Diametric Colourings in Ultrametric Spaces

JAKE R. GAMEROFF

ABSTRACT. Let Γ_X denote the family of compact subsets of (X, d). A colouring $\chi : \Gamma_X \to [k]$ is *diametric* if every pair of compact subsets with equal diameters receive the same colour. A free ultrafilter $\mathcal{F}$ is called *diametrically Ramsey* if every diametric colouring admits a set $M \in \mathcal{F}$ whose compact subsets are monochrome. We show that every infinite ultrametric space contains a sequence (x_n) such that any free ultrafilter containing (x_n) is diametrically Ramsey, thereby extending a result of Protasov and Protasova [1].

1. Introduction

Ramsey Theory explores the underlying structure emerging in "large enough" complex systems. For example, Frank Ramsey proved in [2] that for each $k \in \mathbb{N}$ there is a sufficiently large $n \in \mathbb{N}$ such that in any red-blue colouring of the edges of the complete graph K_n there is a set of k vertices joined by edges of the same colour.

Another seminal result is Van der Waerden's theorem, which states that for all positive integers $r, k \in \mathbb{N}$ there is a large enough $n \in \mathbb{N}$ such that if we colour the integers in $[n] = \{1, 2, \ldots, n\}$ using k colours, one can always find a set of r monochromatic integers in arithmetic progression [3].

Motivated by these classical results, we study the structural properties of infinite spaces using a Ramsey-theoretic lens. In particular, we will colour a class of subsets of the space and search for a set whose subsets in this class are all the same colour. We formalise this now.

Fix an infinite metric space (X, d) and let $k \in \mathbb{N}$ be a postive integer. For a family $\mathcal{A}$ of subsets of X, a *colouring* of $\mathcal{A}$ is any mapping $\chi : \mathcal{A} \to [k]$. We would like to find a set $M \subseteq X$ and a colour $\varphi \in [k]$ such that $\chi(N) = \varphi$ for every subset $N \subseteq M$ with $N \in \mathcal{A}$. We then say that M is *monochrome* over $\mathcal{A}$. A natural first choice is $\mathcal{A} = [X]^2$, the class of two-element subsets of X.

In this context, the "large" objects in X which we will work with are free ultrafilters. A *filter* $\mathcal{F}$ on X is a collection of subsets of X satisfying the following for all subsets $A, B \subseteq X$:

(1) If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$;

(2) If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
(3) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.

A filter $\mathcal{F}$ is called an *ultrafilter* if it is not properly contained in another filter. We call $\mathcal{F}$ *free* if $\bigcap \mathcal{F} = \emptyset$. Intuitively, ultrafilters are maximal filters and free filters are "spread out" throughout the space, so we view *free ultrafilters* as large objects in X.

Given that these are the large objects in focus, it is natural to ask whether there is a free ultrafilter $\mathcal{F}$ such that for every colouring of $[X]^2$ there is a monochrome set $M \in \mathcal{F}$. It turns out that this question is undecidable in ZFC even with $X = \mathbb{N}$, though the claim is true if we accept the continuum hypothesis [1]. Consequently, we must define more restrictive classes of colourings.

Towards this end, Protasov and Protasova introduce the following theory in [1]. A colouring $\chi : [X]^2 \to [k]$ of the two-element subsets of X is called *isometric* if $\chi(\{x_1, y_1\}) = \chi(\{x_2, y_2\})$ whenever $d(x_1, y_1) = d(x_2, y_2)$. A free ultrafilter $\mathcal{F}$ is called *metrically Ramsey* if for every isometric colouring of $[X]^2$ there is a monochrome set $M \in \mathcal{F}$.

It turns out that in the particular case of ultrametric spaces, metrically Ramsey free ultrafilters are not too hard to work with. Recall that an *ultrametric* d on a set X is a metric satisfying the *ultrametric inequality*, which states that for all $x, y, z \in X$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

Protasov and Protasova leverage the properties of the ultrametric to prove the following theorem [1].

Theorem 1.1. *Every infinite ultrametric space X contains a sequence (x_n) such that any free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is metrically Ramsey.*

As a follow up, one may ask if similar structure exists when colouring a larger class of subsets of X. The positive answer to this question is the keynote of this paper. In this connection, we will analyse the family Γ_X of all compact subsets of X.

We generalise isometric colourings along these lines. The map $\chi : \Gamma_X \to [k]$ is called a *diametric colouring* if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of compact subsets of X with $\operatorname{diam} A_1 = \operatorname{diam} A_2$. Accordingly, a subset $M \subseteq X$ is monochrome if its compact subsets are the same colour.

Given this, we say that a free ultrafilter $\mathcal{F}$ on X is *diametrically Ramsey* if for every diametric colouring χ there is a monochrome set $M \in \mathcal{F}$. Since finite sets are compact, Γ_X contains $[X]^2$ so that every diametric colouring is isometric.

In this context, our main result is the following.

Theorem 1.2. *Every infinite ultrametric space X contains a sequence (x_n) such that any free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is diametrically Ramsey.*

Building towards the main result, the following two sections review some elementary properties of ultrametric spaces and filters. We then prove Theorem 1.2 in Section 4.

2. Ultrametric Analysis

In this brief section, we introduce some notable examples of ultrametric spaces and survey some of their fundamental properties.

2.1. Examples: The space $\mathbb{N}^{\mathbb{N}}$, graphs, ε-chains.

The simplest example of an ultrametric on a set X is the discrete metric d, where $d(x, y)$ is 1 if $x \neq y$ and 0 otherwise. A slightly more complicated example is the ultrametric given by $d(n, m) = \max\{1 + 1/n, 1 + 1/m\}$ and $d(n, n) = 0$, where n, m are distinct positive integers in $\mathbb{N}$.

However, there are much more interesting constructions. These include ultrametrics on the Baire space $\mathbb{N}^{\mathbb{N}}$, connected graphs, and uniformly disconnected metric spaces, which we will now construct.

2.1.1. *The Baire space.* We will first discuss the *Baire space* $\mathbb{N}^{\mathbb{N}}$, which is the space of all sequences of natural numbers.

For two distinct sequences $x = (x_n)$, $y = (y_n)$ in $\mathbb{N}$, we define $m(x, y) = \min\{k \in \mathbb{N} : x_k \neq y_k\}$ to be the first index at which x and y do not coincide. Set $d(x, y) = m(x, y)^{-1}$ with $d(x, x) = 0$. We claim that d is an ultrametric on the Baire space.

Proof. The symmetry and non-negativity of d are immediate. Furthermore, $d(x, x) = 0$ by definition, and if $d(x, y) = 0$ then $x = y$ since $m(x, y)^{-1} > 0$ for distinct x, y. To prove the ultrametric inequality for d, fix $x, y, z \in \mathbb{N}^{\mathbb{N}}$. We assume x, y, z are distinct sequences, otherwise the inequality is clear. Observe that

$$\begin{aligned}
&d(x, y) \leq \max\{d(x, z), d(z, y)\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \max\{m(x, z)^{-1}, m(z, y)^{-1}\} \\
&\Leftrightarrow m(x, y)^{-1} \leq \min\{m(x, z), m(z, y)\}^{-1} \\
&\Leftrightarrow m(x, y) \geq \min\{m(x, z), m(z, y)\}.
\end{aligned}$$

Clearly $m(x, y) \geq \min\{m(x, z), m(z, y)\}$. Otherwise, $m(x, y) < m(x, z)$ and $m(x, y) < m(z, y)$. Letting $\ell = m(x, y)$, we see that $x_\ell \neq y_\ell$, but since both $m(x, z), m(z, y) \geq \ell + 1$, we have $x_\ell = z_\ell = y_\ell$, a contradiction. Therefore, d is an ultrametric. $\square$

2.1.2. *Graphs.* The following construction is inspired by Leclerc's elegant work in [4]. Let G be a connected graph with positive edge-weights. For an edge e in G, we denote its weight by $w(e)$.

A *walk* in G is a finite sequence of adjacent vertices. Given a walk x, we will denote by e_x an edge in the walk with maximum weight. We say that a walk x between two vertices u, v is a *minimax walk* if there is no other walk x' between u, v whose max-weight edge is lighter than e_x. Equivalently, $w(e_x) \leq w(e_{x'})$ for every walk x' between u, v.

If we think of an edge's weight as the difficulty level of travelling from one of its ends to the other, the minimax walk from u to v minimises the most challenging part of the journey.

We can use this notion to define an ultrametric on $V(G)$. Specifically, given two distinct vertices $u, v \in V(G)$ with minimax walk x, we set $d(u, v) = w(e_x)$ with $d(u, u) = 0$. The proof that d is an ultrametric is as follows.

Proof. That d is symmetric and non-negative is clear. By definition, $d(u, u) = 0$ and if $d(u, v) = 0$ then $u = v$, as otherwise there is an edge in G with weight 0 even though its edges have only positive weights.

To prove the ultrametric inequality, fix $u, v, w \in V(G)$. We may assume that u, v, w are distinct, else the inequality follows. Let $x_{u,w} = (v_1, v_2, \ldots, v_m)$ and $x_{w,v} = (v_m, v_{m+1}, \ldots, v_n)$ denote minimax walks between u, w and w, v respectively, and let $x = (v_1, v_2, \ldots, v_n)$ be their union. Then the max-weight edge in x has weight $\max\{d(u, w), d(w, v)\}$. Since x is a walk between u, v, we have $d(u, v) \leq \max\{d(u, w), d(w, v)\}$, as claimed. $\square$

2.1.3. *Uniformly disconnected spaces.* The theory below is based on Guy and Semmes' work in [5] and Heinonen's ideas in [6].

Let (X, d) be any metric space and $\varepsilon > 0$ be given. An *ε-chain* between the pair $x, y \in X$ is any finite sequence $x = x_0, x_1, \ldots, x_n = y$ with

$$\max_{1 \leq i \leq n} d(x_{i-1}, x_i) \leq \varepsilon \cdot d(x, y).$$

In this case, we say that x, y are *ε-connected.* Observe that if x, y are ε-connected then they are ε'-connected for every $\varepsilon' \geq \varepsilon$; and if x, y can not be ε-connected then they can not be ε'-connected for every $\varepsilon' \leq \varepsilon$.

The space X is called *uniformly disconnected* if there is an $\varepsilon > 0$ such that no two distinct points in X can be ε-connected. It is not hard to prove that uniform disconnectivity is stronger than total disconnectivity. To give some intuition, we note that with the Euclidean metric the middle thirds Cantor set is uniformly disconnected and the set $\{1/n : n \in \mathbb{N}\}$ is not [6].

Let (X, d) be a uniformly disconnected metric space. For distinct points $x, y \in X$ we define $c(x, y)$ to be the infimum over all $\varepsilon > 0$ such that x and y are $d(x, y)^{-1} \cdot \varepsilon$-connected, with $c(x, x) = 0$.

Proof. Clearly c is symmetric and non-negative. By definition, $c(x, x) = 0$, and if $c(x, y) = 0$ then $x = y$, otherwise x, y are $d(x, y)^{-1} \cdot \varepsilon$-connected for every $\varepsilon > 0$, contradicting the uniform disconnectivity of X.

The last thing to prove is the ultrametric inequality for c. To this end, fix $x, y, z \in X$ and let $\varepsilon > 0$ be given. We may assume the points x, y, z are distinct, otherwise the claim is immediate. By definition of the infimum, there exist $\gamma_1, \gamma_2 > 0$ with $\gamma_1 \le c(x, z) + \varepsilon$ and $\gamma_2 \le c(z, y) + \varepsilon$ such that x, z are $d(x, z)^{-1} \cdot \gamma_1$-connected and z, y are $d(z, y)^{-1} \cdot \gamma_2$-connected. Hence we obtain sequences

- $x = x_0, x_1, \dots x_m = z$ with $\max_{1 \le i \le m} d(x_{i-1}, x_i) \le \gamma_1$; and
- $z = x_m, x_{m+1}, \dots, x_n = y$ with $\max_{m+1 \le i \le n} d(x_{i-1}, x_i) \le \gamma_2$.

Now set $\gamma = \max\{\gamma_1, \gamma_2\}$ and note that x, y are $d(x, y)^{-1} \cdot \gamma$-connected since

$$\max_{1 \le i \le n} d(x_{i-1}, x_i) \le \gamma = \frac{\gamma}{d(x, y)} \cdot d(x, y).$$

So

$$\begin{aligned} c(x, y) \le \gamma &= \max\{\gamma_1, \gamma_2\} \\ &\le \max\{c(x, z) + \varepsilon, c(z, y) + \varepsilon\} \\ &= \max\{c(x, z), c(z, y)\} + \varepsilon, \end{aligned}$$

and sending $\varepsilon \to 0$ yields $c(x, y) \le \max\{c(x, z), c(z, y)\}$. □

2.2. **Properties of ultrametric spaces.** The ultrametric inequality is much stronger than the usual triangle inequality. Because of this, ultrametric spaces have some interesting properties which we explore now (see [7] for a comprehensive overview).

First, it turns out that every triangle is isosceles.

Lemma 2.1. *If x, y, z are distinct points in an ultrametric space X and $d(x, z) < d(z, y)$ then $d(x, y) = d(z, y)$.*

Proof. Since $d(x, z) < d(z, y)$, the ultrametric inequality implies

$$d(x, y) \le \max\{d(x, z), d(z, y)\} = d(z, y).$$

Furthermore, $d(x, z) < d(x, y)$ else

$$d(z, y) \le \max\{d(x, z), d(x, y)\} = d(x, z)$$

is a contradiction. Thus, $d(z, y) \le \max\{d(z, x), d(x, y)\} = d(x, y)$. Combining inequalities, we conclude that $d(x, y) = d(z, y)$. □

We now examine open balls in ultrametric spaces. An *open ball* (or simply a ball) of radius $r > 0$ centred about $x \in X$ is the set

$$B_r(x) = \{y \in X : d(x, y) < r\}.$$

A subset $U \subseteq X$ is called *open* if it is a union of open balls. A set is *closed* if its complement is open.

Open balls in ultrametric spaces have vastly unintuitive properties. For example, every point in a ball is its centre.

Lemma 2.2. *Let $B_r(x)$ be an open ball in X. Then $B_r(x) = B_r(y)$ for every point $y \in B_r(x)$.*

Proof. Let y be any point in $B_r(x)$. So $d(x, y) < r$, and if $t \in B_r(y)$ then $d(y, t) < r$. Then $t \in B_r(x)$ since $d(x, t) \leq \max\{d(x, y), d(y, t)\} < r$. Hence $B_r(y) \subseteq B_r(x)$, and the reverse inclusion follows symmetrically as $x \in B_r(y)$. $\square$

Another important result is that if two balls intersect then one of them contains the other.

Lemma 2.3. *Let $B_r(x)$ and $B_q(y)$ be a pair of intersecting balls in X with $r \leq q$. Then $B_r(x) \subseteq B_q(y)$.*

Proof. Since the two balls intersect, there is a point $t \in B_r(x) \cap B_q(y)$. Thus, Lemma 2.2 implies that $B_r(t) = B_r(x)$ and $B_q(t) = B_q(y)$, and since $r \leq q$ we obtain $B_r(x) = B_r(t) \subseteq B_q(t) = B_q(y)$, as required. $\square$

Open balls also interact with general subsets of X in a unique way.

Lemma 2.4. *Let $A \subseteq X$ and consider any ball $B_r(x)$ in X. If $B_r(x)$ intersects A then $A \cap B_r(x)$ is a ball in the space (A, d).*

Proof. Since $A \cap B_r(x) \neq \emptyset$ there is a point $t \in A \cap B_r(x)$. Let $B_A = \{a \in A : d(t, a) < r\}$ be a ball in A. We show that $A \cap B_r(x) = B_A$.

Note from Lemma 2.2 that t is the centre of $B_r(x)$ so that $A \cap B_r(x) = A \cap B_r(t)$. Then if $a \in A \cap B_r(x)$ we have $a \in A \cap B_r(t)$ so that $d(t, a) < r$ and hence $a \in B_A$. On the other hand, if $a \in B_A$ then from the ultrametric inequality we obtain

$$d(x, a) \leq \max\{d(x, t), d(t, a)\} < r,$$

since $t \in B_r(x)$. Hence $a \in A \cap B_r(x)$ and we're done. $\square$

Furthermore, it turns out that all open balls are also closed in X.

Lemma 2.5. *If $B_r(x)$ is an open ball then $X \setminus B_r(x)$ is open in X. In particular, $B_r(x)$ is closed.*

Proof. Assume $X \setminus B_r(x)$ is not open. So there is a point $t \in X \setminus B_r(x)$ which is not an interior point of $X \setminus B_r(x)$. That is, for every $q > 0$ the ball $B_q(t)$ is not contained in $X \setminus B_r(x)$, meaning $B_q(t) \cap B_r(x) \neq \emptyset$. In particular, if $q = r$ one deduces from Lemma 2.3 that $B_r(x) = B_q(t)$. But then $t \in X \setminus B_r(x) = X \setminus B_q(t)$ is a contradiction. Therefore, $X \setminus B_r(x)$ is open, so $B_r(x)$ is closed. □

Lemma 2.5 implies that every ultrametric space X is *zero-dimensional*, meaning that every open set in X is a union of clopen (both open and closed) sets. Using this and the fact that ultrametric spaces are Hausdorff, it follows that every ultrametric space is totally disconnected.

Our final application of the ultrametric inequality is to the diameter of subsets of the space.

Lemma 2.6. *For a subset $A \subseteq X$ with $a \in A$, the quantity*

$$\operatorname{diam} A = \sup\{d(a, x) : x \in A\}.$$

Proof. Set $u = \sup\{d(a, x) : x \in A\}$ and fix $x, y \in A$. Then

$$d(x, y) \leq \max\{d(a, x), d(a, y)\} \leq u$$

so that u is an upper bound of $D = \{d(x, y) : x, y \in A\}$. Now suppose u' is an upper bound of D with $u' < u$. Then for each $x \in A$ we have $d(a, x) \leq u' < u$ contradicting our choice of u. Hence, $u = \sup D$. □

A nice consequence of Lemma 2.6 is that $\operatorname{diam} B_r(x) \leq r$.

3. Filters and Ultrafilters

In this short section, we introduce a sequence of lemmas building up to Lemma 3.3, which is fundamental to the proof of the main result. We start with some elementary examples of filters and ultrafilters, and we conclude with Lemma 3.4, which is yet another elegant application of Lemma 3.3. The development of these lemmas is due to the theory covered in the notes of Koppelberg [8] and the work of Brian [9].

Recall from the introduction that a *filter* $\mathcal{F}$ on X is a family of subsets of X with $\emptyset \notin \mathcal{F}$, $X \in \mathcal{F}$, and which is closed under the superset inclusion and finite intersections. A filter $\mathcal{F}$ is an *ultrafilter* if no filter properly contains it, and we call $\mathcal{F}$ *free* if $\bigcap \mathcal{F} = \emptyset$.

It turns out that one needs the axiom of choice to prove that free ultrafilters exist in $\mathbb{N}$. Likewise, Zorn's lemma is required to prove that every filter is contained in an ultrafilter. For a more in-depth treatment, the interested reader is encouraged to see Section 3 in [10].

We briefly look at some simple examples of filters and ultrafilters as presented in [8]. Then, we will move on to the proof of Lemma 3.1.

- For a non-empty subset $A \subseteq \mathbb{N}$, the family $\mathcal{F} = \{B \subseteq \mathbb{N} : A \subseteq B\}$ is a filter. Note that $\mathcal{F}$ is not free since each set in $\mathcal{F}$ contains A. If A is a singleton, then $\mathcal{F}$ is an ultrafilter [10].

- A subset $A \subseteq \mathbb{N}$ is called *cofinite* if $\mathbb{N} \setminus A$ is finite. The collection of cofinite subsets of $\mathbb{N}$ is a filter, called the Fréchet filter. The Fréchet filter is not an ultrafilter, but it is free since $\mathbb{N} \setminus \{m\}$ is cofinite for every integer $m \in \mathbb{N}$. In fact, a filter is free if and only if it contains the Fréchet filter (see Proposition 2 in [11]).
- For a topological space X and a point $x \in X$, we call a set $N \subseteq X$ a *neighbourhood* of x if there is an open subset of N containing x. Then the family $\mathcal{U}(x)$ of all neighbourhoods of x is a filter, and $\mathcal{U}(x)$ is an ultrafilter if and only if $\{x\}$ is open in X.

With these examples covered, we are ready for our first lemma.

Lemma 3.1. *If $\mathcal{A}$ is a non-empty family of non-empty subsets of X such that any finite intersection of sets in $\mathcal{A}$ is non-empty, then there is a filter $\mathcal{F}$ which contains $\mathcal{A}$.*

Proof. First let $\mathcal{F}' = \mathcal{A} \cup \mathcal{I}$, where $\mathcal{I}$ is the set of all finite intersections of sets in $\mathcal{A}$. Then $\mathcal{F}'$ is closed under finite intersections. Then let $\mathcal{F} = \mathcal{F}' \cup \mathcal{S}$, where $A \in \mathcal{S}$ if and only if A contains a set in $\mathcal{F}'$.

We first note that $\emptyset \notin \mathcal{F}$ since the intersection of any finite sub-collection of $\mathcal{A}$ is non-empty. Likewise, $X \in \mathcal{F}$ by the non-emptyness of $\mathcal{A}$ and the construction of $\mathcal{S}$. Clearly $\mathcal{F}$ is closed under the superset inclusion. It remains to show that if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.

If $A, B \in \mathcal{F}$, the only non-trivial case needing consideration is, without loss of generality, when $A \in \mathcal{S}$. So there are sets $A', B' \in \mathcal{F}'$ with $A' \subseteq A$ and $B' \subseteq B$ (if $B \in \mathcal{F}'$ then $B' = B$). Since $\mathcal{F}'$ is closed under finite intersections, $A' \cap B' \in \mathcal{F}'$. Then, from $A' \cap B' \subseteq A \cap B$ it follows that $A \cap B \in \mathcal{S} \subseteq \mathcal{F}$. Hence $\mathcal{F}$ is closed under finite intersections, as needed. □

Lemma 3.2. *A filter $\mathcal{F}$ is an ultrafilter if and only if for every subset $A \subseteq X$ either $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.*

Proof. For "$\Rightarrow$", let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. So every set in $\mathcal{F}$ intersects both A and A^c. Using this and the fact that $\mathcal{F}$ is a filter, it follows that any finite sub-collection of $\mathcal{F} \cup \{A\}$ has a non-empty intersection. Then, Lemma 3.1 implies that there is a filter containing $\mathcal{F} \cup \{A\}$, and so $\mathcal{F}$ is properly contained in a filter, a contradiction.

For "$\Leftarrow$", suppose $\mathcal{F}$ is a filter with the property that $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$ for every subset $A \subseteq X$. If $\mathcal{F}$ is not an ultrafilter then there is another filter $\mathcal{F}'$ which properly contains $\mathcal{F}$. Hence there is a subset $E \subseteq X$ with $E \in \mathcal{F}'$ and $E \notin \mathcal{F}$. Thus $E^c \in \mathcal{F}$ and hence $E^c \in \mathcal{F}'$. But then $\mathcal{F}'$ contains the empty set since it is closed under finite intersections and $\emptyset = E \cap E^c \in \mathcal{F}'$, which is a contradiction. □

We will say that a filter $\mathcal{F}$ has the *Ramsey property* if whenever $\bigcup_{j=1}^{n} A_j \in \mathcal{F}$ there is a $j \in [n]$ with $A_j \in \mathcal{F}$. In this connection, we have the following surprising result.

Lemma 3.3. *A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}$ has the Ramsey property.*

Proof. It suffices to prove the claim for $n = 2$. Let $\mathcal{F}$ be an ultrafilter and suppose for a contradiction that $A = A_1 \cup A_2$ is in $\mathcal{F}$ but $A_1 \notin \mathcal{F}$ and $A_2 \notin \mathcal{F}$. From Lemma 3.2, we have $A_1^c \in \mathcal{F}$ and $A_2^c \in \mathcal{F}$ so that $A^c = A_1^c \cap A_2^c \in \mathcal{F}$. Consequently, $\emptyset = A \cap A^c \in \mathcal{F}$ is a contradiction.

Conversely, assume $\mathcal{F}$ has the Ramsey property. If $\mathcal{F}$ is not an ultrafilter, then Lemma 3.2 asserts that there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. But $X \in \mathcal{F}$ and $X = A \cup A^c$, so we must have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$, contradicting our choice of A. $\square$

For a family $\mathcal{F}$ of subsets of X, we define its *dual* $\mathcal{F}^*$ to be the collection of all subsets of X which intersect every set in $\mathcal{F}$.

A simple example of families and their duals is as follows [9]. We call a subset $A \subseteq \mathbb{N}$ *thick* if it contains intervals of arbitrary lengths, and we call A *syndetic* if the space between its intervals is bounded. That is, A is syndetic if there is an integer $N \in \mathbb{N}$ such that every interval of length N contains a point in A. Then, the dual of the family of syndetic subsets of $\mathbb{N}$ is the class of thick sets in $\mathbb{N}$.

The following result, due to Glasner, relates ultrafilters to their duals [12]. We provide its proof here for clarity.

Lemma 3.4. *A filter $\mathcal{F}$ is an ultrafilter if and only if $\mathcal{F}^*$ is a filter.*

Proof. For "$\Rightarrow$", assume that $\mathcal{F}$ is an ultrafilter, and fix $A, B \in \mathcal{F}^*$. To prove that $A \cap B \in \mathcal{F}^*$, it suffices to fix a set $E \in \mathcal{F}$ and prove that E intersects $A \cap B$. Note that since $A \in \mathcal{F}^*$ we have $E \cap A \neq \emptyset$. Hence we may write

$$E = (E \cap A) \cup (E \setminus A).$$

Certainly $E \setminus A \notin \mathcal{F}$, as otherwise $A \in \mathcal{F}^*$ implies $(E \setminus A) \cap A \neq \emptyset$. Since $\mathcal{F}$ is an ultrafilter, Lemma 3.3 implies that $\mathcal{F}$ has the Ramsey property so that $E \cap A \in \mathcal{F}$. Since $B \in \mathcal{F}^*$, it follows as needed that $E \cap (A \cap B) \neq \emptyset$.

Conversely, for "$\Leftarrow$", let $\mathcal{F}^*$ be a filter. Suppose for a contradiction that $\mathcal{F}$ is not an ultrafilter. Then Lemma 3.2 implies that there is a subset $A \subseteq X$ with $A \notin \mathcal{F}$ and $A^c \notin \mathcal{F}$. Thus, every set in $\mathcal{F}$ intersects both A and A^c. That is, for every $B \in \mathcal{F}$ we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$. By definition, then, $A \in \mathcal{F}^*$ and $A^c \in \mathcal{F}^*$. But $\mathcal{F}^*$ is a filter, so $\emptyset = A \cap A^c \in \mathcal{F}^*$ is a contradiction, and the proof is complete. $\square$

4. Main Result

4.1. **A technical lemma.** Protasov and Protasova introduce the following lemma, constructing a sequence (x_n) with a useful property [1]. We use the existence of this sequence to prove the main result. Hence, for the sake of completeness, we start this section with its statement and proof. We conclude this section with a proof of Theorem 1.2.

Lemma 4.1. *Let (X, d) be an infinite metric space. Then there is a sequence $(x_n)_{n=0}^{\infty}$ of distinct points in X such that either*

(1) The sequence $(d(x_0, x_n))_{n=1}^{\infty}$ is strictly monotone; or
(2) For every $n \geq 0$ and $i, j \geq n$ the distances $d(x_n, x_i) = d(x_n, x_j)$.

Proof. For a point $x \in X$ and a subset $A \subseteq X$, define

$$D(x, A) = \{d(x, y) : y \in A\}.$$

We first assume that there is a point $x_0 \in X$ such that $D(x_0, X)$ is infinite. Hence, there is a countably infinite subset $A \subseteq X$ with $x_0 \notin A$ and $d(x_0, x) \neq d(x_0, y)$ for every $x, y \in A$. Define the sequence $\xi = \{d(x_0, x) : x \in A\}$, and note that since ξ is a sequence of reals with distinct points, it has a strictly monotone subsequence $(d(x_0, x_n))_{n=1}^{\infty}$ as required.

Otherwise, $D(x, X)$ is finite for every $x \in X$. Write $A_0 = X$ and fix a point $x_0 \in A_0$. Then there is a distance $d_1 \in D(x_0, A_0)$ such that $A_1 = \{y \in A_0 : d(x_0, y) = d_1\}$ is infinite. Indeed, otherwise one may write $D(x_0, A_0) = \{\ell_1, \ell_2, \ldots, \ell_m\}$ and note that

$$X = \bigcup_{j=1}^{m} \{y \in A_0 : d(x_0, y) = \ell_j\}$$

is finite, which contradicts our choice of X. Observe that $D(x_0, A_1) = \{d_1\}$ and $x_0 \notin A_1$, since otherwise $d_1 = 0$ and so $|A_1| = 1 < \infty$.

For each $n \geq 2$, we choose x_n and define A_n inductively as follows. By the same reasoning as above, there is a distance $d_n \in D(x_{n-1}, A_{n-1})$ such that $A_n = \{y \in A_{n-1} : d(x_{n-1}, y) = d_n\}$ is infinite (otherwise $|A_{n-1}| < \infty$ contradicts the inductive hypothesis), and we have $x_{n-1} \notin A_n$ since $d_n \neq 0$. Note also by construction that $D(x_{n-1}, A_n) = \{d_n\}$ and $A_{n-1} \supsetneq A_n$. By induction, the sequence (x_n) satisfies (2). $\square$

4.2. **Main result.** Recall that the map $\chi : \Gamma_X \to [k]$ is called a *diametric colouring* if $\chi(A_1) = \chi(A_2)$ for every pair A_1, A_2 of compact subsets of X with $\operatorname{diam} A_1 = \operatorname{diam} A_2$. A subset $M \subseteq X$ is called *monochrome* if its compact subsets receive the same colour; that is, there is a colour $\varphi \in [k]$ such that $\chi(\Gamma_M) = \{\varphi\}$. A free ultrafilter $\mathcal{F}$

on X is called *diametrically Ramsey* if for every diametric colouring χ there is a monochrome set $M \in \mathcal{F}$.

We are ready to prove Theorem 1.2. Its statement is the following: *Every infinite ultrametric space X contains a sequence (x_n) such that any free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is diametrically Ramsey.*

Proof of Theorem 1.2. Let χ be any diametric colouring on X and consider any free ultrafilter $\mathcal{F}$ containing the sequence $(x_n)_{n=0}^{\infty}$ as obtained in Lemma 4.1.

Let $h : (x_n) \to \mathbb{R}^+$ be a fixed map. How we define h depends on (x_n), so we do this later on. Moreover, let $f : \mathbb{R}^+ \to [k]$ be a mapping satisfying $f(h(x_n)) = \chi(A)$ whenever $A \in \Gamma_X$ is a compact subset of X with $\operatorname{diam} A = h(x_n)$. Note that f is well-defined since χ is diametric. Finally, put $c = f \circ h$.

Write $(x_n) = c^{-1}([k]) = \bigcup_{j=1}^{k} c^{-1}(\{j\})$ and note that since $(x_n) \in \mathcal{F}$, there is a colour $\varphi \in [k]$ such that $c^{-1}(\{\varphi\}) \in \mathcal{F}$. Indeed, $\mathcal{F}$ is an ultrafilter so Lemma 3.3 implies that $\mathcal{F}$ has the Ramsey property.

We set $M = c^{-1}(\{\varphi\})$ and complete the proof by showing that M is monochrome. Specifically, we show that $\chi(\Gamma_M) = \{\varphi\}$, meaning that $\chi(N) = \varphi$ for every compact subset N of M.

Fix $N \in \Gamma_M$ and note that since N is compact, there is a pair of points $x_i, x_j \in N$ with $i < j$ and $d(x_i, x_j) = \operatorname{diam} N$. We now consider the conditions on (x_n) as described in the cases of Lemma 4.1 and define h accordingly to complete the proof.

Case 1. We first assume that case (1) of Lemma 4.1 holds, namely that (x_n) is a sequence of distinct points in X such that $(d(x_0, x_n))_{n=1}^{\infty}$ is strictly monotone. In this case, h will indicate the distance between x_0 and a term x_n, given by $h(x_n) = d(x_0, x_n)$.

If h is strictly increasing, then we have $d(x_0, x_i) < d(x_0, x_j)$. Hence Lemma 2.1 implies that $d(x_i, x_j) = d(x_0, x_j)$, since d is an ultrametric. Otherwise h is strictly decreasing, so $d(x_0, x_i) > d(x_0, x_j)$ and hence $d(x_i, x_j) = d(x_0, x_i)$, using Lemma 2.1 once more. Possibly swapping the symbols i, j, we may assume that the former case holds. Since $x_j \in M = c^{-1}(\{\varphi\})$, $\operatorname{diam} N = d(x_i, x_j)$, and using the construction of f, we have

$$\varphi = c(x_j) = f(d(x_0, x_j)) = f(d(x_i, x_j)) = \chi(N),$$

so N has colour φ as needed.

Case 2. We now assume that case (2) of Lemma 4.1 applies to (x_n). Thus, for each $n \in \mathbb{N}$ and $i, j \geq n$ we have $d(x_n, x_i) = d(x_n, x_j)$. Define $h : (x_n) \to \mathbb{R}^+$ by $h(x_n) = d(x_n, x_{n+1})$, and note that $h(x_n) = d(x_n, x_j)$ for every $j > n$. Using the same reasoning as above, N receives colour

φ since

$$\varphi = c(x_i) = f(d(x_i, x_{i+1})) = f(d(x_i, x_j)) = \chi(N).$$

This completes the proof, since M is monochrome in both cases. □

5. Concluding Remarks and Questions

We have shown that every infinite ultrametric space contains a sequence (x_n) such that any free ultrafilter $\mathcal{F}$ with $(x_n) \in \mathcal{F}$ is diametrically Ramsey. An interesting follow up question regards the assumption that $(x_n) \in \mathcal{F}$. The case where $(x_n) \in \mathcal{F}$ is settled, but what can we say about $\mathcal{F}$ if it contains no sequence as described in Lemma 4.1?

Perhaps the strong properties of both ultrametric spaces and ultrafilters can be leveraged more precisely to prove that $\mathcal{F}$ is diametrically Ramsey. Ideally, it would be most elegant if one could show that *every* free ultrafilter on an ultrametric space X is diametrically Ramsey, but such a proof may be challenging (or even impossible)! ☺

Acknowledgements. I would like to extend my deepest thanks to my Directed Reading Program (DRP) advisor, Aaron Shalev, for his invaluable guidance and support. His expertise, dedication, and insightful feedback have not only strengthened this work substantially, but they have also deepened my understanding of the subject.

I would also like to thank the DRP commitee for their significant efforts in organising the program and fostering an environment that encourages rigorous academic enquiry and growth.

References

[1] I. Protasov and K. Protasova, "Metrically Ramsey ultrafilters," *Matematychni Studii*, vol. 49, no. 2, pp. 115–121.

[2] F. P. Ramsey, "On a problem of formal logic," *Proceedings of the London Mathematical Society*, vol. s2-30, no. 1, pp. 264–286, 1930.

[3] B. L. V. der Waerden, "Beweis einer baudetschen vermutung," *Nieuw Arch. Wiskunde*, vol. 15, pp. 212–216, 2018.

[4] B. Leclerc, "Description combinatoire des ultramétriques," *Mathématiques et sciences humaines*, vol. 73, pp. 5–37, 1981.

[5] D. Guy and S. Semmes, *Fractured Fractals and Broken Dreams: Self-similar Geometry Through Metric and Measure*, ch. 15, pp. 156–171. Oxford, 1997.

[6] J. Heinonen, *Lectures on analysis on metric spaces*, ch. 14, pp. 109–118. Springer, 2001.

[7] W. H. Schikhof, *Ultrametric calculus: an introduction to p-adic analysis*, ch. 1, pp. 46–55. Cambridge University Press, 1985.

[8] S. Koppelberg, *Filters and ultrafilters.* Freie Universität, 2011. `www.mi.fu-berlin.de/math/groups/ag-logik/Lehre/UST-chapter02.pdf`.

[9] W. R. Brian, "Ramsey shadowing and minimal points," *Proceedings of the American Mathematical Society*, vol. 144, pp. 2697–2703, 2016.
[10] C. Higgins, *Ultrafilters in set theory.* University of Chicago, 2018. www.math.uchicago.edu/ may/REU2018/REUPapers/Higgins.pdf.
[11] C. Stanford, *Filters and ultrafilters.* University of California, Davis, 2015. www.web.cs.ucdavis.edu/ cdstanford/doc/2015/GISP-lecture17.pdf.
[12] S. Glasner, "Divisible properties and the Stone-Čech compactification," *Canadian Journal of Mathematics*, vol. 32, no. 4, p. 993–1007, 1980.

Email address: jake.gameroff@mail.mcgill.ca