DRP Write-up: Categories at the Heart of Algebra

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Acknowledgements:

This winter, I learned about category theory by reading the book *"Category Theory in Context"* by Emily Riehl and with the help of my dear mentor Asa Kohn. I express my gratitude towards him for supporting me despite having other duties as graduate student, not to mention the TA strike. I also thank the DRP organizers for making this experience possible and for pairing me with Asa Kohn. Through this DRP, I have been introduced to a branch of mathematics that I have never heard about, and I have had a taste of a math graduate student's life. This write-up summarizes what I have learned this semester. I hope to make it accessible to an undergraduate audience while preserving the mathematical rigor throughout, so that my dear readers may also learn about the fascinating topic of categories in the same way that I have through this DRP.

Introduction:

As an undergraduate student, one learns about various mathematical structures such as sets, groups, vector spaces, graphs, and so on. More specifically, one studies functions between sets in Analysis, homomorphism of groups and linear transformation between vector spaces in Algebra, edges between vertices of a graph in Combinatorics, just to name a few. In fact, all the above can be viewed as categories. And therein lies the beauty of category theory. Even when confronted with a new mathematical structure, one can apply concepts of category theory to understand it with ease. And the reason for this is that in category theory, one considers *objects* (sets, groups, vector spaces, vertices) as atomic entities and focuses primarily on *morphisms* (functions, homomorphisms, linear transformation, edges) between those objects. With that, we are ready to define categories:

A *category* is a collection of objects ($A, B, C,\ldots$) and morphisms/arrows between those objects ($f, g, h,\ldots$) such that:

- There is an identity morphism $1_A: A \to A$ which maps an object to itself for each object in the category. Also, for a morphism $f: A \to B$, we have $1_A f = f 1_B = f$.
- Whenever there are morphisms $f: A \to B$ and $g: B \to C$, there is the composition morphism $(g \circ f): A \to C$, namely, f and g are composable.
- For any composable morphisms f, g, h, there is an associativity rule: $f(gh) = (fg)h$.

One question that arises from the above definition is: When is A "$=$"B ? Just like with groups or vector spaces, we consider that A "$=$"B when A and B are isomorphic to one another. And one defines an isomorphism between objects as follows:

An *isomorphism* $f: A \to B$ is such that there exists a morphism $g: B \to A$ such that $fg = 1_B$ and $gf = 1_A$. Then, we say that A, B are *isomorphic*, which we denote by $A \cong B$.

Moreover, we call a morphism $f: A \to A$ an *endomorphism* (domain is equal to codomain). If f happens to be an isomorphism, then we call it an *automorphism*.

Duality:

One of the core concepts of category theory is that of *dual* of a category. Indeed, the *dual* of a category helps us to understand more about the category we are interested in. More specifically, a statement about a category C is true if and only if the *dual* statement for the dual category is also true. We call this the *duality principle*. In category theory, one defines the *dual* of a category as follows:

The dual/opposite category C^{op} of a category C is such that:

- C and C^{op} have the same objects.
- For every morphism $f: A \to B$, we define a morphism $f^{op}: B \to A$.

This definition shows how C^{op} and C "mirror" each other, as we simply "reverse" the arrows when going from one to the other. Note that for every object A in C^{op}, we have the identity morphism ${1_A}^{op}: A \to A$. Moreover, we can derive a composition rule as follows:

Consider the morphisms $f: A \to B$ and $g: B \to C$ in a category D. By our definition of category, we also have a morphism $(g \circ f): A \to C$. Now consider the dual category D^{op}. In this category, we have morphisms $f^{op}: B \to A$ and $g^{op}: C \to B$. But note that by composing g^{op} and f^{op}, we get the morphism $f^{op} \circ g^{op}: C \to A$. It follows that $(g \circ f)^{op} = f^{op} \circ g^{op}$.

Now, how can one use the duality principle to prove statements about categories? In Algebra and Analysis, one studies injective and surjective functions. In fact, injectivity and surjectivity are dual properties (as shown later), so one can prove injectivity of a function in a space by proving surjectivity of this function in the dual space. Category theory generalizes the notions of injectivity and surjectivity as follows:

A morphism $f: X \to Y$ in a category is said to be:

- A *monomorphism* if for any two morphisms $g, h: W \to X$, $fh = fg \Leftrightarrow h = g$
- An *epimorphism* if for any two morphisms $g, h: Y \to Z$, $hf = gf \Leftrightarrow h = g$

Note that the definitions of monomorphism and injectivity are equivalent in the category of sets. Indeed, given two maps $x, x': 1 \to X$ in the category of sets, with the domain being the singleton set, and a monomorphism $f: X \to Y$, $fx = fx' \Rightarrow x = x'$.

Similarly, the definitions of epimorphism and surjectivity are equivalent in the category of sets. Indeed, given an epimorphism $f: X \to Y$ and maps $g, h: Y \to Z$, we have $hf = gf \Rightarrow h = g$. But saying that $hf = gf$ is only saying that g, h are equal on the image of f. So, it must be that the image of f is the domain Y of g, h, namely, f must be surjective.

We first show that injectivity and surjectivity are indeed dual statements. Namely, we show that $f: X \to Y$ is a monomorphism in a category C if and only if $f^{op}: Y \to X$ is an epimorphism in the dual category C^{op}.

Let $f: X \to Y$ be a monomorphism in a category C. Then, for morphisms $g, h: W \to X$, we have that $fh = fg \Leftrightarrow h = g \Leftrightarrow h^{op} = g^{op}$.

But $fh = fg \Leftrightarrow (fh)^{op} = (fg)^{op} \Leftrightarrow h^{op} \circ f^{op} = g^{op} \circ f^{op}$.

It follows that $h^{op} \circ f^{op} = g^{op} \circ f^{op} \Leftrightarrow h^{op} = g^{op}$, so f^{op} is indeed an epimorphism. □

Now, we use these notions to prove part b) of the following lemma using duality:

a) If $f: X \to Y$ and $g: Y \to Z$ are monomorphisms, then so is $(g \circ f): X \to Z$.

b) If $f: X \to Y$ and $g: Y \to Z$ are epimorphisms, then so is $(g \circ f): X \to Z$.

Proof of a):

Let $h, k: W \to X$ be such that $gfh = gfk$. Then, since g is a monomorphism, we have that $fh = fk$. Since f is a monomorphism, we then have that $h = k$. So, we have shown that $(gf)h = (gf)k \Leftrightarrow h = k$, so $(g \circ f): X \to Z$ is indeed a monomorphism. □

Proof of b):

By duality, $f: X \to Y$ and $g: Y \to Z$ are epimorphisms if and only if $f^{op}: Y \to X$ and $g^{op}: Z \to Y$ are monomorphisms.

By part a), it follows that $f^{op} \circ g^{op}: Z \to X$ is a monomorphism, thus, $(g \circ f)^{op}$ is a monomorphism.

Then, applying duality again, we get that $g \circ f: X \to Z$ is an epimorphism. □

Now that we have introduced what categories are and the concept of duality, it is time to study the maps between categories, which we call functors.

Functors:

Functors are to categories what morphisms are to objects in a category. Namely, we can define a *covariant functor* $F: C \to D$ between categories C, D such that:

- Every object $c \in C$ has an image $Fc \in D$ by the functor F.
- Every morphism $f: x \to y$ in a category C has an image $Ff: Fx \to Fy$ by the functor F.
- Composition: $F(gf) = (Fg)(Ff)$.
- Identity: For every $c \in C$, $F(1_c) = 1_{Fc}$

Note that we call $F: C \to D$ a covariant functor to differentiate it from a *contravariant functor* from C to D, which is a functor $G: C \to D$ such that:

- Every object $c \in C$ has an image $Gc \in D$ by the functor G.
- Every morphism $f: x \to y$ in the category C has an image $Gf: Gy \to Gx$ by G.
- Composition: $G(gf) = (Gf)(Gg)$.
- Identity: For every $c \in C$, $G(1_c) = 1_{Gc}$

One of the most fundamental properties of functors is the fact that they preserve isomorphisms. Namely, if $f: x \to y$ is an isomorphism in C and $F: C \to D$ is a covariant functor, then the image $Ff: Fx \to Fy$ of f is an isomorphism in D. The proof of this lemma follows from the axioms of functors and goes as follows:

Let $F: C \to D$ be a covariant functor between the categories C and D, and let $f: x \to y$ be an isomorphism in C. Then, there exists a map $g: y \to x$ such that $fg = 1_y$ and $gf = 1_x$. Now, we claim that $(Fg)(Ff) = 1_{Fx}$ and $(Ff)(Fg) = 1_{Fy}$ and thus that Ff is also an isomorphism with inverse Fg. Indeed, by functoriality axioms, we have that:

$$(Fg)(Ff) = F(gf) = F(1_x) = 1_{Fx}$$

$$(Ff)(Fg) = F(fg) = F\left(1_y\right) = 1_{Fy}$$

This concludes the proof. □

Now, we can apply our knowledge to study functors between categories. In this section, for simplicity and conciseness, we will restrict ourselves to functors $F: BG \to C$ where the domain BG is a group regarded as a one object category (the group is the object G, and the morphisms are the elements of that group), and the codomain is a general category C.

Let $X = FG \in C$ be the image of the group object in C. Then, we can say that F is a left action of G on X, where X is under the action of the images $Fg: X \to X$ of the morphisms $g: BG \to BG$ (namely, $g \in G$). In contrast, we can say that $F: BG^{op} \to C$ is a right action of G on X. But categorically speaking, every $g \in G$ is an isomorphism in the one object category BG, since in a group, every element has its inverse. So, by the above lemma, for every $g \in G$, $Fg: X \to X$ is an isomorphism in the category C, with inverse $F(g^{-1})$, where g^{-1} is the inverse of g in G (in fact, Fg is an automorphism, since the domain and codomain are the same). In summary, every morphism in the image of the functor F is an automorphism in the category C.

We have seen in the introductory section that two objects in a category can be said to be *"equal"* when there exists an isomorphism between the objects. But when can two categories be considered *"equal"* ? The final section of this write-up will attempt to answer that question by introducing the notion of equivalence of categories.

Equivalence of categories:

An intuitive answer to the above question would be to say that two categories C and D are *"equal"* when given a functor $F: C \to D$, one can find a functor $G: D \to C$ such that $FG = 1_D$ and $GF = 1_C$, where 1_C and 1_D are the identity functors for C and D respectively. This definition would be analogous to the one given in introduction for two isomorphic objects. But there is still one problem with this definition: what does it mean for two functors to be *"equal"*, namely, what do the equalities "$FG = 1_D$" and "$GF = 1_C$" really mean? This provides the motivation to introduce isomorphisms between functors, or more generally, maps between functors, also called natural transformations.

Given categories C, D and functors $F, G: C \to D$, one can define a *natural transformation* $\alpha: F \to G$ as follows:

- For every object $c \in C$, there is a map/arrow $\alpha_c: Fc \to Gc$ in D. Those arrows define the *components* of the natural transformation α.
- Given a morphism $f: c \to c'$ in C, and morphisms $Ff: Fc \to Fc'$ and $Gf: Gc \to Gc'$ in D, $\alpha_{c'} \circ Ff = Gf \circ \alpha_c$.

Now, we can define an isomorphism between functors as follows:

A natural isomorphism $\alpha: F \to G$ is such that all components α_c of α are isomorphisms. Then, we can consider the functors F, G to be isomorphic, which we denote by $F \cong G$.

Now, we are finally ready to answer our question by defining equivalence of categories:

Two categories C, D are said to be equivalent if, given a functor $F: C \to D$, one can find a functor $G: D \to C$ such that $FG \cong 1_D$ and $GF \cong 1_C$, namely, one can find a natural transformation $\alpha: FG \to 1_D$ and $\alpha': GF \to 1_C$. Then, we can write $C \cong D$.

One natural property is that equivalence of categories defines an equivalence relation. The axiom of reflexivity, $C \cong C$, is obvious, one just considers the identity functor $1_C: C \to C$ which is its own inverse. The axiom of symmetry, $C \cong D \Leftrightarrow D \cong C$ is also obvious from the definition. It remains to show the transitivity axiom, namely, $C \cong D, D \cong E \Rightarrow C \cong E$.

By definition, $C \cong D$ if and only if there exist functors $F: C \to D$ and $G: D \to C$ such that we have $FG \cong 1_D$ and $GF \cong 1_C$.

Likewise, $D \cong E$ if and only if there exist functors $H: D \to E$ and $J: E \to D$ such that we have $HJ \cong 1_E$ and $JH \cong 1_D$.

We claim that $(GJ) \circ (HF) \cong 1_C$ and that $(HF) \circ (GJ) \cong 1_E$, which would prove that $C \cong E$. Indeed, we have by the associativity rule that:

$$(GJ) \circ (HF) \cong G \circ (JH) \circ F \cong G \circ 1_D \circ F \cong GF \cong 1_C$$

$$(HF) \circ (GJ) \cong H \circ (FG) \circ J \cong H \circ 1_D \circ J \cong HJ \cong 1_E$$

This concludes the proof that equivalence of categories defines an equivalence relation. □

The following theorem provides another definition of equivalence of categories. Though we will not prove this theorem, the proof being lengthy and relying on a more advanced technique called *diagram chasing*, we will define all its key terms and prove a lemma used in the proof. Do not worry if you do not understand the theorem when first reading it, we will take the time to define all the important terms. Without further ado, here is the theorem, as stated in *"Category Theory in Context"* by Emily Riehl:

"A functor defining an equivalence of categories is full, faithful, and essentially surjective on objects. Assuming the axiom of choice, any functor with these properties defines an equivalence of categories."

This theorem states three properties of functors which, combined, are equivalent to our above definition of equivalence of categories. The properties are as follows:

A functor $F: C \to D$ is said to be:

- *Full* if for any given $c, c' \in C$, the map $\sigma: Hom(c, c') \to Hom(Fc, Fc')$ is surjective.
- *Faithful* if for any given $c, c' \in C$, the map $\sigma: Hom(c, c') \to Hom(Fc, Fc')$ is injective.
- *Essentially surjective on objects* if for any object $d \in D$, there exists some $c \in C$ such that $Fc \cong d$.

In the above definitions, $Hom(c, c')$ refers to the set of morphisms from c to c', and $Hom(Fc, Fc')$ refers to the set of morphisms from Fc to Fc'. The image of a morphism $f: c \to c'$ by the map σ is simply $\sigma f = Ff$.

Though we will not prove this theorem, we will prove the following lemma, which is used in the proof of the theorem, as stated in *"Category Theory in Context"* by Emily Riehl:

"Any morphism $f: a \to b$ and fixed isomorphisms $a \cong a'$ and $b \cong b'$ determine a unique morphism $f': a' \to b'$ so that any of—or, equivalently, all of—the following four diagrams commute:"

$$
\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\
b \arrow[r, "\cong"'] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b \arrow[r, "\cong"'] & b'
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[d, "f"'] & a' \arrow[l, "\cong"'] \arrow[d, "f'"] \\
b & b' \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
a \arrow[r, "\cong"] \arrow[d, "f"'] & a' \arrow[d, "f'"] \\
b & b' \arrow[l, "\cong"]
\end{tikzcd}
$$

For the proof, let $I_A: a \to a'$ and $I_B: b \to b'$ be the isomorphisms represented in the above diagrams. The first diagram then defines f' as follows: $f' = I_B \circ f \circ (I_A)^{-1}$.

It follows that $f' \circ I_A = I_B \circ f$, thus, the second diagram also commutes.

Moreover, $(I_B)^{-1} \circ f' = f \circ (I_A)^{-1}$, so the third diagram also commutes.

Finally, $(I_B)^{-1} \circ f' \circ I_A = f$, so the fourth diagram also commutes. □

Conclusion:

This write-up, while summarizing what I have learned this winter through the DRP project, has introduced fundamental ideas of category theory such as duality, functors, or natural transformations. Once again, I cannot thank enough the author Emily Riehl of the book *"Category Theory in Context",* my mentor Asa Kohn, and the organizers of the DRP for what I have learned this winter. I hope that you, dear readers, have been able to learn from this write-up like I have through the DRP.

Bibliography:

Riehl, Emily. *Category Theory in Context*. Courier Dover Publications, 2017.