Occurring Everywhere, Comprehensible Nowhere: An Exploration Into Fractal Geometry

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ABSTRACT. The developments in fractal geometry during the 20^{th} century have achieved a feat quite rare in modern mathematics: they are known by normal people. That is, some of the public at least has a vague idea of what a fractal is. But as we venture down the spiraling, self-similar nature of these beautiful behemoths, a variety of complications arise that challenge conventional understandings. This paper is subsequently divided into two parts. The first is an overview of the basic mathematics behind two of the most popular fractals: the Mandelbrot Set and the Sierpinski Triangle. The second introduces a construction that illustrates some of the seemingly paradoxical properties that fractals can have, along with a connection to the coastline problem.

CONTENTS

1. FRACTALS: THE BASICS

If one were to ask a normal person what they knew about fractals, there is a decent chance that they would mention something about beautiful images or certain fractals they may have come across, such as the Mandelbrot Set. It is also likely that they could tell you little about the mathematical formulation of these fractals or various properties they may hold. It is thus necessary to lay down the fundamental definitions and the key mathematical ideas behind what makes a fractal. This section will do so through explorations into two popular fractals: the aformentioned Mandelbrot set and the Sierpinski Triangle.

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1.1. **The Mandelbrot Set: An Exploration Into Self-Similarity.**

The Mandelbrot Set is a bizarre creation. The beautiful image (see Figure 1) makes one think of the fractal as a piece of art, not a set per se. Yet many fractals are merely a collection of numbers or points. The Mandelbrot Set exemplifies this and is constructed in the following way:

First, let $c \in \mathbb{C}$. Consider the function $f_c(z) = z^2 + c$. Start at $z = 0$, and then iterate the function ad infinitum. That is, consider $f_c(0), f(f_c(0)), f(f(f_c(0)))$, and so on. A point c is in the Mandelbrot set if the sequence of iterations does not diverge to infinity! Thus, the Mandelbrot Set is just a collection of points that satisfy a certain property. Indeed, we get the beautiful image by color coding regions based upon how soon the sequence of absolute values of the function iterations cross a threshold value, e.g. two iterations until the modulus (length) of the complex number is larger than 3.

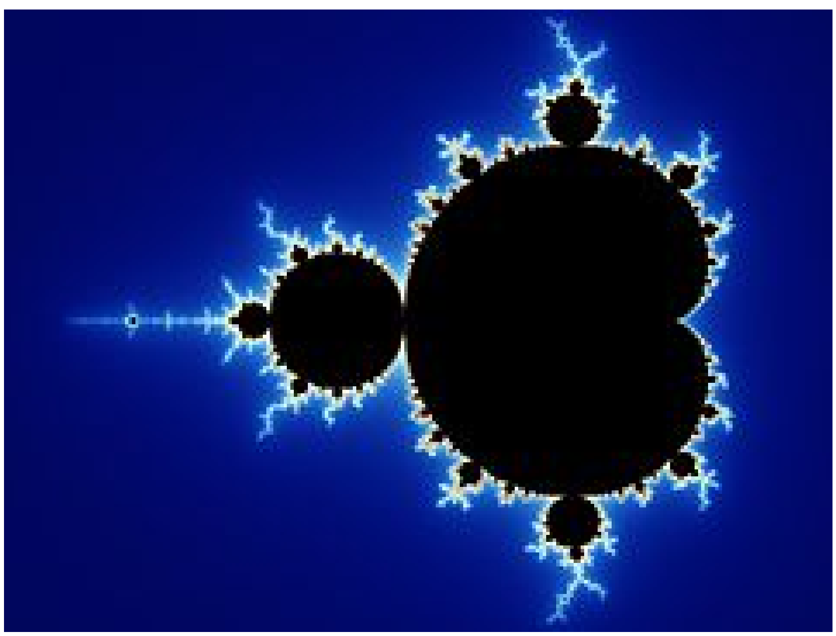
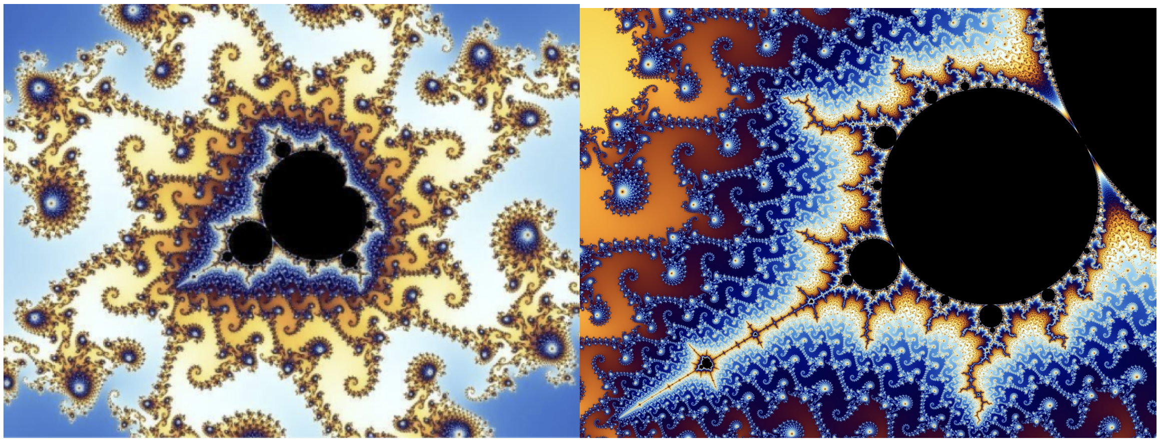


FIGURE 1. The Mandelbrot Set

What we get is an image that exhibits an important feature of fractals, that being self-similarity. Informally, self-similarity means that when you 'zoom in' on one part of the fractal, the fractal starts to 'repeat itself'. If we consult the following figures, we can see how when zooming in on the boundary of the Mandelbrot set, the image start to occur again, although in a slightly different way.



(A) (B)

FIGURE 2. Zoomed In Images of the Mandelbrot Set

With this intuitive understanding in mind, we can now give a formal definition of self-similarity.

Definition 1.1 (Self-Similarity)**.** *Let X be a compact, topological space. We say that X is self-similar if there exists an indexing set S and a set* $\{f_s : s \in S\}$ *of non-surjective homeomorphisms such that* $X = \bigcup_{s \in S} f_s(X)$.

Recall that a homeomorphism is a bijective, continuous function between topological spaces with a continuous inverse. Basically, what this formal definition is saying is that a fractal is self-similar if we can write it as a collection of bijective, continuous transformations of itself. Although seemingly complicated, this definition makes sense: the fractal is constructed from transformations of itself, so some parts of the fractal can 'look like' the whole or other parts.

1.2. **The Sierpinski Triangle: How Can We Make Fractals?**

The next object we will look at is the Sierpinski Triangle (see Figure 3). One reason to study it is due to its interesting construction. Indeed, it is common for fractals to be made in the way the Sierpinski Triangle is, where we begin with an initial object, the initiator, and replace parts of the object with new objects, the generators. If we refer to Figure 4, we see that the Sierpinski Triangle is made by starting with a normal triangle and then removing the 'middle' triangle, repeating this process ad infinitum for each remaining triangle at each stage. The Sierpinski Triangle is the part of the original triangle that is left over.

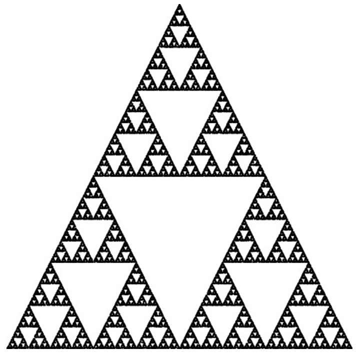


FIGURE 3. Sierpinski Triangle

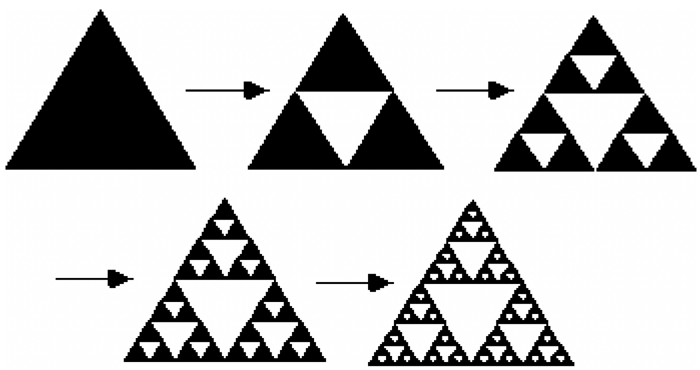


FIGURE 4. Generation Process for the Sierpinski Triangle

Before concluding this section, I would like to ask the following question: What exactly is a fractal? We discussed the property of self-similarity and the initiator-generator process that can be used when talking about fractals, but we actually have not defined what a fractal is!

To do so, we will need to distinguish between two different definitions: the topological dimension and the Hausdorff dimension of a space. This is because a fractal is defined in relation to these two ideas of dimension. As a motivation, consider the Sierpinski Triangle: What is its dimension? On the one hand, it is a geometric object embedded in $\mathbb{R}^2$ and could be two dimensional. But at the same time, we can also view it as one-dimensional given that the continual removal of triangles effectively removes a notion of area from the object, making it akin to a collection of lines more than a two-dimensional object. This is all informal and general, so to hone down precise metrics of dimensionality, mathematicians have defined a variety of definitions of dimension. These may give different values for the dimension, however.

We consider a topological space X that can be written as the union of a collection of open sets (an open cover), e.g., if I is an index set and $\{O_i\}$ is the family of open sets, then $X = \bigcup_{i \in I} O_i$. Furthermore, we define the order of an open cover as the smallest n for which every point in the space belongs to at most n open sets in the cover. Finally, recall that a refinement of an open cover is another open cover for which each open set in the refinement is contained in some open set of the original cover.

We can now state the definition of the topological dimension:

Definition 1.2 (Lebesque Covering Dimension (Topological Dimension))**.** *The Lebesque Covering Dimension of a topological space X is the minimum value d such that every finite open cover of X has an open refinement with order d+1. If no such d exists, the space is said to have an infinite covering dimension.*

As a brief aside, we can think of the order of open covers as a kind of maximum thickness, and a refinement is a more precise cover of a space. Thus, the Lebesque Covering Dimension is such that there is at least one refinement of every open cover that has a 'maximum thickness' of $d + 1$.

We will now compare this with the Hausdorff dimension. To do so, however, we first have to define the Hausdorff metric:

Definition 1.3 (Hausdorff Metric)**.** *Let X be a metric space. Consider a subset $S \subset X$, and let $d \in [0, \infty)$. Further, let I be a countable index set, and consider an arbitrary countable open cover $\{U_i\}$ of S. Then we can construct the following outer measure: $H^d_\delta(S) = \inf\{\sum_{i=1}^{\infty}(diam(U_i))^d : \bigcup_{i\in I} U_i \supseteq S, diam(U_i) < \delta\}$. To construct the d-dimensional Hausdorff metric, we consider the limit of the above quantity as δ collapses to 0: $H^d(S) = \lim_{\delta\to 0} H^d_\delta(S)$.*

Before proceeding to the definition of the Hausdorff dimension, a quick explanation of the Hausdorff metric is in order. We first establish a countable open cover of the metric space. We calculate the diameter of each part of the cover such that the diameter is less than a pre-specified miniscule δ and calculate the infinite sum over these diameters, raised to the dimensional power d. We take the infimum over all possible covers for the prespecified delta and d. To get the d-dimensional Hausdorff metric, we let δ go to 0. For each d, this represents a kind of 'minimum metric' on the space, so to speak. Now that we understand what's happening in this definition, we can now define the Hausdorff dimension:

Definition 1.4 (Hausdorff Dimension)**.** *The Hausdorff dimension $dim_H(X)$ of the metric space X is $dim_H(X) = inf\{d \geqslant 0 : H^d(X) = 0\}$.*

The Hausdorff dimension d is just the infimum over all d such that the Hausdorff metric is 0. That is, it is the minimum power needed to raise the miniscule diameters to in order to ensure the Hausdorff metric is 0. Note that this number does not have to be an integer. As we can see, the Hausdorff dimension is incredibly complicated! Actually calculating it is quite challenging, so one approach to (over) approximate this dimension is as follows:

Definition 1.5 (Box-Counting Dimension)**.** *Let (X,d) be a metric space. Suppose we can cover a fractal S* $\subset (X, d)$ *with a set of boxes. WLOG, say the side length of each box is r, and there are N boxes that cover the fractal. The Box-Counting Dimension is*

$$dim_{box}(S) = \lim_{r \to 0} \frac{ln(N)}{ln(\frac{1}{r})}. \tag{1.1}$$

For our purposes, we can now define a fractal the same way Benoit Mandelbrot defined a fractal:

Definition 1.6 (Fractal)**.** *A fractal is a set whose Hausdorff dimension strictly exceeds its topological dimension.*

2. Not-So Perfect Squares

We have just covered a lot of content. We reviewed some of the approaches to generating fractals and some overarching ideas and definitions of the math behind fractal geometry, namely self-similarity, topological dimension, and the Hausdorff dimension. We saw how self-similarity worked within the Mandelbrot Set and how the Sierpinski Triangle raises questions of dimension, what a fractal even is, and how fractals are often constructed. In this section, I explore an example of a construction I thought of when talking with my other members of the Directed Reading Program.[1] This example explores some of the paradoxes that can arise when thinking of the infinite nature of fractals and how it connects to a major area of focus in the literature on fractals, the Coastline Problem.

The construction is as follows. First, we start with a unit square defined over the interval $[0, 1]$. This will act as the initiator. Note that this square has an area of 1 and a perimeter of 4. For the first iteration, we divide this unit square into four smaller squares in each quadrant of the original unit square (see the figure). Remove the top left and bottom right squares. We are now left with two squares, one defined over the interval $[0, \frac{1}{2}]$ and the other defined over $[\frac{1}{2}, 1]$, both with an area of $\frac{1}{4}$ and a perimeter of 2. Note that this implies the total area of this second construction is now one-half but that the total perimeter is still 4.

We can repeat this process of removing top left and bottom right squares from each newly generated square. Doing so continually adds to an ascending staircase of squares, with each square getting smaller and smaller. Nonetheless, it is easy to show that while the total area collapses toward zero, the perimeter is constantly equal to 4. That is, we have constructed an object that continually loses area but maintains the same perimeter, along with the object 'collapsing towards' the line $y = x$ defined over the unit interval.

[1] I thank them for their help in refining the construction.

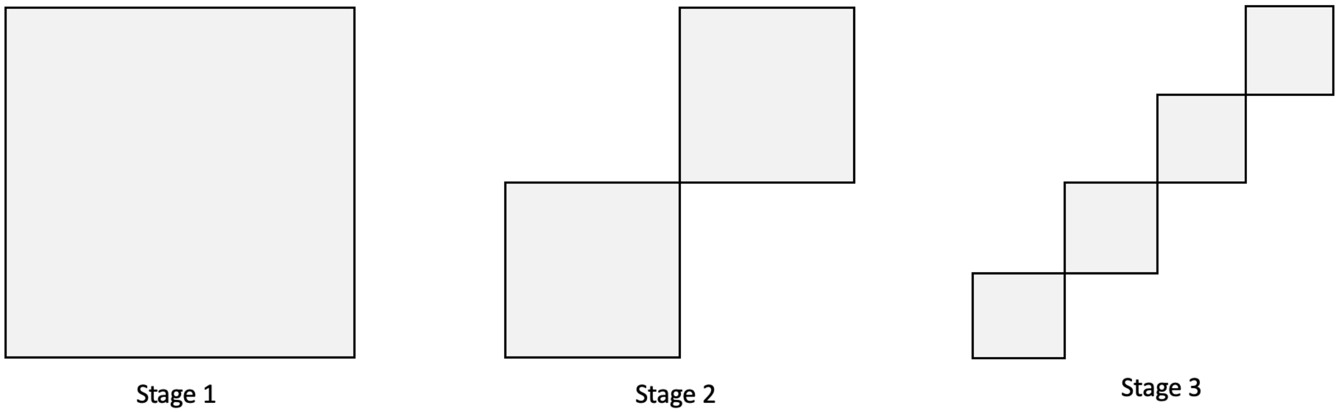


FIGURE 5. Generator Process

To elucidate this example I present a quick proof of these two interesting properties of the construction.[2]

Theorem 2.1. *$\forall n \in \mathbb{N}$, let P_n denote the sum of the perimeters and A_n the sum of the areas of all squares at the n^{th} stage. Then $\forall n \in \mathbb{N}$, $P_n = 4$ and $\lim_{n\to\infty} A_n = 0$.*

Proof. To prove the claim concerning the perimeter, we will manually calculate the perimeter of each square at an arbitrary stage and sum over the number of squares at that stage. We see that the length of each side of the square at each stage is $\frac{1}{2^{n-1}}$ since the length falls by a half at each stage and starts at 1 for $n = 1$. This means that the perimeter of each square is $\frac{4}{2^{n-1}}$. However, we also see that the number of squares grows at an 'exponential' rate, with the number of squares doubling after each stage (each square becomes two smaller ones), so the number of squares at any stage is 2^{n-1}. Thus, $P_n = \frac{4}{2^{n-1}} * 2^{n-1} = 4$.

For the area, since the side length at each stage is $\frac{1}{2^{n-1}}$, then the area of each square is $\frac{1}{2^{2(n-1)}}$ and $A_n = \frac{1}{2^{2(n-1)}} * 2^{n-1} = \frac{1}{2^{n-1}}$ so $\lim_{n\to\infty} A_n = 0$ trivially. $\square$

Recall the intuition that this construction collapses toward the line $y = x$. However, we have not defined our notion of 'collapses', or for a set of points to 'approach' another set. Before moving onto the next section, I briefly introduce a concept that can be used in the course of studying fractals to prove that one set can 'converge' to another set. This concept is called Kuratowski convergence.[3]

[2]In Euclidean geometry, the area of a square is defined as the square's length times its width. The perimeter is likewise defined as the sum of the side lengths of the square.

[3]Although not used on this construction, it is still important to see how the construction of fractals or fractal-like objects can be a limiting process. We thus need to be equipped with abstract notions of convergence.

Definition 2.2 (Kuratowski Convergence)**.** *Let X be an arbitrary set, and (X,d) be a metric space. If we let* $x \in X$ *and* $A \subset X$ *be a non-empty subset of X, then we can define the distance between the point and the subset as:*

$$d(x, A) = \inf_{y \in A}\{d(x, y)\}, \forall x \in X. \tag{2.1}$$

Moreover, let $(A_n)_{n=1}^{\infty}$ *be a sequence of subsets of X. Then we can define the two following quantities, the Kuratowski Limit Inferior and Kuratowski Limit Superior, respectively:*

$$LiA_n = \{x \in X : \limsup_{n\to\infty}\{d(x, A_n)\} = 0\}, \tag{2.2}$$

$$LsA_n = \{x \; inX : \liminf_{n\to\infty}\{d(x, A_n)\} = 0\}. \tag{2.3}$$

Note, we are taking the lim sup and lim inf, not a limit of a sequence of suprememums and infimums. If these two quantities are equal to each other, that set is called the Kuratowski limit of A_n*, denoted* $\underset{n\to\infty}{Lim}A_n$*. That is,*

$$\underset{n\to\infty}{Lim}A_n = LiA_n = LsA_n. \tag{2.4}$$

Now that the technicalities are out of the way, let's discuss what exactly this construct shows. We can continually remove and keep squares from an original initiator square such that we preserve the perimeter but shrink the area. Intuitively, the resulting staircase shrinks towards the line $y = x$, and it might be possible to prove this with different notions of convergence, including the one just defined.[4] Yet the line has length $\sqrt{2}$. So we can in principle find subsets of the plane that are arbitrarily as close to the line $y = x$ as we want, but that have a larger 'length'! This is similar to the Coastline Problem in that, when delving into geometric objects that are the result of an infinite generation process from some starting point, we can get quite weird results concerning its length, volume, and so on, even if the resulting objects are nearly indistinguishable from other, more simple objects. Here, two objects can converge but have different lengths.[5] In the Coastline Problem, we will see that fractal-like objects in nature can get ever more complex as our measurements get more precise, potentially extending the measured length of a coastline to infinity, implying that the coastline of many countries do not have a well-defined length!

[4] One can also show this by seeing how, for a given ϵ tube around the line, we can find a step along the generation such that the staircase lies entirely within that tube. This is similar to uniform converge of the staircase.

[5] After looking into this construction, me and other DRP members in my group found that it was coincidentally similar to the staircase paradox from geometry. For those unfamiliar with this, the staircase paradox starts with the top half of the unit square and thus concerns itself with the fact that at each stage, the length of the top half of our staircase is 2 but converges to a line with length $\sqrt{2}$.

2.1. **The Coastline Problem.**

But what is the Coastline Problem exactly? The name derives from an analysis of the length of the coastline of Britain by Benoit Mandelbrot and of other coastlines by other mathematicians before him. We can start with some simple questions: How long is the coastline of Britain? We need to measure the coastline, but how so?

One approach goes as follows. Suppose we had a ruler and started at a certain spot on the coast. We then used this ruler as an approximation of the uneven surfaces and added up the total length this ruler gave as we went around the entire coast. Now, further suppose we wanted a finer measurement tool than a ruler. Imagine we had a smaller ruler that could get into some of the cracks and crevices that the larger ruler glossed over and repeated the process with this more precise instrument. Evidently, we would get more detail and the length of the coastline would increase! The Coastline Problem states, in an informal sense, that if we try to acquire continually smaller rulers, the measured length of the coastline could grow without bound. For a visual understanding, see the following figure.

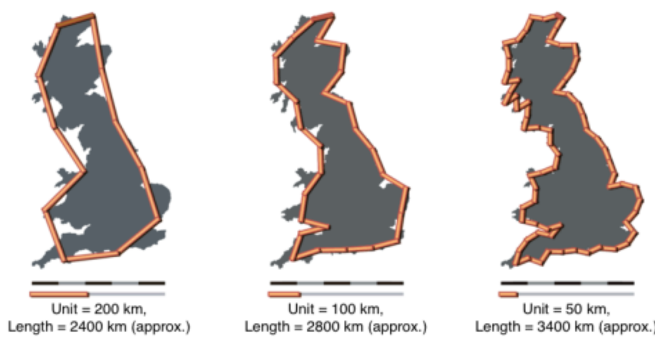


FIGURE 6. The Coastline Problem Visualized

To wrap up this aside, we should note that the coastline problem is more of a theoretical issue than a practical one. There are a variety of ways to approximate coastline lengths in real life that are useful for administrative governance purposes. The issue theoretically is in the potentially infinite nature of many of the fractal-like elements of a coastline, but in reality a coastline has to be finite to some level, so it has to have some kind of finite length.[6] The problem in a practical sense boils down to increased precision in measurement leading to unbounded increases in length and exemplifies the difficulty and care needed when working with fractals.

[6]Philosophically, this relates to the debate over the existence of so-called ‘actual infinities’. For instance, we can ask if infinite sets can ‘exist’ in reality. This is an unresolved debate and has wide-ranging implications, beyond the measurability of a coastline. Interestingly, the denial of absolute infinities is used in the Kalam Cosmological argument for the existence of God!

3. Conclusion

In this exploration into fractal geometry, we have defined some of the fundamental concepts of the field, namely self-similarity, topological dimension, Hausdorff dimension, the box-counting dimension, and related topics. We saw how these concepts played out in two popular fractals, namely the Mandelbrot Set and the Sierpinski Triangle, while at the same time seeing the different ways fractals can be made. To reinforce the confusing and paradoxical nature of many fractals, I presented a construction I made and analyzed some of its properties before presenting an overview of the Coastline Problem. From all this, something should be self-evident: while fractal geometry is confusing, it is quite an intellectually lucrative field! From beautiful art to philosophical dilemmas and re-evaluations of our understanding of nature, the fruits of studying fractals and the math behind them is widespread and rich.

References

(1) Mandelbrot, Benoit B. ”The Fractal Geomtry of Nature”. New York: W.H. Freeman and Company, 1982.

(2) Barnsley, Michael. ”Fractals Everywhere”. San Diego: Academic Press, 1993. 2nd Edition.