curvature $\hat{R}$ of $\hat{g}$ is given by:

$$\begin{aligned}\hat{R} &= G^{1-p}\left(4\frac{n-1}{n-2}\Delta + R\right)G \\ &= G^{1-p}\left(4\frac{n-1}{n-2}\Delta + R\right)\left(4(n-1)\operatorname{vol}(S^{n-1})\Gamma_{x_0}\right) \\ &= 4(n-1)\operatorname{vol}(S^{n-1})G^{1-p}\left(4\frac{n-1}{n-2}\Delta + R\right)\Gamma_{x_0}\end{aligned}$$

However, recall that Γ_{x_0} is the Green's function for the operator $4(n-1)/(n-2)\Delta + R$. It follows by definition that:

$$\hat{R} = 4(n-1)\operatorname{vol}(S^{n-1})G^{1-p}\ \delta_{x_0} \tag{5.13}$$

Since $x_0 \notin \hat{M}$, we conclude that $\hat{R} = 0$ on $\hat{M}$. □

Definition 5.6 (Asymptotically Flat Manifolds)**.** Let (N, g) be a Riemannian manifold of dimension $n \geq 3$. (N, g) is asymptotically flat of order $\tau > 0$ if there exists a decomposition $N = N_0 \cup N_\infty$ such that N_0 is compact and N_∞ is diffeomorphic to $\mathbb{R}^n \backslash B_r$ for some $r > 0$, satisfying:

$$g_{ij} = \delta_{ij} + O(|z|^{-\tau}) \qquad \partial_k g_{ij} = O\left(|z|^{-\tau-1}\right) \qquad \partial_\ell \partial_k g_{ij} = O\left(|z|^{-\tau-2}\right) \tag{5.14}$$

as $|z| \to \infty$ where $\{z_i\}$ are the coordinates induced by the diffeomorphism from N_∞ to $\mathbb{R}^n$. The coordinates $\{z_i\}$ are called asymptotic coordinates.

Although it appears as though this definition depends on the choice of asymptotic coordinates, it can be shown that the asymptotically flat structure is determined solely by the metric alone.

Proposition 5.5. Let (M, g) be a compact, Riemannian manifold of dimension n, and let $(\hat{M}, \hat{g})$ be the image manifold of a stereographic projection of M from some point $x_0 \in M$. Then $(\hat{M}, \hat{g})$ is asymptotically flat. In particular, $(\hat{M}, \hat{g})$ is asymptotically flat of order 1 if $n = 3$, of order 2 if $n \geq 4$, and of order $n-2$ if M is conformally flat near x_0.

Definition 5.7 (Inverted Conformal Normal Coordinates)**.** Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, let p be a point in M, and let $\{x^i\}$ be a conformal normal coordinate system on a neighbourhood U of p. An inverted conformal normal coordinate system $\{z^i\}$ defined on $U \backslash \{p\}$ is given by:

$$z^i = |x|^{-2}x^i \tag{5.15}$$

Additionally, the basis vector fields in inverted conformal normal coordinates on $U \backslash \{p\}$ are given by:

$$\frac{\partial}{\partial z^i} = |z|^{-2}\left(\delta_{ij} - 2|z|^{-2}z^i z^j\right)\frac{\partial}{\partial x^i} \tag{5.16}$$

Definition 5.8 (General Relativity)**.** Let (X, g) be a pseudo-Riemannian manifold of dimension n. In general relativity, spacetime is a specific four-dimensional pseudo-Riemannian manifold called a Lorentzian manifold, whose metric g satisfies the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{5.17}$$

where G is the universal gravitational constant, c is the speed of light, and T is the energy-momentum tensor of the system.

The Einstein field equations are analogous to the equation in Newtonian gravity:

$$\nabla^2 \Phi = 4\pi G \rho \tag{5.18}$$

where Φ is the gravitational potential and ρ is the mass density of the system.

Definition 5.9 (Einstein-Hilbert Action)**.** Let (X, g) be a pseudo-Riemannian manifold of dimension n. The Einstein-Hilbert action S is given by the functional:

$$S(g) = \frac{c^4}{16\pi G} \int_X R \; dV_g \tag{5.19}$$

where R is the scalar curvature of g.

The first variation of the Einstein-Hilbert action yields the vacuum Einstein field equations ($T = 0$). For simplicity, we will neglect all constants:

$$S(g) = \int_X R \; dV_g \tag{5.20}$$

Lemma 5.6. Let (X, g) be a pseudo-Riemannian manifold of dimension n, and let h be a smooth, symmetric 2-tensor. Consider a family of metric tensors g_t parametrized by a single variable t such that:

$$h = \frac{d}{dt} g_t \bigg|_{t=0} \tag{5.21}$$

Suppose first that h is compactly supported. It follows by varying the Einstein-Hilbert action and by the divergence theorem that, at $t = 0$:

$$\frac{d}{dt} S(g_t) \bigg|_{t=0} = \int_X h^{ij} \left(R_{ij} - \frac{1}{2} g_{ij} R \right) dV_g \tag{5.22}$$

Suppose now that (X, g) is asymptotically flat, and let $\{z^i\}$ be a system of asymptotic coordinates on X_∞. It follows by varying the Einstein-Hilbert action and by integrating over a large sphere S_R as $R \to \infty$ in the asymptotic end that, at $t = 0$:

$$\frac{d}{dt} S(g_t) \bigg|_{t=0} = \int_X h^{ij} \left(R_{ij} - \frac{1}{2} g_{ij} R \right) dV_g - \lim_{R \to \infty} \int_{S_R} \xi \; dV_g \tag{5.23}$$

where

$$\xi = \left(\frac{\partial h_{ij}}{\partial z^i} - \frac{\partial h_{ii}}{\partial z^j} \right) \left(1 + O(|z|^{-1})\right) \tag{5.24}$$

By observation, the boundary term is the first variation of a geometric invariant, which we call mass.

Definition 5.10 (Mass-Density Vector Field)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, and let $\{z^i\}$ be a system of asymptotic coordinates on N_∞. The mass-density vector field μ defined on N_∞ is given by:

$$\mu = \left(\frac{\partial g_{ij}}{\partial z^i} - \frac{\partial g_{ii}}{\partial z^j}\right)\frac{\partial}{\partial z^j} \tag{5.25}$$

Definition 5.11 (Mass)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, and let $\{z^i\}$ be a system of asymptotic coordinates on N_∞. If the limit exists, the mass $m(g)$ of (M, g) is given by:

$$m(g) = \lim_{r\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_r} \mu \lrcorner\, dz \tag{5.26}$$

where μ is the mass-density vector field. The symbol $\lrcorner$ denotes the interior product.

Proposition 5.7. Let (N, g) be an asymptotically flat Riemannian manifold, and let g_t be a one-parameter family of metrics with $h = dg_t/dt$ at $t = 0$. Then, it follows from Lemma 5.6 that, at $t = 0$:

$$\frac{d}{dt}\left(S(g_t) + m(g_t)\right)\bigg|_{t=0} = \int_N h^{ij}\left(R_{ij} - \frac{1}{2}g_{ij}R\right) dV_g \tag{5.27}$$

5.2 Analytic Preliminaries

All definitions are taken from [LP87].

Definition 5.12 (Weighted Lebesgue Spaces)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, let $\{z^i\}$ be a system of asymptotic coordinates on N_∞, and let $\rho(z) = |z|$ on N_∞ be extended to a smooth, positive function on all of N. Let $q \geq 1$, and $\beta \in \mathbb{R}$. The weighted Lebesgue space $L^q_\beta(N)$ is the set of locally integrable functions u on N whose norm $\|u\|_{0,q,\beta}$ is finite. The β-weighted q-norm $\|\cdot\|_{0,q,\beta}$ is given by:

$$\|u\|_{0,q,\beta} = \left(\int_N |\rho^{-\beta}u|^q \rho^n \, dV_g\right)^{\frac{1}{q}} \tag{5.28}$$

Definition 5.13 (Weighted Sobolev Spaces)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, let $\{z^i\}$ be a system of asymptotic coordinates on N_∞, and let $\rho(z) = |z|$ on N_∞ be extended to a smooth, positive function on all of N. The weighted Sobolev space $W^{k,q}_\beta(N)$ is the set of functions $u \in L^q(N)$ whose weak derivatives $|\nabla^i u|$ up to order k have a finite $L^q_{\beta-i}(N)$ norm. The β-weighted Sobolev norm $\|\cdot\|_{k,q,\beta}$ is given by:

$$\|u\|_{k,q} = \sum_{i=0}^{k} \|\nabla^i u\|_{0,q,\beta-i} \tag{5.29}$$

$$= \sum_{i=0}^{k}\left(\int_N |\rho^{-(\beta-i)}\nabla^i u|^q \rho^n \, dV_g\right)^{\frac{1}{q}} \tag{5.30}$$

where the covariant derivatives $\nabla^i u$ are taken in a weak sense.

Definition 5.14 (Weighted C^k Spaces)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, let $\{z^i\}$ be a system of asymptotic coordinates on N_∞, and let $\rho(z) = |z|$ on N_∞ be extended to a smooth, positive function on all of N. The weighted C^k space $C^k_\beta(N)$ is the set of k-times continuously differentiable functions u on N whose norm $\|u\|_{C^k_\beta}$ is finite. The β-weighted $C^k_\beta(N)$ norm $\|\cdot\|_{C^k_\beta}$ is given by:

$$\|u\|_{C^k_\beta} = \sum_{i=0}^{k} \sup_N \left(\rho^{-(\beta-i)}|\nabla^i u|\right) \tag{5.31}$$

Definition 5.15 (Weighted Hölder Spaces)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, let $\{z^i\}$ be a system of asymptotic coordinates on N_∞, and let $\rho(z) = |z|$ on N_∞ be extended to a smooth, positive function on all of N. The weighted Hölder space $C^{k,\alpha}_\beta(N)$, where $0 < \alpha \leq 1$, is the set of functions $u \in C^k_\beta(N)$ whose norm $\|u\|_{C^{k,\alpha}_\beta}$ is finite. The β-weighted Hölder norm $\|\cdot\|_{C^{k,\alpha}_\beta}$ is given by:

$$\|u\|_{C^{k,\alpha}_\beta} = \|u\|_{C^k_\beta} + \sup_{x\neq y\in N} \left(\frac{|\nabla^k u(x) - \nabla^k u(y)|}{|x-y|^\alpha} \left(\min\{\rho(x), \rho(y)\}\right)^{-(\beta-k-\alpha)}\right) \tag{5.32}$$

where the supremum is taken over all points y contained within a normal coordinate neighbourhood of x for any $x \in N$.

5.3 Proof Sketch of the Positive Mass Theorem

Definition 5.16 (A Special Set of Metrics $\mathcal{M}_\tau$)**.** Let (N, g) be an asymptotically flat Riemannian manifold of dimension n and of order $\tau > (n-2)/2$, and let $\{z^i\}$ be a system of asymptotic coordinates on N_∞. $\mathcal{M}_\tau$ is the set of all C^∞ Riemannian metrics on N such that, in asymptotic coordinates $\{z^i\}$ on N_∞:

$$g_{ij} - \delta_{ij} \in C^{1,\alpha}_{-\tau}(N_\infty) \qquad\qquad R \in L^1(N) \tag{5.33}$$

where δ is the Kronecker delta, and R is the scalar curvature of g.

Additionally, in asymptotic coordinates on N_∞, the scalar curvature R of a metric $g \in \mathcal{M}_\tau$ satisfies:

$$R = g^{jk}\left(\frac{\partial \Gamma^i_{jk}}{\partial z^i} - \frac{\partial \Gamma^i_{ij}}{\partial z^k} + \Gamma^i_{i\ell}\Gamma^\ell_{jk} - \Gamma^i_{k\ell}\Gamma^\ell_{ij}\right) \tag{5.34}$$

$$= \frac{\partial}{\partial z^j}\left(\frac{\partial g_{ij}}{\partial z^i} - \frac{\partial g_{ii}}{\partial z^j}\right) + O(|z|^{-(2\tau+2)}) \tag{5.35}$$

Lemma 5.8. Let (N, g) be an asymptotically flat Riemannian manifold of dimension n and of order $\tau > (n-2)/2$. The mass functional $m(g)$ is infinitely differentiable on $\mathcal{M}_\tau$.

Theorem 5.9. Let (N, g) be an asymptotically flat Riemannian manifold of dimension n and of order $\tau > (n-2)/2$. Suppose that $g \in \mathcal{M}_\tau$. Then the mass $m(g)$ depends only on the metric g and not the choice of coordinates.

Theorem PMT (Positive Mass Theorem)**.** Let (N, g) be a Riemannian manifold of dimension $n \geq 3$ which is asymptotically flat to order $\tau > (n-2)/2$ with non-negative scalar curvature R. Then $m(g) \geq 0$ with $m(g) = 0$ if and only if (N, g) is isometric to $(\mathbb{R}^n, \delta)$.

The proof of the positive mass theorem follows directly from the three following lemmas (which, as it turns out, I didn't have time to sketch).

Lemma 5.10. Let (N, g) be an asymptotically flat Riemannian manifold of dimension n, and let $\{z^i\}$ be a system of asymptotic coordinates on N_∞. Suppose that the metric g is of the following form:

$$g_{ij}(z) = (1 + K|z|^{2-n})\delta_{ij} + \Phi_{ij}(z) \tag{5.36}$$

where K is some constant, δ is the Euclidean metric, and $\Phi_{ij} \in C^5_{1-n}(N_\infty)$.

If the scalar curvature R of g is non-negative, then the mass $m(g)$ is non-negative and:

$$m(g) = (n-1)(n-2)K \tag{5.37}$$

Lemma 5.11. Let (N, g) be a Riemannian manifold of dimension $n \geq 3$ which is asymptotically flat to order $\tau > (n-2)/2$ with non-negative scalar curvature R. Then $m(g) \geq 0$.

Lemma 5.12. Let (N, g) be a Riemannian manifold of dimension $n \geq 3$ which is asymptotically flat to order $\tau > (n-2)/2$ with non-negative scalar curvature R. If $m(g) = 0$, then N is isometric to $\mathbb{R}^n$ equipped with the Euclidean metric δ.

6 Theorem C (Schoen)

Notation. Let (M, g) be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be some function. We write $f(r) = O''(r^k)$ if f satisfies the following properties:

$$f(r) = O(r^k) \qquad \nabla f(r) = O(r^{k-1}) \qquad \nabla^2 f(r) = O(r^{k-2}) \tag{6.1}$$

Theorem 6.1. Let (M, g) be a Riemannian manifold of dimension $n \geq 2$, let $(\hat{M}, \hat{g})$ be the image manifold of M obtained by stereographic projection from some point $p \in M$. Let $\{x^i\}$ be a conformal normal coordinate system on a neighbourhood U of p, and let $\{z^i\}$ be an inverted conformal normal coordinate system on $U\backslash\{p\}$. Define:

$$\gamma = |x|^{n-2}G \tag{6.2}$$

where G is as defined in Equation (5.12).

Then, using Equation (5.16), the metric $\hat{g}$ in inverted conformal normal coordinates is given by:

$$\begin{aligned} \hat{g}_{ij}(z) &= \gamma^{p-2}|z|^4 g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) && (6.3)\\ &= \gamma^{p-2}\left(\delta_{ik} - 2\frac{z^i z^k}{|z|^2}\right)\left(\delta_{j\ell} - 2\frac{z^j z^\ell}{|z|^2}\right) g_{k\ell}\left(|z|^{-2}z\right) && (6.4)\\ &= \gamma^{p-2}\left(\delta_{ij} + O''(|z|^{-2})\right) && (6.5) \end{aligned}$$

For large values of $|z|$, it follows from Equation (6.5) that:

$$\hat{g}^{\rho\rho} = \gamma^{2-p} \qquad \det(\hat{g}) = \gamma^{2p} \tag{6.6}$$

Additionally, if $n = 3, 4, 5$, or if M is conformally flat in a neighbourhood of p, the functions G, and consequently γ, have the following asymptotic expansions:

$$G(x) = |x|^{2-n} + C + O''(r) \qquad\qquad \gamma(z) = 1 + C|z|^{2-n} + O''(|z|^{1-n}) \tag{6.7}$$

for some constant C.

Definition 6.1 (Spherical Density Function)**.** Let $(M, \widetilde{g})$ be a Riemannian manifold of dimension n, let $p \in M$, and let $\{x^i\}$ be a set of normal coordinates associated with the metric $\widetilde{g}$ in some neighbourhood of p. Let $r = |x|$, and denote by ∂B_r the geodesic $(n-1)$-sphere of radius r centered around p. The ratio of the $\widetilde{g}$-volume of ∂B_r around p to its Euclidean volume is given by the spherical density function:

$$h(r) = \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} d\widetilde{\omega}_r \tag{6.8}$$

where $d\widetilde{\omega}_r = r^{n-1} d\widetilde{\omega}$ is the volume form on ∂B_r induced by $\widetilde{g}$.

Proposition 6.2 (Distortion Coefficient)**.** Let (M, g) be a compact, Riemannian manifold of dimension $n \neq 6$. Let $(\hat{M}, \hat{g})$ be the Riemannian manifold obtained from (M, g) by stereographic projection, and let $\{z^i\}$ be a set of inverted conformal normal coordinates on $\hat{M}$. The asymptotic expansion of the spherical density function as $|z| \to \infty$ is given by:

$$h(|z|) = 1 + \left(\frac{\mu}{k}\right) |z|^{-k} + O''(|z|^{-(k+1)}) \tag{6.9}$$

for some constant k which depends on n.

The constant μ, computed in conformal normal coordinates $\{z^i\}$, is called the distortion coefficient $\hat{g}$. Its geometric meaning of μ at infinity is analogous to that of the scalar curvature at a finite point.

Proof. Let $\{x^i\}$ be a set of normal coordinates associated with the metric $\widetilde{g}$ on M, and let $r = |x|$. Observe that $\nabla r / |\nabla r|$, where ∇ here denotes the gradient, is the unit normal vector to ∂B_r. Then:

$$\begin{aligned}
h(r) &= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} d\widetilde{\omega}_r \\
&= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} \frac{\nabla r}{|\nabla r|} \lrcorner\, dV_{\widetilde{g}} \\
&= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} \frac{\nabla r}{|\nabla r|} \lrcorner\, (\det(\widetilde{g}))^{\frac{1}{2}}\ dx \\
&= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} \frac{\nabla r}{\sqrt{\langle \nabla r, \nabla r\rangle}} \lrcorner\, (\det(\widetilde{g}))^{\frac{1}{2}}\ dx \\
&= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} \frac{\nabla r}{\sqrt{\widetilde{g}(\nabla r, \nabla r)}} \lrcorner\, (\det(\widetilde{g}))^{\frac{1}{2}}\ dx \\
&= \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} (\widetilde{g}^{rr})^{-\frac{1}{2}}\, \nabla r \lrcorner\, (\det(\widetilde{g}))^{\frac{1}{2}}\ dx
\end{aligned}$$

Observe that $\nabla r = \widetilde{g}^{ij} r^{-1} x^j \partial_i$, where the inverse metric $\widetilde{g}^{ij}$ was included to ensure that the result is a vector field. Then:

$$h(r) = \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} (\widetilde{g}^{rr})^{-\frac{1}{2}}\, r^{-1} \widetilde{g}^{ij} x^j \partial_i \lrcorner\, (\det(\widetilde{g}))^{\frac{1}{2}}\ dx$$

Since $\partial_i \lrcorner\, dx = r^{-1}x^i \; d\omega_r$ on ∂B_r, the expression reduces to:

$$h(r) = \frac{1}{r^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_r} (\widetilde{g}^{rr} \det(\widetilde{g}))^{\frac{1}{2}} \; d\omega_r \tag{6.10}$$

Consider now an image manifold $(\hat{M}, \hat{g})$ with inverted normal conformal normal coordinates obtained by stereographic projection from some point p in M, and let $\rho = |z|$. In these coordinates, the spherical density function is given by:

$$h(\rho) = \frac{1}{\rho^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_\rho} (\hat{g}^{\rho\rho} \det(\hat{g}))^{\frac{1}{2}} \; d\omega_\rho$$

Since $\hat{g}^{\rho\rho} = \gamma^{2-p}$ and $\det(\hat{g}) = \gamma^{2p}$ (Theorem 6.1):

$$\begin{aligned} h(\rho) &= \frac{1}{\rho^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_\rho} \left(\gamma^{2-p}\gamma^{2p}\right)^{\frac{1}{2}} \; d\omega_\rho \\ &= \frac{1}{\rho^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_\rho} \gamma^{\frac{p+2}{2}} \; d\omega_\rho \end{aligned} \tag{6.11}$$

Using the expansion for γ given in Theorem 6.1:

$$h(\rho) = \frac{1}{\rho^{n-1}\,\mathrm{vol}(S^{n-1})} \int_{\partial B_\rho} (1 + C\rho^{2-n} + O''(\rho^{1-n}))^{\frac{p+2}{2}} \; d\omega_\rho$$

Applying the binomial expansion and integrating over the sphere of radius ρ:

$$h(\rho) = 1 + C\rho^{-k} + O''(\rho^{-(k+1)}) \tag{6.12}$$

for $k = n - 2$ and for some constant C.

We define the distortion coefficient μ to be this leading order coefficient C multiplied by k. □

Lemma 6.3. Let (M, g) be a compact, Riemannian manifold of dimension $n \neq 6$. Let $(\hat{M}, \hat{g})$ be the Riemannian manifold obtained from (M, g) by stereographic projection, let $\{z^i\}$ be a set of inverted conformal normal coordinates on $\hat{M}$, and let $\rho = |z|$. Then:

$$\frac{4}{2}\frac{n-1}{n-2} \int_{\partial B_\rho} (\partial_\rho \gamma)\, \mathrm{vol}(\partial B_\rho)^{-1} \; d\omega_\rho = h'(\rho) + O(\rho^{-(2k+1)}) \tag{6.13}$$

$$= -\mu\rho^{-k-1} + O(\rho^{-(k+2)}) \tag{6.14}$$

Proof. Recall that the Euclidean volume of an $(n-1)$-sphere of radius ρ is given by:

$$\mathrm{vol}(\partial B_\rho) = \rho^{n-1}\,\mathrm{vol}(S^{n-1}) \tag{6.15}$$

From Equation (6.11):

$$\begin{aligned} h'(\rho) &= \frac{d}{d\rho}\left(\frac{1}{\mathrm{vol}(\partial B_\rho)} \int_{\partial B_\rho} \gamma^{\frac{p+2}{2}} \; d\omega_\rho\right) \\ &= \frac{d}{d\rho}\left(\int_{\partial B_\rho} \gamma^{\frac{p+2}{2}}\, \mathrm{vol}(\partial B_\rho)^{-1} \; d\omega_\rho\right) \end{aligned}$$

Since $\mathrm{vol}(\partial B_\rho)^{-1} d\omega_\rho$ is a homogeneous $(n-1)$-form of degree zero, we may differentiate under the integral sign in the following way:

$$\begin{aligned} h'(\rho) &= \int_{\partial B_\rho} \partial_\rho \left(\gamma^{\frac{p+2}{2}}\right) \mathrm{vol}(\partial B_\rho)^{-1}\ d\omega_\rho \\ &= \int_{\partial B_\rho} \frac{p+2}{2} \gamma^{\frac{p}{2}} (\partial_\rho \gamma)\, \mathrm{vol}(\partial B_\rho)^{-1}\ d\omega_\rho \\ &= \frac{4}{2}\frac{n-1}{n-2} \int_{\partial B_\rho} \gamma^{\frac{p}{2}} (\partial_\rho \gamma)\, \mathrm{vol}(\partial B_\rho)^{-1}\ d\omega_\rho \end{aligned}$$

where we used the fact that $p = 2n/(n-2)$.

Using the asymptotic expansion for γ given in Theorem 6.1:

$$\begin{aligned} h'(\rho) &= \frac{4}{2}\frac{n-1}{n-2} \int_{\partial B_\rho} (1 + C\rho^{2-n} + O''(\rho^{1-n}))^{\frac{p}{2}} (\partial_\rho \gamma)\, \mathrm{vol}(\partial B_\rho)^{-1}\ d\omega_\rho \\ &= \frac{4}{2}\frac{n-1}{n-2} \int_{\partial B_\rho} (\partial_\rho \gamma)\, \mathrm{vol}(\partial B_\rho)^{-1}\ d\omega_\rho + O(\rho^{-(k+2)}) \end{aligned}$$

By direct computation and using the definition of μ:

$$h'(\rho) = -\mu \rho^{-k-1} + O(\rho^{-(k+2)})$$

The desired result follows immediately. □

Lemma 6.4. Let (M, g) be a compact, Riemannian manifold of dimension n, and let $p \in M$. Let $(\hat{M}, \hat{g})$ be the stereographic projection of M from p, and let μ be the distortion coefficient computed in inverted conformal normal coordinates $\{z^i\}$. If $n < 6$ or if M is conformally flat in a neighbourhood of p, then $\mu = m(\hat{g})/2$.

Proof. Suppose that $n < 6$ or that M is conformally flat in a neighbourhood of $p \in M$. Using Theorem 6.1 and the fact that the scalar curvature $\hat{R}$ of $\hat{g}$ is identically zero on $\hat{M}$ (Proposition 5.4), we may conclude that $\hat{g} \in \mathcal{M}_\tau$ with $\tau > (n-2)/2$. The mass $m(\hat{g})$ can then be defined.

Let $\{z^i\}$ be a set of inverted conformal normal coordinates on $\hat{M}_\infty$. Recall that the mass $m(g)$ is given by:

$$\begin{aligned} m(\hat{g}) &= \lim_{r\to\infty} \mathrm{vol}(S^{n-1})^{-1} \int_{\partial B_r} \left(\frac{\partial g_{ij}}{\partial z^i} - \frac{\partial g_{ii}}{\partial z^j}\right) \frac{\partial}{\partial z^j} \lrcorner\, dz \\ &= \lim_{r\to\infty} \mathrm{vol}(S^{n-1})^{-1} \left(\int_{\partial B_r} \frac{\partial g_{ij}}{\partial z^i} \frac{\partial}{\partial z^j} \lrcorner\, dz - \int_{\partial B_r} \frac{\partial g_{ii}}{\partial z^j} \frac{\partial}{\partial z^j} \lrcorner\, dz\right) \end{aligned}$$

Let $\rho = |z|$. On the $(n-1)$-sphere ∂B_ρ, we have that:

$$\begin{aligned} \frac{\partial}{\partial z^j} \lrcorner\, dz &= \frac{z^j}{|z|}\ d\omega_\rho \\ &= \frac{z^j z^k}{|z|^2} \frac{\partial}{\partial z^k} \lrcorner\, dz \end{aligned}$$

The mass is then given by:

$$\begin{aligned} m(\hat{g}) &= \lim_{r\to\infty} \mathrm{vol}(S^{n-1})^{-1} \left(\int_{\partial B_r} \frac{\partial g_{ij}}{\partial z^i} \frac{z^j z^k}{|z|^2} \frac{\partial}{\partial z^k} \lrcorner\, dz - \int_{\partial B_r} \frac{\partial g_{ii}}{\partial z^k} \frac{\partial}{\partial z^k} \lrcorner\, dz \right) \\ &= \lim_{\rho\to\infty} \mathrm{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} \left(\frac{z^j z^k}{|z|^2} \frac{\partial g_{ij}}{\partial z^i} - \frac{\partial g_{ii}}{\partial z^k} \right) \frac{\partial}{\partial z^k} \lrcorner\, dz \end{aligned}$$

where we relabelled the index $j \to k$ in the second integral term.

Consider the $(n-2)$-form η given by:

$$\eta = z^j z^k \hat{g}_{ij} \partial_i \lrcorner\, \partial_k \lrcorner\, dz \tag{6.16}$$

Observe that the exterior derivative of η is given by:

$$d\eta = \left(z^j z^k \partial_i \hat{g}_{ij} - z^j z^i \partial_i \hat{g}_{kj} + z^k \hat{g}_{ii} - n z^j \hat{g}_{kj} \right) \partial_k \lrcorner\, dz \tag{6.17}$$

Since the boundary of ∂B_ρ is empty, by Stokes' Theorem:

$$\int_{\partial B_\rho} d\eta = 0 \tag{6.18}$$

Observe now that:

$$\begin{aligned} \hat{g}_{\rho\rho} &= \hat{g}(\partial_\rho, \partial_\rho) \\ &= \frac{z^k z^j}{|z|^2} \hat{g}_{kj} \end{aligned} \qquad \begin{aligned} \partial_\rho \hat{g}_{\rho\rho} &= \partial_\rho \left(\frac{z^k z^j}{|z|^2} \hat{g}_{kj} \right) \\ &= \frac{z^i z^j z^k}{|z|^3} \frac{\partial \hat{g}_{kj}}{\partial z^i} \end{aligned}$$

It follows then that:

$$m(\hat{g}) = \lim_{\rho\to\infty} \mathrm{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} \left(\partial_\rho(\hat{g}_{\rho\rho} - \hat{g}_{ii}) + \rho^{-1}(n\hat{g}_{\rho\rho} - \hat{g}_{ii}) \right) d\omega_\rho$$

Recall that, in inverted normal coordinates, $\hat{g}_{\rho\rho} = \gamma^{p-2}$ and $\det(\hat{g}) = \gamma^{2p} = 1 + O(\rho^{2-n})$. Then:

$$n\partial_\rho \hat{g}_{\rho\rho} = n\partial_\rho(\gamma^{p-2}) \tag{6.19}$$

$$= n(p-2)\gamma^{p-3}\partial_\rho \gamma \tag{6.20}$$

Using the fact that $p = 2n/(n-2)$ and undoing a chain rule:

$$\begin{aligned} n\partial_\rho \hat{g}_{\rho\rho} &= 2p\gamma^{p-3}\partial_\rho\gamma \\ &= \gamma^{p-2}\partial_\rho(\log(\gamma^{2p})) \\ &= \gamma^{p-2}\partial_\rho(\log(\det(\hat{g}))) \\ &= \gamma^{p-2}\det(\hat{g})^{-1}\partial_\rho(\det(\hat{g})) \end{aligned}$$

Recall that since $(\hat{M}, \hat{g})$ is asymptotically flat of order $\tau > (n-2)/2$, we have that $\hat{g}_{ij} = \delta_{ij} + O(\rho^{-\tau})$. Then, after applying the binomial expansion, we find that $\det(\hat{g})^{-1} = (1+O(\rho^{-\tau}))^{-1}$. Furthermore, after applying the determinant formula and using the expansion of $\hat{g}_{ij}$ in terms of δ_{ij} we obtain $\partial_\rho(\det(\hat{g})) = \partial_\rho(\hat{g}_{ii}) + O(\rho^{-(\tau+1)})$.

Finally, using the asymptotic expansion for γ given in Theorem 6.1:

$$n\partial_\rho \hat{g}_{\rho\rho} = \partial_\rho \hat{g}_{ii} + O(\rho^{-(2\tau+1)}) \tag{6.21}$$

Due to asymptotic flatness, we also have that $\hat{g}_{ii} = n$ (in Einstein summation notation) and $\hat{g}_{\rho\rho} = 1$ at $\rho = \infty$. Then, by integrating along the radial direction ρ from infinity, we obtain on $\hat{M}_\infty$:

$$n\hat{g}_{\rho\rho} = \hat{g}_{ii} + O(\rho^{-2\tau}) \tag{6.22}$$

Then, using Equation (6.22), the second term in the integrand is $O(\rho^{-(2\tau+1)})$ and vanishes in the limit as $\rho \to \infty$:

$$\begin{aligned} m(\hat{g}) &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} \left(\partial_\rho(\hat{g}_{\rho\rho} - \hat{g}_{ii}) + \rho^{-1}O(\rho^{-2\tau})\right) d\omega_\rho \\ &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} \partial_\rho(\hat{g}_{\rho\rho} - \hat{g}_{ii})\; d\omega_\rho \\ &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} 4\frac{1-n}{n-2}\partial_\rho\gamma\; d\omega_\rho \end{aligned}$$

Additionally, using Equations (6.21) and (6.20):

$$\begin{aligned} m(\hat{g}) &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} \Big(\partial_\rho(\hat{g}_{\rho\rho} - n\hat{g}_{\rho\rho}) + O(\rho^{-(2\tau+1)})\Big)\; d\omega_\rho \\ &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} (1-n)\partial_\rho \hat{g}_{\rho\rho}\; d\omega_\rho \\ &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} (1-n)(p-2)\gamma^{p-3}\partial_\rho\gamma\; d\omega_\rho \end{aligned}$$

Using the asymptotic expansion for γ and the fact that $p = 2n/(n-2)$:

$$\begin{aligned} m(\hat{g}) &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} 4\frac{1-n}{n-2}\left(1 + C\rho^{2-n} + O(\rho^{1-n})\right)^{p-3} \partial_\rho\gamma\; d\omega_\rho \\ &= \lim_{\rho\to\infty} \operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} 4\frac{1-n}{n-2}\gamma^{p-3}\partial_\rho\gamma\; d\omega_\rho \end{aligned}$$

Recall from Lemma 6.3 that:

$$\rho^{1-n}\operatorname{vol}(S^{n-1})^{-1} \int_{\partial B_\rho} 4\frac{n-1}{n-2}\partial_\rho\gamma\; d\omega_\rho = -2\mu\rho^{-k-1} + O(\rho^{-(k+2)}) \tag{6.23}$$

Since $k = n-2$, it follows by taking the limit that $m(\hat{g}) = 2\mu$. $\square$

Theorem 6.5. Let $n \in \mathbb{N}$, and consider the Sobolev extremal functions on $\mathbb{R}^n$ given by:

$$u_\alpha(x) = \left(\frac{|x|^2 + \alpha^2}{\alpha}\right)^{\frac{2-n}{2}} \tag{6.24}$$

The Yamabe constant of the n-sphere $\lambda(S^n)$ is given by:

$$\lambda(S^n) = 4n(n-1)\|u_\alpha\|_p^{p-2} \tag{6.25}$$

where $p = 2n/(n-2)$ is the critical Sobolev exponent.

Proof. Let Δ denote the Euclidean Laplacian. It can be shown by direct computation that:

$$\Delta u_\alpha = n(n-2)u_\alpha^{p-1} \tag{6.26}$$

where $p = 2n/(n-2)$ is the critical Sobolev exponent.

Multiplying both sides by u_α and integrating:

$$\int_{\mathbb{R}^n} u_\alpha \Delta u_\alpha \, dx = n(n-2)\int_{\mathbb{R}^n} u_\alpha^p \, dx$$

Applying integration by parts on the left-hand side:

$$\int_{\mathbb{R}^n} |\nabla u_\alpha|^2 \, dx = n(n-2)\int_{\mathbb{R}^n} u_\alpha^p \, dx$$

Then:

$$\|\nabla u_\alpha\|_2^2 = n(n-2)\|u_\alpha\|_p^p$$

Recall that, on $\mathbb{R}^n$, the functions u_α satisfy Equation (4.6) from Theorem 4.3:

$$4\frac{n-1}{n-2}\|\nabla u_\alpha\|_2^2 = \lambda(S^n)\|u_\alpha\|_p^2$$

It directly follows that:

$$\lambda(S^n) = 4n(n-1)\|u_\alpha\|_p^{p-2}$$

□

Proposition 6.6. Let (M, g) be a compact, Riemannian manifold of dimension n. Suppose that $n = 3$, 4 5, or suppose M is locally conformally flat. Fix a large radius $R > 0$, and let $\{z^i\}$ be a set of inverted conformal normal coordinates associated with the conformal metric $\widetilde{g}$. Define a set of smooth, positive functions $\varphi_\alpha : M \to \mathbb{R}$ by:

$$\varphi_\alpha(z) = \begin{cases} u_\alpha(z), & |z| \geq R \\ u_\alpha(R), & |z| \leq R \end{cases} \tag{6.27}$$

where

$$u_\alpha(z) = \left(\frac{|z|^2 + \alpha^2}{\alpha}\right)^{\frac{2-n}{2}} \tag{6.28}$$

Then, as $\alpha \to \infty$, there exist positive constants k and C such that:

$$Q_g(\varphi_\alpha) \leq \lambda(S^n) - C\mu\alpha^{-k} + O(\alpha^{-(k+1)}) \tag{6.29}$$

Proof. Let $(\hat{M}, \hat{g})$ be the image manifold of a stereographic projection of M from some point $x_0 \in M$. It follows by Proposition 5.5 that $\hat{M}$ is asymptotically flat. Then $\hat{M} = \hat{M}_0 \cup \hat{M}_\infty$.

By definition of asymptotically flat manifolds, recall that $\hat{M}_\infty$ is diffeomorphic to $\mathbb{R}^n \backslash B_r$ for some radius $r > 0$. Let the radius R defined in Equation (6.27) be this radius r.

Recall that the energy $E(\varphi_\alpha)$ on $\hat{M}$ is given by:

$$\begin{aligned} E(\varphi_\alpha) &= \int_{\hat{M}} \left(4\frac{n-1}{n-2} |\nabla \varphi_\alpha|^2 + \hat{R}\varphi_\alpha^2 \right) dV_{\hat{g}} \\ &= 4\frac{n-1}{n-2} \left(\int_{\hat{M}_0} |\nabla \varphi_\alpha|^2 \, dV_{\hat{g}} + \int_{\hat{M}_\infty} |\nabla \varphi_\alpha|^2 \, dV_{\hat{g}} \right) + \int_{\hat{M}} \hat{R}\varphi_\alpha^2 \, dV_{\hat{g}} \end{aligned}$$

By Proposition 5.4, the scalar curvature $\hat{R}$ of $\hat{g}$ is zero:

$$E(\varphi_\alpha) = 4\frac{n-1}{n-2} \left(\int_{\hat{M}_0} |\nabla \varphi_\alpha|^2 \, dV_{\hat{g}} + \int_{\hat{M}_\infty} |\nabla \varphi_\alpha|^2 \, dV_{\hat{g}} \right)$$

Let $\rho = |z|$. Using the definition of φ_α (Equation (6.27)) and the fact that u_α is only a function of ρ:

$$\begin{aligned} E(\varphi_\alpha) &= 4\frac{n-1}{n-2} \left(\int_{\hat{M}_0} |\nabla \left(u_\alpha(R) \right)|^2 \, dV_{\hat{g}} + \int_{\hat{M}_\infty} |\nabla u_\alpha(z)|^2 \, dV_{\hat{g}}(z) \right) \\ &= 4\frac{n-1}{n-2} \int_{\hat{M}_\infty} \hat{g}^{\rho\rho} (\partial_\rho u_\alpha)^2 \, dV_{\hat{g}} \\ &= 4\frac{n-1}{n-2} \int_{\hat{M}_\infty} \hat{g}^{\rho\rho} (\partial_\rho u_\alpha)^2 \sqrt{\det(\hat{g})} \, dz \end{aligned}$$

It follows from Theorem 6.1 that $\hat{g}^{\rho\rho} = \gamma^{2-p}$ and $\det(\hat{g}) = \gamma^{2p}$:

$$\begin{aligned} E(\varphi_\alpha) &= 4\frac{n-1}{n-2} \int_{\hat{M}_\infty} \gamma^{2-p} (\partial_\rho u_\alpha)^2 \sqrt{\gamma^{2p}} \, dz \\ &= 4\frac{n-1}{n-2} \int_{\hat{M}_\infty} (\partial_\rho u_\alpha)^2 \gamma^2 \, dz \end{aligned} \tag{6.30}$$

Consider now the following integral:

$$4\frac{n-1}{n-2} \int_{B_L - B_R} (\partial_\rho u_\alpha)^2 \gamma^2 \, dz \tag{6.31}$$

where $B_L - B_R$ denotes the annulus $\{R \leq |z| \leq L\}$.

Let Δ_0 denote the Euclidean Laplacian. Applying integration by parts to Equation (6.31):

$$\begin{aligned} 4\frac{n-1}{n-2} \int_{B_L - B_R} (\partial_\rho u_\alpha)^2 \gamma^2 \, dz = 4\frac{n-1}{n-2} & \int_{B_L - B_R} u_\alpha \Delta_0 u_\alpha \gamma^2 \, dz \\ & - 4\frac{n-1}{n-2} \int_{B_L - B_R} u_\alpha \partial_\rho u_\alpha \partial_\rho (\gamma^2) \, dz \\ & - 4\frac{n-1}{n-2} \int_{\partial B_L \cup \, \partial B_R} u_\alpha \partial_\rho u_\alpha \gamma^2 \partial_\rho \lrcorner \, dz \end{aligned} \tag{6.32}$$

Consider the third integral term in Equation (6.32). Observe that:

$$\begin{aligned} u_\alpha \partial_\rho u_\alpha &= \left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{\frac{2-n}{2}} \frac{(2-n)\rho}{\alpha} \left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{-\frac{n}{2}} \\ &= \frac{(2-n)\rho}{\alpha} \left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n} \end{aligned}$$

Then, for a fixed α, we find that $u_\alpha \partial_\rho u_\alpha$ is $O(L^{3-2n})$ on ∂B_L. Since γ is bounded and the volume form $\partial_\rho \lrcorner\, dz$ on ∂B_L contributes $O(L^{n-1})$, the integral over ∂B_L is $O(L^{2-n})$ for a fixed α and vanishes as $L \to \infty$ since $n > 2$.

Similarly, we find that the integral over ∂B_R is $O(\alpha^{-n})$ because $u_\alpha \partial_\rho u_\alpha$ is $O(\alpha^{-n})$ on ∂B_R.

Consider now the first integral term in Equation (6.32). Evaluating the Laplacian of u_α by direct computation:

$$\begin{aligned} 4\frac{n-1}{n-2}\int_{B_L-B_R} u_\alpha \Delta_0 u_\alpha \gamma^2 \, dz &= 4\frac{n-1}{n-2}\int_{B_L-B_R} n(n-2) u_\alpha u_\alpha^{p-1} \gamma^2 \, dz \\ &= 4n(n-1)\int_{B_L-B_R} u_\alpha^{p-2} (u_\alpha \gamma)^2 \, dz \end{aligned}$$

Let $\alpha = 1/(1-2/p)$ and $\beta = p/2$ be conjugate exponents. By Hölder's inequality:

$$\begin{aligned} 4\frac{n-1}{n-2}\int_{B_L-B_R} u_\alpha \Delta_0 u_\alpha \gamma^2 \, dz &\le 4n(n-1)\left(\int_{B_L-B_R} u_\alpha^{\alpha(p-2)} \, dz\right)^{\frac{1}{\alpha}} \left(\int_{B_L-B_R} (u_\alpha\gamma)^{2\beta} \, dz\right)^{\frac{1}{\beta}} \\ &= 4n(n-1)\left(\int_{B_L-B_R} u_\alpha^{p} \, dz\right)^{1-\frac{2}{p}} \left(\int_{B_L-B_R} (u_\alpha\gamma)^{p} \, dz\right)^{\frac{2}{p}} \\ &\le 4n(n-1)\left(\int_{\hat{M}} u_\alpha^{p} \, dz\right)^{1-\frac{2}{p}} \left(\int_{\hat{M}} u_\alpha^p\gamma^{p} \, dz\right)^{\frac{2}{p}} \\ &\le 4n(n-1)\|u_\alpha\|_p^{p-2} \left(\int_{\hat{M}} u_\alpha^p\gamma^{p} \, dz\right)^{\frac{2}{p}} \end{aligned}$$

Recall that $\det(\hat{g}) = \gamma^{2p}$ (Theorem 6.1). Then, by Theorem 6.5:

$$\begin{aligned} 4\frac{n-1}{n-2}\int_{B_L-B_R} u_\alpha \Delta_0 u_\alpha \gamma^2 \, dz &\le 4n(n-1)\|u_\alpha\|_p^{p-2} \left(\int_{\hat{M}} u_\alpha^p \sqrt{\det(\hat{g})} \, dz\right)^{\frac{2}{p}} \\ &= 4n(n-1)\|u_\alpha\|_p^{p-2} \left(\int_{\hat{M}} u_\alpha^p \, dV_{\hat{g}}\right)^{\frac{2}{p}} \\ &= 4n(n-1)\|u_\alpha\|_p^{p-2}\|\varphi\|_p^2 \\ &= \lambda(S^n)\|\varphi\|_p^2 \end{aligned}$$

Consider finally the second integral term. Denote by $d\omega_\rho$ the standard volume form on ∂B_ρ. Then, letting $L \to \infty$:

$$4\frac{n-1}{n-2}\lim_{L\to\infty}\int_{B_L-B_R} u_\alpha \partial_\rho u_\alpha \partial_\rho(\gamma^2) \, dz = 4\frac{n-1}{n-2}\int_R^\infty u_\alpha \partial_\rho u_\alpha \int_{\partial B_\rho} \partial_\rho(\gamma^2) \, d\omega_\rho d\rho \qquad (6.33)$$

Using the asymptotic expansion for γ given in Theorem 6.1:

$$\begin{aligned}
4\frac{n-1}{n-2}\int_{\partial B_\rho} \partial_\rho(\gamma^2)\ d\omega_\rho &= 8\frac{n-1}{n-2}\int_{\partial B_\rho} \gamma\partial_\rho\gamma\ d\omega_\rho \\
&= 8\frac{n-1}{n-2}\int_{\partial B_\rho} \left(1 + C\rho^{2-n} + O(\rho^{1-n})\right)\partial_\rho\gamma\ d\omega_\rho \\
&= 8\frac{n-1}{n-2}\int_{\partial B_\rho} \partial_\rho\gamma\ d\omega_\rho + O(\rho^{-2n+3}) \\
&= 4\frac{4}{2}\frac{n-1}{n-2}\int_{\partial B_\rho} \partial_\rho\gamma\ d\omega_\rho + O(\rho^{-(2k+1)})
\end{aligned}$$

where we used the fact that $k = n-2$.

Since $n \neq 6$ or if M is conformally flat near $p \in M$, we may apply Lemma 6.3:

$$\begin{aligned}
4\frac{n-1}{n-2}\int_{\partial B_\rho} \partial_\rho(\gamma^2)\ d\omega_\rho &= 4\left(h'(\rho) + O(\rho^{-(2k+1)})\right)\rho^{n-1}\operatorname{vol}(S^{n-1}) \\
&= -4\left(\mu\rho^{-(k+1)} + O(\rho^{-(k+2)})\right)\rho^{n-1}\operatorname{vol}(S^{n-1}) \\
&= -4\mu\rho^{-(k+1)}\operatorname{vol}(S^{n-1})\rho^{n-1} + \rho^{n-1}O(\rho^{-(k+2)})
\end{aligned}$$

Then the second integral term (continuing from Equation (6.33)) is given by:

$$\begin{aligned}
4\frac{n-1}{n-2}\lim_{L\to\infty}\int_{B_L - B_R} u_\alpha\partial_\rho u_\alpha\partial_\rho(\gamma^2)\ dz &= -4\mu\operatorname{vol}(S^{n-1})\int_R^\infty u_\alpha\partial_\rho u_\alpha\rho^{-(k+1)}\rho^{n-1}\ d\rho \\
&\quad - \int_R^\infty u_\alpha\partial_\rho u_\alpha O(\rho^{-(k+2)})\rho^{n-1}\ d\rho \\
&= -4\mu\alpha^{-1}(2-n)\operatorname{vol}(S^{n-1})\int_R^\infty \rho^{-k}\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\rho^{n-1}\ d\rho \\
&\quad - \alpha^{-1}\int_R^\infty O(\rho^{-(k+1)})\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\rho^{n-1}\ d\rho
\end{aligned}$$

Using a similar reasoning as in Lemma 4.5, the substitution $\sigma = \rho/\alpha$ shows that if $2-n < k < n$, the following integral is bounded:

$$C^{-1}\alpha^{-k+1} \leq \int_R^\infty \rho^{-k}\left(\frac{\rho^2+\alpha^2}{\alpha}\right)^{1-n}\rho^{n-1}\ d\rho \leq C\alpha^{-k+1}$$

for some constant C.

It follows then that:

$$\begin{aligned}
4\frac{n-1}{n-2}\lim_{L\to\infty}\int_{B_L - B_R} u_\alpha\partial_\rho u_\alpha\partial_\rho(\gamma^2)\ dz &\leq -C\mu\alpha^{-1}\alpha^{-k+1} + \alpha^{-1}O(\alpha^{-k}) \\
&\leq -C\mu\alpha^{-k} - O(\alpha^{-(k+1)})
\end{aligned}$$

for some new constant C.

Combining the above results:

$$E(\varphi_\alpha) \le \lambda(S^n)\|\varphi_\alpha\|_p^2 - C\mu\alpha^{-k} + O(\alpha^{-(k+1)}) \tag{6.34}$$

for some positive constants C and k.

Dividing by $\|\varphi_\alpha\|_p^2$, we obtain the desired result:

$$Q_g(\varphi_\alpha) \le \lambda(S^n) - C\mu\alpha^{-k} + O(\alpha^{-(k+1)}) \tag{6.35}$$

for some new constant C. □

Theorem C (Schoen)**.** Suppose that (M, g) is a compact, Riemannian manifold of dimension $n = 3$, 4, or 5 or is locally, conformally flat. Then $\lambda(M) < \lambda(S^n)$ unless M is conformal to S^n.

Proof. Suppose that (M, g) is a compact, Riemannian manifold of dimension $n = 3$, 4, or 5 or is locally, conformally flat. Without loss of generality, suppose that $\lambda(M) > 0$. This is because the inequality $\lambda(M) < \lambda(S^n)$ is trivial since $\lambda(S^n) > 0$.

Fix some $p \in M$, and denote by $(\hat{M}, \hat{g})$ the stereographic projection of M from p. If $(\hat{M}, \hat{g})$ is isometric to $(\mathbb{R}^n, \delta)$, then (M, g) is necessarily conformal to (S^n, g_0) and has constant scalar curvature. Suppose then that $(\hat{M}, \hat{g})$ is not isometric $(\mathbb{R}^n, \delta)$.

Recall that $(\hat{M}, \hat{g})$ is an asymptotically flat manifold of order $\tau > (n-2)/2$ and that the scalar curvature $\hat{R}$ of $\hat{g}$ is zero (Proposition 5.4). Since $(\hat{M}, \hat{g})$ is not isometric $(\mathbb{R}^n, \delta)$ and the scalar curvature is non-negative, we may apply the positive mass theorem (Theorem PMT) to conclude that the mass $m(\hat{g})$ is positive. By Lemma 6.4, it follows that the distortion coefficient μ is positive as well.

Let $\{\varphi_\alpha\}$ be the sequence of smooth, positive functions as defined in Proposition 6.6. Then, as $\alpha \to \infty$, φ_α satisfies:

$$Q_g(\varphi_\alpha) \le \lambda(S^n) - C\mu\alpha^{-k} + O(\alpha^{-(k+1)}) \tag{6.36}$$

for some positive constants C and k.

Since $\mu > 0$, it follows immediately by the definition of infimum that $\lambda(M) < \lambda(S^n)$ for a sufficiently large α. □

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