Symmetry Methods
for
Differential Equations

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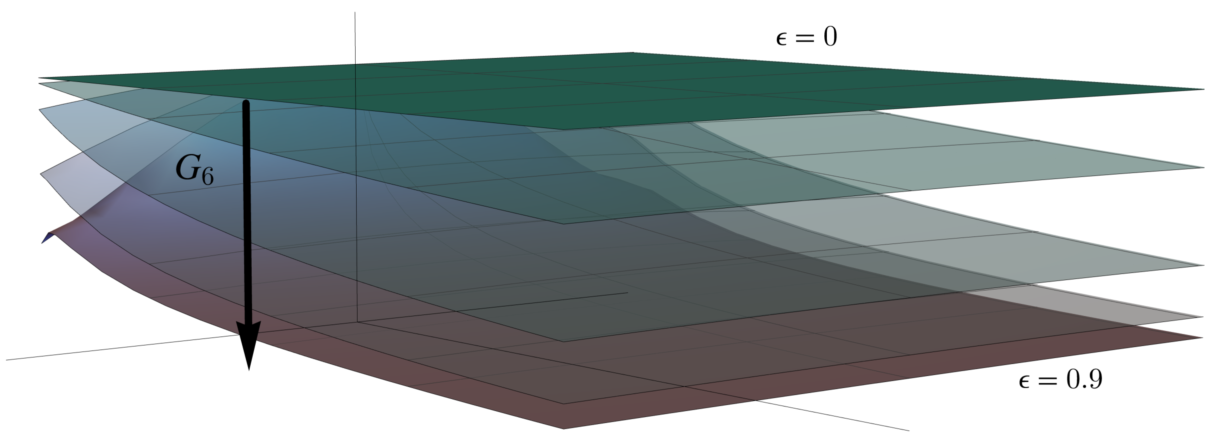


A family of solutions to the heat equation $u_t = u_{xx}$ obtained from the constant solution via a particular Lie group transformation; see example 4.1.

Abstract

We give an overview of continuous symmetry methods for classifying, solving, and generally working with differential equations. We begin with an overview of the theory of Lie groups and algebras, before turning to several applications.

Contents

1 Introduction

A differential equation is, in short, a statement about the relationship between an unknown function and its derivatives,

$$F(x, y, \ldots; u; u_x, u_y; u_{xy}, u_{xx}, u_{yy}, \ldots) = 0,$$

with a solution a function $u = f(x, y, \ldots)$ that satisfies this relation. Such equations arise frequently throughout many areas of math and beyond. Studying such equations can take many different approaches.

The manner that we will be discussing here will be less concerned with actually extrapolating or proving the existence of solutions, but rather describing potential solutions qualitatively. Namely, our goal is to describe the "symmetries" of solutions to differential equations, without necessarily having a solution on hand.

Let's consider the following system of differential equations

$$\dot{x} = -y,$$
$$\dot{y} = x.$$

Let us (without much real motivation, admittedly) consider the effect of *rotating* this entire curve in the plane, about the origin. More specifically, let $\theta \in [0, 2\pi)$ be some angle, and map each point (x, y) on the curve to the new point $(\tilde{x}, \tilde{y}) := (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$.

Remark now that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{x}, \tilde{y}) &= (\dot{x} \cos\theta - \dot{y} \sin\theta, \dot{x} \sin\theta + \dot{y} \cos\theta) \\ &= (-y \cos\theta - x \sin\theta, -y \sin\theta + x \cos\theta) \qquad \text{(since } \dot{x}, \dot{y} \text{ a solution)} \\ &= (-\tilde{y}, \tilde{x}), \end{aligned}$$

that is, $(\tilde{x}, \tilde{y})$ is *also* a solution to our original differential equation! We say, fittingly, that the system admits a rotational symmetry group.

Indeed, since the map $g_\theta : (x, y) \mapsto (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$ maps to a new solution for *any* θ, we actually have an entire *family* of symmetries $G := \{g_\theta\}_{\theta \in [0,2\pi)}$. Moreover, endowing this set with the typical function composition $\circ$, one can show that G is a *group*, though rather different than many one may typically consider; it is not discrete.

This is a first example of a *Lie group*, a group in the traditional sense, but with the extra condition of "continuity" in its elements, in a sense to be made precise to follow. It turns out that such structures are key to "formally" defining what it means for a differential equation to be "symmetric" in a well-defined way.

The kind of symmetries that we will be studying can be thought of as particular changes of coordinates that leave a differential equation (at least locally) unchanged. The theory we'll develop to make this idea more concrete will be very geometric in its language. It turns out that studying such properties of a differential equation is quite powerful, and can often aid in finding solutions to a differential equation with a little added inspiration (despite this not being our primary intention!).

The content in this paper largely follows the methodologies presented by [Olv86].

2 Notation

We notate $(x, u) \in M := X \times U := \mathbb{R}^p \times \mathbb{R}^q$, $p, q \in \mathbb{N}$ as the space of independent × dependent variables, with $x = (x^1, \ldots, x^p), u = (u^1, \ldots, u^q)$. A function $f = f(x, u^{(n)})$ relies on x, u, and up to (and including) the nth order partial derivatives of u. $D_x u$ and $\frac{\partial u}{\partial x}$ denote the total and partial derivatives of u with respect to x, respectively; we will occasionally use ∂_x to represent the partial derivative operator.

When we write

$$G(x)|_{F(x)=0},$$

one should interpret $G(x)$ being evaluated at every x such that $F(x) = 0$.

3 Lie Groups and their Algebras

In this section, we begin by explaining some of the machinery we will be using throughout. Our end goal is to formalize how we can talk about the symmetries of a differential equation - namely, given a solution, how can we continuously map to another?

The manner that we approach this question is by abstracting our notion of symmetry via a group (namely, a "Lie group"), and discussing how the group acts on a manifold. We will show how we can view the action of the group as a vector field on the manifold, defining a "Lie algebra" as such. This allows us to consider real-valued functions defined on the manifold, and study how the group acts on it by considering instead how it changes under the flow of the vector field. This will give us the underlying machinery to ultimately consider differential functions and study their symmetries with the same perspective.

3.1 Lie Groups

3.1.1 First Definitions

Definition 3.1 (Manifold). *An r-dimensional manifold M is a topological space that "resembles" r-dimensional Euclidean space.*

More technically, every point in M has a neighborhood homeomorphic *to a subspace of $\mathbb{R}^r$.*

We will not be too concerned in our applications about any more rigorous notions of manifolds than "kind of like $\mathbb{R}^r$", Moreover, we will, unless otherwise stated, be working with smooth manifolds, where the described homeomorphism(s) are smooth. Typical examples of smooth manifolds are the sphere S^2 (two dimensional), the circle S^1 (one dimensional), and of course $\mathbb{R}^r$ itself,

Definition 3.2 (Lie Group)**.** *A* r*-parameter Lie group*[1] *G is a set with both the structure of a group and a smooth r-dimensional manifold, in such a way that the group operation and inversion are smooth maps.*

This definition can be slightly confusing initially; the group and manifold structure are not obviously compatible at first sight. We immediately consider some examples to hopefully elucidate this definition.

Example 3.1. *Take $G := \mathbb{R}$ under addition, with inversion given by $a^{-1} = -a$. Both of these operations are smooth and thus G is a 1-parameter Lie group, viewed as the 1-dimensional manifold $\mathbb{R}$.*

Example 3.2. *Let*

$$SO(2) := \{A \in M_2(\mathbb{R}) : \det(A) = 1, A^t A = AA^t = I_2\}$$
$$\equiv \{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : 0 \le \theta < 2\pi\}$$

be the "special orthogonal group" of 2×2 matrices. This is a 1-parameter Lie group that we can view as the unit circle

$$S^1 := \{(\cos\theta, \sin\theta) : 0 \le \theta < 2\pi\},$$

a 1-dimensional manifold; given any $\theta \in S^1$, there is a unique corresponding element in $SO(2)$. This defines the manifold structure on $SO(2)$.

We'll discuss $SO(2)$ quite a lot in examples to follow, as it arises naturally as the rotation group in the plane.

3.1.2 Group Actions

[1] We often wish to consider only local Lie groups, corresponding to "local symmetries", where instead of G being a manifold it is rather a set of connected open sets with smooth operations. We won't specify the technical differences in our following definitions between local and global Lie groups unless vital.

Definition 3.3 (Group of Transformations)**.** *Let M a smooth manifold, G a Lie group, and an open set $U \subseteq G$ such that e (identity) $\in U$. Let*

$$\psi : \mathcal{U} := U \times M \to M$$

be a smooth function such that:

(i) *if $(h, x) \in \mathcal{U}$, $(g, \psi(h, x)) \in \mathcal{U}$, and $(g \cdot h, x) \in \mathcal{U}$, then $\psi(g, \psi(h, x)) = \psi(g \cdot h, x)$;*

(ii) *$\psi(e, x) = x$ for all $x \in M$;*

(iii) *if $(g, x) \in \mathcal{U}$, then so is $(g^{-1}, \psi(g, x)) \in \mathcal{U}$ with $\psi(g^{-1}, \psi(g, x)) = x$.*

We say G is a local group of transformations on M, with group transformation ψ with a domain $\mathcal{U}$. If $\mathcal{U} = G \times M$, then we say we have a global group of transformations. We will usually omit the ψ and denote such a group action as $g \cdot x \equiv \psi(g, x)$ when ψ is clear from context.

Example 3.3. *Consider $G = \mathbb{R}$ and $M = \mathbb{R}$, and for $x \in M$, $g \in \mathbb{R}$, define*

$$g \cdot x = x + g.$$

That is, G acts on M by translation. We could also define, more generally, for $x = (x^1, \ldots, x^n) \in M = \mathbb{R}^n$

$$g \cdot x = (x^1 + g, \ldots, x^n + g)$$

or even

$$g \cdot x = (x^1, \ldots, x^i + g, \ldots, x^n)$$

for any $1 \leq i \leq n$. The second of these will appear often in applications, when, say, a differential equation in two independent variables admits translation symmetries in both.

Example 3.4. *Consider $G = SO(2)$ as defined in example 3.2 and $M = \mathbb{R}^2$. For $p = (x, y) \in M$ and $g \in G$, define*

$$g \cdot p = \text{ rotate } p \text{ by } g = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

It is readily seen that this defines a group of transformations of G on M; indeed, this is *a global group of transformations.*

Example 3.5. *For a perhaps more exotic example, consider $G = \mathbb{R}$ and $M = \mathbb{T}^2 = S^1 \times S^1$, the (2-dimensional) torus. We can write coordinates on M as $(\theta, \phi) \in M$; define*

$$g \cdot (\theta, \phi) = (\theta + g, \phi + c \cdot g) \mod 2\pi,$$

for some real constant c. This is again a global group of transformations.

Consider the orbit[2] of $x = (0, 0)$ (for simplicity's sake):

$$O_x = \{(g, c \cdot g) \mod 2\pi : g \in \mathbb{R}\}$$
$$= \{(\theta, c \cdot \theta) : 0 \leq \theta < 2\pi\}.$$

Now, remark that $(0, 0) = (\theta, c \cdot \theta) \mod 2\pi \iff \theta = m \cdot 2\pi$ and c rational. Concretely, this means that the actual, geometric orbit described by O_x is "closed" (ie, loops back on itself) only if c is an integer. If c is irrational, then the orbit will never loop back on itself and instead form a "dense" subspace of $\mathbb{T}^2$ (see fig. 1).

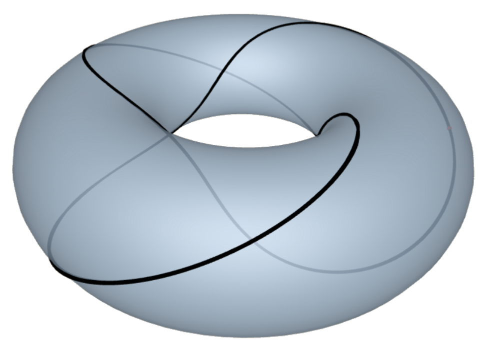


Figure 1: A rational flow on the torus; an irrational flow is not pictured since it would essentially be, visually, nothing but a black-painted torus.

With this in mind, and hopefully some sense for the idea of what makes a group action of a Lie group distinct from that of a "discrete" group, we can start to discuss the connection to group actions on functions, and ultimately (as is our goal) on differential equations.

[2]Recall, the orbit of some $x \in M$ is $O_x := \{g \cdot x : g \in G\}$. If G only acts locally, we need to refine this definition to only include elements $g \cdot x$ where this product is well-defined.

3.1.3 Action on Functions

Let $f : X := \mathbb{R}^p \to U := \mathbb{R}^q$ be a smooth function. Denote the *graph* of f as

$$\Gamma_f := \{(x, f(x)) : x \in X \text{ s.t. } f(x) \text{ defined } \} \subseteq X \times U.$$

Let G be a Lie group acting on $X \times U$ and $g \in G$. Assuming the graph Γ_f is contained in the domain of definition of g, we have

$$g \cdot \Gamma_f = \{(\tilde{x}, \tilde{u}) := g \cdot (x, u) : (x, u) \in \Gamma_f\},$$

the result of g acting on Γ_f.[3] Now, if there existed another function $\tilde{u} = \tilde{f}(\tilde{x})$ for which $\Gamma_{\tilde{f}} = g \cdot \Gamma_f$, then we would simply have that the result of g acting on f is this function, $\tilde{f}$. However, such a $\tilde{f}$ does not exist in general. For example, the graph obtained from rotating the parabola $f(x) = x^2$ (defined on all of $\mathbb{R}$) by $\pi/2$ about the origin cannot possibly be the graph of a function on all of $\mathbb{R}$, since it would necessarily have multiple outputs for a given input.[4]

However, we can circumvent this issue by appropriately shrinking our domain of consideration, and for g sufficiently close to the identity in G, we can guarantee the existence of such a $\tilde{f}(\tilde{x})$; we thus write

$$\tilde{f}(\tilde{x}) = g \cdot f(x).$$[5]

Another important factor to note is that when Γ_f is transformed, both the x and u are transformed, and we are thus working with new (transformed) coordinates $(\tilde{x}, \tilde{u})$. Thus, to find a function $\tilde{f}$, it must be a function of $\tilde{x}$, and not of x. In practice, this means solving for $\tilde{u}$ in terms of just $\tilde{x}$. We illustrate:

Example 3.6. *Let $u = f(x) = mx + b : \mathbb{R} \to \mathbb{R}$ be a straight line in the plane acted on by $SO(2)$. Letting $\theta \in SO(2)$, we find*

$$\theta \cdot (x, u) = \theta \cdot (x, mx+b) = (x\cos\theta - (mx+b)\cdot\sin\theta, x\sin\theta + (mx+b)\cdot\cos\theta) = (\tilde{x}, \tilde{u}),$$

[3] We will, in general, use tildes over variables/functions/etc. to indicate that they have been transformed by a group action.

[4] "Fails the vertical line test", if you will.

[5] It's important to note that this notation is not the same as the group action notation, as it doesn't make any sense to say that "g acts on f"; this is simply saving the time of defining and transforming the graph of f every time.

identically to example 3.4 upon replacing y with u; indeed, the only difference between this example and that one is that now u (resp. $\tilde{u}$) is taken as a function of x (resp. $\tilde{x}$). We have thus

$$\tilde{x} = x\cos\theta - (mx+b)\cdot\sin\theta$$
$$\tilde{u} = x\sin\theta + (mx+b)\cdot\cos\theta,$$

ie a system of two equations, with the goal of finding $\tilde{u}$ as a function of $\tilde{x}$; hence we must eliminate the dependence on x in each. After computation, we find

$$\tilde{u} = \frac{\sin\theta + m\cos\theta}{\cos\theta - m\sin\theta}\tilde{x} + \frac{b}{\cos\theta - m\sin\theta},$$

and so, taking $\tilde{f}(\tilde{x}) = \tilde{u}$, this is precisely the result of rotating a line by an angle of θ. Note that $\tilde{f}$, as should be expected, is still linear in $\tilde{x}$, with slope $\tilde{m} := \frac{\sin\theta + m\cos\theta}{\cos\theta - m\sin\theta}$ and intercept $\tilde{b} := \frac{b}{\cos\theta - m\sin\theta}$. In addition, $\tilde{f}$ well-defined, for sufficiently small θ, for any m.

Indeed, the only issue we may run into is when the denominator $\cos\theta - m\sin\theta$ in $\tilde{u}$ equals 0, which is precisely when

$$\theta = \arctan^{-1} m.$$

Geometrically, this is when the slope of the transformed function $\tilde{f}(\tilde{x})$ goes to infinity.

This example worked relatively well, but suppose we wanted to apply the same idea but to, say, an exponential, a more complicated polynomial, or even just a general function. This is not, in general, an easy task, and involves applying the inverse function theorem to the transformed graph to solve explicitly, and results in messy, complicated formulas, even in the relatively simple case of the rotation group. We'll come back to this soon.

Now that we can find how a Lie group transforms a given function, we proceed to extend this notion to differential equations. Precisely:

Definition 3.4 (Symmetry group of a differential equation)**.** *Let $\Delta[x, u^{(n)}] = 0$ be a system[6] of differential equations defined on $(x,u) \in X \times U = \mathbb{R}^p \times \mathbb{R}^q$. We say a Lie group G acting on some open subset of $X \times U$ is a* symmetry group *of Δ if for any solution[7] $u = f(x)$, $u = g \cdot f(x)$ is also a solution for any $g \in G$ such that $g \cdot f$ defined.*

More succinctly, G a symmetry group of Δ *if*

$$\Delta\Big|_{g\cdot f(x)} = 0 \impliedby \Delta\Big|_{f(x)} = 0$$

for any $g \in G$.

Consider for instance the classic ODE

$$\Delta = u' - u = 0$$

with solution $u = ce^x$ for any real constant c. Supposing I scale c smoothly, then I can generate a "smoothly varying family of solutions", hence scaling a solution to Δ gives another. Letting $(x, u) \mapsto (x, c \cdot u)$ be the action of $\mathbb{R}$ on the space of definition of Δ, then we see that this is indeed a symmetry in the sense of our above definition:

$$\Delta\Big|_{c\cdot e^x} = \frac{\mathrm{d}}{\mathrm{d}x}ce^x - ce^x = 0.$$

Given a solution to Δ, then, we can determine whether a given group is a symmetry group for Δ. Moreover, if it is, then we can find other solutions by applying the group action to our known solution.

This is quite helpful in rigorously defining what it means to be a symmetry, but tells us nothing about:

1. How do we find a symmetry group, particularly if we don't have a solution to the DE in the first place?
2. How do we find *all* symmetry groups?

We'll answer both of these (in some sense) to follow. As a brief spoiler, one may notice that we haven't exactly made use of all the machinery of Lie groups that we initially introduced. Indeed, we've only made use of the continuity of their group actions to ensure continuously varying symmetries, and haven't touched the underlying manifold structure. We'll make use of this in the next section.

Before we move on, however, we briefly discuss a partial answer to the first question.

[6] We'll use "system of differential equations" and "differential equation" somewhat interchangeably in our discussion.

[7] "Solution" will be taken to mean "smooth solution" (on some appropriately-resized domain).

First, to characterize how a differential equation changes under a particular group, we must see how a group action transforms the derivative of a general function. More specifically, let G be a Lie group acting on $u = f(x)$; since f is transformed to $\tilde{f}$ by G, then naturally the derivatives of f must also be transformed. But how precisely? We would like to have a closed-form way to write $u_x \mapsto \tilde{u}_x$ just as we know how to write $(x, u) \mapsto (\tilde{x}, \tilde{u})$.

The key to answering this question is through the use of "representative functions", ie functions with nth order derivatives that can easily be elucidated. For instance, in the $p = q = 1$ case, we would take the Taylor polynomial

$$f(x) = u^0 + u_x^0(x - x^0) + \frac{u_{xx}^0}{2}(x - x^0)^2 + \cdots$$

up to an arbitrary order of choice, where variables with 0 exponents are essentially dummy variables. Remark that $f(x^0) = u^0, f'(x^0) = u_x^0, f''(x^0) = u_{xx}^0$, and so on. Hence, if we would like to find how the derivatives of $u = f(x)$ change under a group operation, we simply need to find $g \cdot f(x)$ (as we know how to do from above), then take the corresponding derivative of $\tilde{f}(\tilde{x})$ evaluated at x^0.

In short, finding how the nth-order derivative of a function is transformed amounts to finding how an nth-order polynomial is transformed. This is perhaps best understood by example.

Example 3.7. *We recall again $G = SO(2)$ acting on $(x, u) \in \mathbb{R}^2$ from example 3.2, example 3.4, example 3.6. Letting*

$$f(x) = u^0 + u_x^0(x - x^0)$$

be our representative function, remark that f linear in x with slope $m = u_x^0$ and intercept $b = u^0 - u_x^0 x^0$. We know how G transforms linear functions from example 3.6, hence

$$\tilde{f}(\tilde{x}) = \theta \cdot f(x) = \frac{\sin\theta + u_x^0 \cos\theta}{\cos\theta - u_x^0 \sin\theta}\tilde{x} + \frac{u^0 - u_x^0 x^0}{\cos\theta - u_x^0 \sin\theta}$$

and so

$$\tilde{u}_x^0 = \tilde{f}'(\tilde{x}^0) = \frac{\sin\theta + u_x^0 \cos\theta}{\cos\theta - u_x^0 \sin\theta}.$$

It follows thus that

$$u_x \mapsto \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta}.$$

This "technique" of using representative functions extends naturally to higher order derivatives of u, and to cases where $p > 1, q > 1$ by using higher order and multivariate Taylor polynomials respectively. We call the corresponding action on the derivatives $u_x, \ldots$ the *prolongation* of the group action, denoted with a $\mathbf{pr}^{(n)}$ with n the corresponding order; for instance in the case of example 3.7, we would write

$$\mathbf{pr}^{(1)}\theta \cdot (x, u, u_x) = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta}).$$

Note the increase in difficulty for computing the prolongation for higher and higher order derivatives, stemming from the fact that we have to solve corresponding systems of equations involving polynomials of higher and higher degrees. Ultimately, we don't want to have to do this regularly.

3.2 Lie Algebras

To any Lie group, we can associate a canonical *Lie algebra*. We recall that an algebra V (over a given field, which in our case will always be $\mathbb{R}$) is a vector space together with a bilinear product, $[\cdot,\cdot] : V \times V \to V$. More specifically, the Lie algebra associated with a Lie group is identified as the tangent space at the identity of the Lie group. We begin this section by briefly reviewing some general theory on manifolds we'll need before specifying our discussion to the Lie group setting.

3.2.1 Tangent Spaces, Flows as Group Actions

Definition 3.5 (Manifold Review: Tangent space, Vector Fields, Flows)**.** *Let M be an r-dimensional manifold. Let*

$$\varphi = (\varphi^1, \ldots, \varphi^r) : I \to M,$$

where $I \subseteq \mathbb{R}$ some interval, be a parametrized curve on the manifold. At each point $x = (x^1, \ldots, x^r) = \varphi(t)$ on the curve, there exists a tangent vector $\dot\varphi(t) = (\dot\varphi^1(t), \ldots, \dot\varphi^r(t))$; we denote[8]

$$\mathbf{v}|_x := \dot\varphi(t) = \dot\varphi^1(t)\partial_{x^1}|_x + \cdots + \dot\varphi^r(t)\partial_{x^r}|_x.$$

(Remark the (suggestive) notation; for now, one should consider the components ∂_{x^i} as "placeholders" for basis vectors. We adopt this notation to clarify when we are working with tangent vectors versus directly on the manifold (we will show later that such a tangent vector can be considered as a partial differential operator).)

We define the tangent space *of M at x as the vector space*

$$\begin{aligned} TM|_x &:= \{\mathbf{v}|_x = \dot{\varphi}(t) : \varphi(t) \text{ s.t. } \exists t_0 : \varphi(t_0) = x\} \\ &= \{\text{all tangent vectors to all curves passing through } x\} \end{aligned}$$

See fig. 2.

We put

$$TM := \bigcup_{x \in M} TM|_x,$$

called the tangent bundle *of M.*

We define a vector field *on M as a smooth assignment to each point x in M a vector in $TM|_x$, which we denote*

$$\mathbf{v}|_x = \xi^1(x)\partial_{x^1}|_x + \cdots + \xi^r(x)\partial_{x^r}|_x,$$

where each ξ^i smooth in x. An integral curve *of $\mathbf{v}$ is a smooth (parametrized) curve $\varphi(t)$ such that*

$$\dot{\varphi}(t) = \mathbf{v}|_{\varphi(t)}$$

for all t in the domain of definition of φ. Given some point $x_0 \in M$, the integral curve that is "as large as possible" passing through x_0 (ie any other integral curve is contained in it) is called the maximal integral curve *of $\mathbf{v}$ at x_0.* [9]

We often wish to refer, for a fixed vector field $\mathbf{v}$, to multiple maximal integral curves that pass through specified points. We write

$$\psi(t, x) := \phi(t) : \phi(0) = x,$$

or equivalently,

$$\exp(t\mathbf{v})(x)$$

when we wish to emphasize the particular vector field in question. We call this function the flow *of $\mathbf{v}$ at x (parametrized by t).*

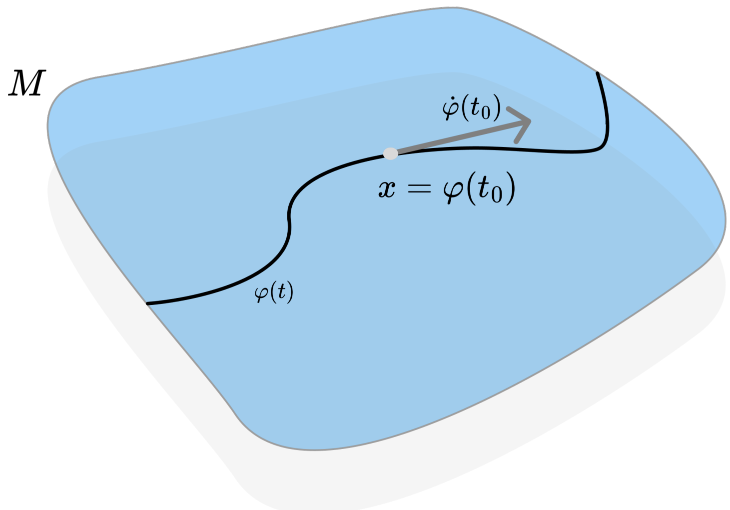


Figure 2: Tangent vector to a point on a curve.

Let M be an r-dimensional smooth manifold, $\mathbf{v}$ a vector field which we denote in local coordinates

$$\mathbf{v}|_x = \sum_{i=1}^{r} \xi^i(x)\partial_{x^i}.$$

Let $f : D \subseteq M \to \mathbb{R}$ be a smooth function on M. Then, for any $x \in D$, we have a corresponding flow of $\mathbf{v}$, $\varphi(t) : I \to M$, and can consider $f \circ \varphi = (\varphi^1, \ldots, \varphi^r) : I \to \mathbb{R}$.

We would like to see how f changes under the flow of $\varphi(t)$; ie we want to find the change with respect to "time" t of $f(\varphi(t))$. Differentiating with respect to t via the chain rule, remark then that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(f(\varphi(t))) &= \sum_{i=1}^{r} \frac{\partial f}{\partial x^i}(\varphi(t)) \cdot \frac{\mathrm{d}\varphi^i(t)}{\mathrm{d}t} \\ &= \sum_{i=1}^{r} \frac{\partial f}{\partial x^i}(\varphi(t)) \cdot \xi^i(\varphi(t)).\end{aligned}$$

But then, remark that this is simply equivalent to treating $\mathbf{v}|_{\phi(t)}$ as a partial differential

[8] We will usually omit the $|_x$ in this notation when it is clear from the context what x is, but it is necessary, technically speaking - we are working in different spaces for each x, which thus carry different basis vectors.

[9] Finding the (maximal) integral curve of a given $\mathbf{v}$ reduces to solving a system of differential equations; namely, in standard notation taking $\varphi = (\varphi^1, \ldots, \varphi^r)$, we solve

$$\dot{\varphi}^i = \xi^i(\varphi^i), \quad i = 1, \ldots, r,$$

which, if we take some arbitrary initial condition, will admit a unique solution by classical ODEs theory.

operator; notably, at $t = 0$ (at $\varphi(0) = x$), we have

$$\mathbf{v}(f)(x) = \sum_{i=1}^{r} \frac{\partial f}{\partial x^i}(x) \cdot \xi^i(x).$$

This hopefully clears up the reason for the notation used for vectors in the tangent space; vector fields act as partial derivative operators in their respective coordinate components on functions[10]. Indeed, it can be shown that $\mathbf{v}|_x$ defines a *derivation* on the space of smooth, real-valued functions defined around x, ie it is linear and obeys the Leibniz rule.

Proposition 3.1. *Let* $\mathbf{v}$ *be a vector field on an* r*-dimensional smooth manifold* M*, and for* $x \in M$ *denote the flow of* $\mathbf{v}$ *as* $\psi(t, x)$*, where* $\psi(0, x) = x$*. Then,* ψ *is equivalent to the (local) action of* $\mathbb{R}$ *(as a Lie group) on* M*.*

Conversely, given a local (one-parameter) group of transformations on M $\phi(t, x)$*, there exists a unique corresponding vector field* $\mathbf{v}$ *on* M *such that its flow coincides with* ϕ*, given by*

$$\mathbf{v}|_x = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi(t, x).$$

In short, there is a one-to-one correspondence with one-parameter groups of transformations on M *and vector fields on* M*.*

Proof Recall that the action, $+$, of $\mathbb{R}$ on M must be a smooth operation obeying the following (cf. definition 3.3):

i) $g + (h + x) = (g + h) + x$

ii) $e + x = x$

iii) $g^{-1} + (g + x) = x$

for any $g, h \in \mathbb{R}, x \in M$. In the language of ψ, we prove each in turn; fix $t_0, t_1 \in I$ and $x \in M$.

[10] An intuitive way to think about this is that a fixed vector (ie, $\mathbf{v}|_x$) on a manifold "tells" a function defined on the manifold how much to "move" in a given direction, the directions given by the basis vectors ∂_{x^i}.

i) We need to show $\psi(t_0, \psi(t_1, x)) = \psi(t_0 + t_1, x)$. By definition

$$\frac{\mathrm{d}}{\mathrm{d}t_0}\psi(t_0, \psi(t_1, x)) = \mathbf{v}|_{\psi(t_1,x)}$$
$$\frac{\mathrm{d}}{\mathrm{d}t_0}\psi(t_0 + t_1, x) = \mathbf{v}|_{\psi(t_1,x)},$$

and in addition, at $t_0 = 0$, the left hand side $\psi(0, \psi(t_1, x)) = \psi(t_1, x)$. Hence, we have two differential equations with the same initial conditions, and since everything is smooth we are guaranteed unique solutions and the two sides must be equal.

ii) This is equivalent to showing $\psi(0, x) = x$, which holds by assumption.

iii) We need to show $\psi(-t_0, \psi(t_0, x)) = x$; by i), we have that this equals $\psi(0, x) = x$.

Finally, ψ is smooth by construction. The second claim follows by similar arguments. ■

In short, this proposition gives us an equivalent manner of working with a local group of transformations acting on a manifold by considering vector fields on the manifold.

We call the vector field $\mathbf{v}$ corresponding to a given group of transformations ψ the *infinitesimal generator* of ψ; consider the Taylor expansion of ψ:

$$\psi(t, x) = \psi(0, x) + t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\psi(t, x) + O(t^2) = x + t \cdot (\xi^1(x), \dots, \xi^p(x)) + O(t^2).$$

The coefficients of the vector field arise as the linear terms in the group action, hence we say that $\mathbf{v}$ "linearizes" the group action. From now on, we'll denote $\exp(t\mathbf{v}) \cdot x \equiv \psi(t, x)$.

Example 3.8. *We consider yet again $SO(2)$ acting on $\mathbb{R}^2$, with group operation given by*

$$\theta \cdot (x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta).$$

We have

$$\left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}(\theta\cdot(x,y)) = \left.\frac{\mathrm{d}}{\mathrm{d}\theta}\right|_{\theta=0}(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$
$$= (-x\sin\theta - y\cos\theta, x\cos\theta - y\sin\theta)\Big|_{\theta=0}$$
$$= (-y, x),$$

hence the corresponding infinitesimal generator is given by

$$\mathbf{v}|_x = -y\partial_x + x\partial_y. \tag{1}$$

In perhaps more familiar notation, this is equivalent to writing $-y\hat{\imath} + x\hat{\jmath}$*, which can readily be seen to correspond to a counterclockwise rotating vector field in the plane; see fig. 3.*

Conversely, if we were given eq. (1), to find the group action we would have to solve

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = x,$$

a linear system of differential equations. Solving with standard methods yields a solution with an arbitrary constant, serving as the parameter of the group action.

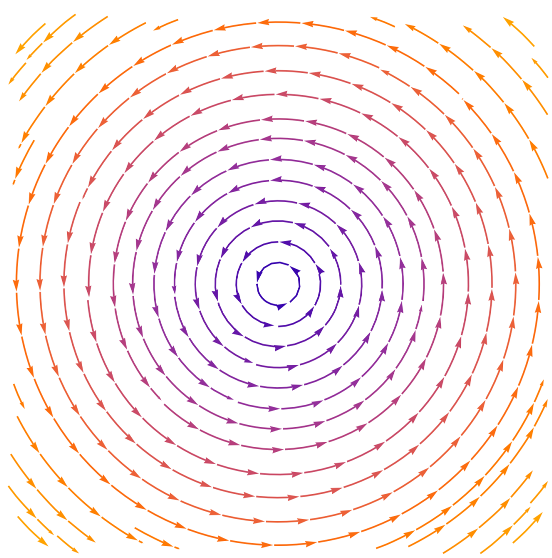


Figure 3: The vector field $-y\partial_x + x\partial_y$ in the plane.

More generally, finding a vector field from a group action amounts to taking a derivative with respect to the group operation parameter, and finding the group action from a vector field amounts to solving a (or even a system of) differential equations.

3.2.2 Lie Algebra Construction

With these properties in mind, we return now to our Lie group setting. Since a Lie group is also a manifold, we can repeat these same constructions of vector fields, flows, etc. However, we have an additional structure now, namely the group itself, and hence we would like to somehow restrict our attention to maintain this structure. Namely, we'd like to only consider vector fields that are, in a sense, "compatible" with the group operation.

Let $(G, \cdot)$ be an r-parameter Lie group. We let

$$R_g : G \to G, \quad R_g(h) := h \cdot g$$

represent right multiplication[11] by some fixed g. Recall that by definition, the group operation is smooth hence so is R_g. In addition, one can show that R_g is a diffeomorphism, and hence $dR_g : TG \to TG$ is a well-defined map on the tangent bundle of G. We say a vector field $\mathbf{v}$ is *right invariant* if, for all $g, h \in G$,

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{hg}.$$

Definition 3.6 (Lie Algebra of a Lie Group)**.** *Given*[12]*a Lie group G, we define*

$$\mathfrak{g} := \{\textit{right-invariant vector fields on } G\} = \{\mathbf{v} : dR_g(\mathbf{v}|_h) = \mathbf{v}|_{hg} \forall h, g \in G\}.$$

That is, $\mathfrak{g}$ contains the "group operation compatible" vector fields defined on G.

Proposition 3.2. $\mathfrak{g} \simeq TG|_e$*; that is, the Lie algebra of G can be identified with tangent space of G at the identity.*

Proof Let $\mathbf{v}$ be a right-invariant vector field on G. Then, at any point $g \in G$,

$$\mathbf{v}|_g = dR_g(\mathbf{v}|_e),$$

[11] A similar construction follows by instead using left multiplication; it's a matter of convention.

[12] One can (and many do) define Lie algebras in their own right without our aforementioned motivations, namely as an algebra with multiplication obeying certain properties. We won't discuss this here.

hence $\mathbf{v}$ uniquely determined by its value at e. On the other hand, given any tangent vector $\mathbf{v}|_e$ at the origin, we can define $\mathbf{v}|_g := dR_g(\mathbf{v}|_e)$ for any $g \in G$, and one can show that

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{hg},$$

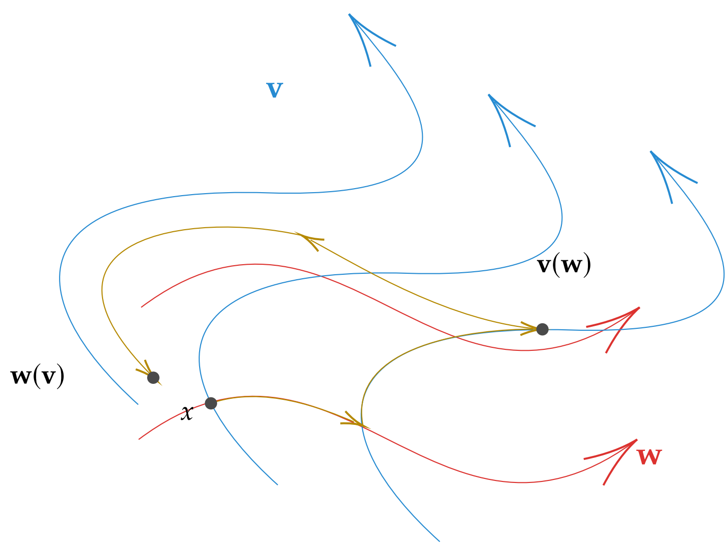
hence $\mathbf{v}$ right-invariant. ∎

Proposition 3.3. $\mathfrak{g}$ *is an algebra, equipped with the skew-symmetric, bilinear* Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, *where* $[\mathbf{v},\mathbf{w}] \in \mathfrak{g}$ *the unique vector field such that*

$$[\mathbf{v},\mathbf{w}](f) = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f))$$

for any $f : M \to \mathbb{R}$.

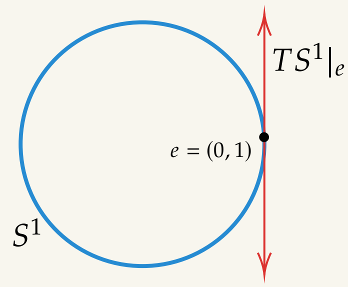
Proof If $\mathbf{v},\mathbf{w}$ right invariant, then $dR(c_1\mathbf{v} + c_2\mathbf{w}) = c_1 dR(\mathbf{v}) + c_2 dR(\mathbf{w}) = c_1\mathbf{v} + c_2\mathbf{w}$ for any scalars c_1, c_2. We won't discuss the Lie bracket too much to follow, so will omit proving the necessary properties. It is very important in a more rigorous classification of Lie algebras and provides extra structure to compare vector fields in our algebra in a meaningful way. It can essentially be seen as the commutator of the flows of the two vector fields $\mathbf{v}$, $\mathbf{w}$:



∎

Example 3.9. *Let $G := SO(2) \simeq S^1$ be the Lie group of rotations in the plane;*

$$\begin{aligned} TG|_e &= \{c_1 \cdot \partial_x|_{(x,y)=(1,0)} + c_2 \cdot \partial_y|_{(x,y)=(1,0)}\} \\ &= \{c_2 \partial_y\} = \text{Span}(\partial_y) \cong \mathbb{R}. \end{aligned}$$



Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting on some manifold M with operation ψ. One can show that the differential of ψ, $d\psi$, is a *homeomorphism* between $\mathfrak{g}$ and the set of vector fields on M, and thus $\mathfrak{g} \cong d\psi(g)$. This, in practice, means we can directly work on M, without reference to local coordinates in G. We will do this to follow without explicitly saying so.

3.2.3 Prolongation of Vector Fields and a Symmetry Condition

With all this setup, we need just one more concept before being able to state the most practically useful theorem so far.

Recall from the previous section the concept of the prolongation of a group action as how a given group acts on the derivatives of arbitrary functions. Analogously, we can define the prolongation of vector fields as the vector fields "extension" to prolonged space including said derivatives. More technically, suppose $\mathbf{v}$ lives on $M \subset X \times U$, our space of independent, dependent variables. Then, the nth prolongation of $\mathbf{v}$, which we denote

$$\mathbf{pr}^{(n)}\mathbf{v}$$

lives on $M^{(n)} = X \times U^{(n)}$; if $X \times U$ has local coordinates (x, u), $U^{(n)}$ has coordinates $(u; u^{(1)}; \ldots, u^{(n)})$; it follows again that $TM^{(n)}$ has coordinates $\{\partial_x; \partial_u; \partial_{u^{(1)}}; \ldots; \partial_{u^{(n)}}\}$.

For example, if $p = 2, q = 1$, we would have $U^{(2)}$ would be the space with coordinates $(u; u_x, u_y; u_{xx}, u_{xy}, u_{yy})$. One can consider these as formal placeholders for functions; a more rigorous construction involves the notion of *jet space* and concepts not necessary for the sake of our applications.

Now, just as we differentiated our group action to find our vector field, we can differentiate our prolonged group action to find the corresponding prolonged vector field. Recall that we had

$$\mathbf{pr}^{(1)}\theta \cdot (x, u, u_x) = (x\cos\theta - u\sin\theta, x\sin\theta + u\cos\theta, \frac{\sin\theta + u_x\cos\theta}{\cos\theta - u_x\sin\theta})$$

as the first prolongation of the action of SO(2). Differentiating with respect to θ at zero, we find

$$\mathbf{pr}^{(1)}\mathbf{v} = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x}.$$

More generally, we have the following formula.

Theorem 3.1 (General Prolongation Formula [Olv86]). *Let*

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x, u)\frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open set $M \subseteq X \times U$. Then,

$$\mathbf{pr}^{(n)}(\mathbf{v}) = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J} \phi_\alpha^J(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha}, \tag{2}$$

where

$$\phi_\alpha^J(x, u^{(n)}) := D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha,$$

where $u_i^\alpha := \frac{\partial u^\alpha}{\partial x^i}$, $u_{J,i}^\alpha := \frac{\partial u_J^\alpha}{\partial x^i}$.

This theorem tells us how to find $\mathbf{pr}^{(n)}(\mathbf{v})$ for any $\mathbf{v}$. We won't prove it here but it's worth noting the proof of this theorem also employs the idea of "representative function" as we used earlier to find the first prolongation of the group action of SO(2) previously, though of course on some general group action.

Despite the fairly scary formulae involved, remark that each term is always linear with respect to ξ^i, ϕ_α. In practice, we wish to determine these coefficients given some conditions, and these formulae give us a system of PDEs to solve which are thus linear, even if the original differential equation we're working with is not.

Example 3.10. *We verify that the first prolongation of* $\mathbf{v} = -u\partial_x + x\partial_u$ *is as determined above using theorem 3.1 directly to elucidate how to actually apply it. We have* $p = q = 1$, *and*

$$\xi \equiv \xi^1 = -u, \quad \phi \equiv \phi_1 = x.$$

We thus have

$$\mathbf{pr}^{(n)}(\mathbf{v}) = \sum_{j=0}^{n} \phi^j \cdot \partial_{u_j},$$

where $u_j := \frac{\partial^j u}{\partial x^j}$. *We find* ϕ^1:

$$\begin{aligned}\phi^1 &= D_x(\phi - \xi \cdot u_x) + \xi u_{xx} \\ &= D_x(x + uu_x) - uu_{xx} \\ &= 1 + u_x^2 + uu_{xx} - uu_{xx} = 1 + u_x^2,\end{aligned}$$

as previously found. For use later, we also find $\mathbf{pr}^{(2)}$; *we have*

$$\phi^2 = D_{xx}(x + uu_x) - uu_{xxx} = 3u_x u_{xx}.$$

With the concept of prolongation of vector fields, we can finally introduce the "infinitesimal symmetry criterion", the "fundamental theorem of continuous symmetries".

Theorem 3.2 (Infinitesimal Symmetry Criterion)**.** *Let* $\Delta[u] = 0$ *be a differential equation defined on* $M^{(n)} \subset X \times U^{(n)}$ *and* G *a local group of transformations on* M. *If*

$$\mathbf{pr}^{(n)}\mathbf{v}(\Delta[u]) = 0 \quad \textit{whenever} \quad \Delta[u] = 0$$

for every infinitesimal generator $\mathbf{v}$ *of* G, *then* G *is a symmetry group of* Δ.

Remark the similarity in the statement with our original definition of symmetry, definition 3.4. This definition, along with the one-to-one relationship between groups of transformations and vector fields from proposition 3.1 almost directly implies this theorem, technical statements about the domain of definition, etc, aside.

4 Applications

Having built up quite a lot of theory, we present now several applications of the theory presented above.

4.1 Symmetries of Differential Equations

This is our main application and directly applies theorem 3.1 and theorem 3.2. Generally, given a differential equation

$$\Delta[u] = 0$$

we wish to answer the question

What are all of the (continuous) symmetries of Δ?

More formally, we wish to find the *Lie algebra of symmetries of* Δ, $\mathfrak{g}$. We begin by defining an arbitrary vector field

$$\mathbf{v} = \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha},$$

and apply theorem 3.2 by solving

$$\mathbf{v}(\Delta) = 0 \impliedby \Delta = 0.$$

In practice, this gives a system of differential equations for the coefficients ξ^i, ϕ_α of the form theorem 3.1.

Once we find all of the independent infinitesimal generators $\mathbf{v}$ of $\mathfrak{g}$, we can exponentiate each to find the corresponding group operations.

Example 4.1 (Symmetry Groups of the Heat Equation)**.** *We present here a worked-out example of the previously described methodology for computing the symmetry groups of the one-dimensional heat equation, namely*

$$u_t = u_{xx}. \tag{3}$$

First, we have that $p = 2, q = 1$, so we take the general form of our vector field to be

$$\mathbf{v} := \xi(t,x,u)\cdot\partial_x + \tau(t,x,u)\cdot\partial_t + \phi(t,x,u)\cdot\partial_u.$$

More precisely, we take $\mathbf{v}$ *to be a general vector field on the space $X \times U \simeq \mathbb{R}^2 \times \mathbb{R}$; we aim to find the forms of the functions ξ, τ, ϕ such that the corresponding group action to* $\mathbf{v}$ *leaves eq. (3) invariant.*

We are working with highest order-derivative 2, so we need to take the second *prolongation, which we'll denote*

$$\mathbf{pr}^{(2)}\mathbf{v} = \mathbf{v} + \phi^{x} \cdot \partial_{u_x} + \phi^{t} \cdot \partial_{u_t} + \phi^{tt} \cdot \partial_{u_{tt}} + \phi^{xx} \cdot \partial_{u_{xx}} + \phi^{tx} \cdot \partial_{u_{tx}}, \tag{4}$$

where we find closed forms for each prolongation coefficient using eq. (2). Remark that each prolonged coefficient is a function of t, x, u, *and* $u^{(n)}$ *up to the degree of the basis element corresponding to the coefficient, (ie,* $\phi^{xx} = \phi^{xx}(t, x, u, u_t, u_x, u_{tt}, u_{xx}, u_{tx})$*) but we omit this for conciseness. Applying eq. (4) to eq. (3), we find (recalling that we can either interpret* $\mathbf{pr}^{(2)}\mathbf{v}$ *as a differential operator or consider, more geometrically, "flowing" our differential equation on the prolonged space and seeing the corresponding infinitesimal change)*

$$\mathbf{pr}^{(2)}\mathbf{v}(u_t - u_{xx}) = \phi^{t} - \phi^{xx} = 0,$$

ie our infinitesimal criterion for invariance is

$$\phi^{t} = \phi^{xx} \quad \text{whenever} \quad u_t = u_{xx}.$$

Hence, we compute ϕ^{t}, ϕ^{xx}*, set them equal to each other, and replace* u_{xx} *by* u_t *wherever it occurs to effectively "solve" in terms of our differential equation.*

Treating our result as a polynomial in derivatives of u*, we then require the coefficients of each distinct term to be identically zero, yielding a (linear!) system of PDEs defining the* τ, ξ, ϕ*. Solving these, we find*

$$\begin{aligned} \xi &= c_1 + c_4 x + 2c_5 t + 4c_6 xt \\ \tau &= c_2 + 2c_4 t + 4c_6 t^2 \\ \phi &= (c_3 - c_5 x - 2c_6 t - c_6 x^2)u + \alpha(x, t), \end{aligned}$$

where $c_1, \ldots, c_6$ *constants and* $\alpha(x, t)$ *any solution to the heat equation (ie* $\partial_{xx}\alpha = \partial_t \alpha$*). From here, setting each* $c_i = \delta_{ij}$ *for* $j = 1, \ldots, 6$ *in turn yields the corresponding* *(independent) vector fields*

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x \\
\mathbf{v}_2 &= \partial_t \\
\mathbf{v}_3 &= u\partial_u \\
\mathbf{v}_4 &= x\partial_x + 2t\partial_t \\
\mathbf{v}_5 &= 2t\partial_x - xu\partial_u \\
\mathbf{v}_6 &= 4tx\partial_x + 4t^2\partial_t - (x^2 + 2t)u\partial_u \\
\mathbf{v}_\alpha &= \alpha(x,t)\partial_u,
\end{aligned}$$

which span the Lie algebra $\mathfrak{g}$ *of symmetries of the heat equation. Remark that* $\mathbf{v}_\alpha$ *itself spans an infinite-dimensional submanifold of* $\mathfrak{g}$.

Now, for each of these vector fields, we can exponentiate to recover the corresponding Lie group. For instance, for $\mathbf{v}_1$, *we need to solve*

$$\frac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 1$$

so

$$\exp(\varepsilon \cdot \partial_x)(x,t,u) = (x + \varepsilon, t, u),$$

which is just the scaling group in x.

As another example, for $\mathbf{v}_5$, *we need to solve*

$$\frac{\mathrm{d}x}{\mathrm{d}\varepsilon} = 2t, \quad \frac{\mathrm{d}u}{\mathrm{d}\varepsilon} = -xu,$$

which gives

$$\exp(2\varepsilon t\partial_x - \varepsilon xu\partial_u)(x,t,u) = (x + 2\varepsilon t, t, u \cdot e^{(-\varepsilon x - \varepsilon^2 t)}).$$

Repeating such calculations for each group, we find the following:

G_i	$\exp(\varepsilon \cdot \mathbf{v}_i)(x,t,u)$	***Physical Interpretation***
1	$(x+\varepsilon, t, u)$	*Space Translation*
2	$(x, t+\varepsilon, u)$	*Time Translation*
3	$(x, t, e^{\varepsilon} u)$	*Linearity (Constants)*
4	$\left(e^{\varepsilon} x, e^{2\varepsilon} t, u\right)$	*Space/Time Scaling*
5	$\left(x+2\varepsilon t, t, u \cdot e^{-\varepsilon x - \varepsilon^2 t}\right)$	*Galilean Boost*
6	$\left(\frac{x}{1-4\varepsilon t}, \frac{t}{1-4\varepsilon x}, u\sqrt{1-4\varepsilon t}\, e^{\frac{-\varepsilon x^2}{1-4\varepsilon t}}\right)$	*(None)*
α	$(x, t, u+\varepsilon\alpha(x,t))$	*Linearity (Additivity)*

In particular, the symmetries G_3, G_α are simply a restatement of the principle of superposition of solutions to linear differential equations.

Given any solution, then, if we apply any of these groups, we will find another. For instance, any constant $u = c$ is a solution, hence, applying G_6 with $\varepsilon = 1$, we find

$$\tilde{u} = \frac{c}{\sqrt{1+4t}} e^{\frac{-\varepsilon x^2}{1+4t}},$$

a far less trivial solution. Indeed, if we apply time translation and set c appropriately, we can actually obtain the so-called "fundamental solution" to the heat equation.

Example 4.2 (Symmetry Groups of Korteweg-deVries (KDV))**.** *We consider*

$$\Delta = u_{xxx} + uu_x + u_t = 0,$$

a nonlinear, third-order PDE that arises in the study of shallow waves; in this case $p = 2, q = 1$. The infinitesimal criterion of invariance gives us

$$\phi^{xxx} + u_x\phi + u\phi^x + \phi^t = 0 \quad \textit{whenever } \Delta = 0.$$

The computation of ϕ^{xxx} using theorem 3.1 is quite tedious, but the steps from the previous example follow identically. We eventually find

$$\xi(t,x,u) = c_1 + c_2 t + c_3 x$$
$$\tau(t,x,u) = c_4 + 3c_3 t$$
$$\phi(t,x,u) = c_2 - 2c_3 u$$

for constants c_1, c_2, c_3, c_4, hence the symmetry Lie algebra of Δ is spanned by the vector fields

$$\begin{aligned}
\mathbf{v}_1 &= \partial_x \\
\mathbf{v}_2 &= \partial_t \\
\mathbf{v}_3 &= t\partial_x + \partial_u \\
\mathbf{v}_4 &= x\partial_x + 3t\partial_t - 2u\partial_u.
\end{aligned}$$

The first two correspond to translation in time and position respectively, the second to a form of Galilean boost, and the last to a form of scaling.

It's worth noting that the two examples above work out quite nicely; the infinitesimal criterion yielded quite simple PDEs to solve, and resulted in fairly reasonable symmetries. Unfortunately, this method does not always work quite so nicely, but it is quite powerful regardless.

Note that theorem 3.1 is actually quite amenable to computational implementation, see for instance [Olv23b] and corresponding software [Olv23a].

4.2 Differential Invariants of a Group Action

In the previous section, we addressed the issue of finding symmetry groups given a differential equation. Conversely, we may be interested in finding the general form of differential equations that admit a given symmetry group.

More precisely, if G is a local group acting on M, we say that a smooth function $\eta : M^{(n)} \to \mathbb{R}$ is a n-th order differential invariant of G if

$$\eta(\mathbf{pr}^{(n)} g \cdot (x, u) = \eta(x, u^{(n)}))$$

for all $(x, u^{(n)}) \in M^{(n)}$ and $g \in G$ such that the left-hand side is well-defined.

Example 4.3. *Let $G = SO(2)$, which has infinitesimal generator*

$$\mathbf{pr}^{(1)}\mathbf{v} = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x}.$$

For $\eta(x, u, u_x)$ to be a second-order invariant, we require then that

$$-u\eta_x + x\eta_u + (1 + u_x^2)\eta_{u_x} = 0,$$

which is just a differential equation for η, treating x, u, u_x as dependent variables, which we can solve with the method of characteristics

$$-\frac{\mathrm{d}x}{u} = \frac{\mathrm{d}u}{x} = \frac{\mathrm{d}u_x}{1 + u_x^2}.$$

Solving each, we find

$$r = \sqrt{x^2 + u^2}, \qquad \xi = \frac{xu_x - u}{uu_x + x},$$

which constitute a complete set of (functionally independent) first-order differential invariants of $SO(2)$; that is, any function F invariant under $SO(2)$ can be written as a function $\tilde{F}(r, \xi)$.

For instance, consider the differential equation

$$\Delta = (u - x)u_x + u + x = 0.$$

This equation has $SO(2)$ as a symmetry group:

$$\begin{aligned}
\mathbf{pr}^{(1)}\mathbf{v}(\Delta) &= -u\frac{\partial \Delta}{\partial x} + x\frac{\partial \Delta}{\partial u} + (1 + u_x^2)\frac{\partial \Delta}{\partial u_x} \\
&= -u(-u_x + 1) + x(u_x + 1) + (1 + u_x^2)(u - x) \\
&= u_x(u + x) - \cancel{u + x} + u_x^2(u - x) + \cancel{u - x} \\
&= u_x((u - x)u_x + u + x) = u_x \cdot \Delta,
\end{aligned}$$

which thus equals 0 whenever $\Delta = 0$.

Alternatively, multiplying $\Delta = 0$ by u_x and rearranging, we find

$$\begin{aligned}
\Delta = (u - x)u_x + u + x &= -(xu_x - u) + uu_x + x \\
&= (-x + uu_x)\left[\frac{xu_x - u}{x + uu_x} - 1\right] \\
&= (-x + uu_x)\left[\xi - 1\right].
\end{aligned}$$

Finally, remark that if change coordinates to polar, ie $x = r\cos\theta$, $u = r\sin\theta$, Δ becomes

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = r,$$

which is quite simple to solve. More generally, certain symmetry groups will provide an obvious change of coordinates in which the differential equation in question will be quite easily solvable.

4.3 Conservation Laws

In this section, we briefly discuss conservation laws, a topic naturally related to symmetries per Noether's Theorem, which we will present briefly.

Definition 4.1 (Conserved Quantity)**.** *Let $F(x, u^{(n)}) = 0$ be a differential equation. A* conserved quantity *or* conservation law *of F is a function $\varphi(x, u^{(n)})$ such that*

$$D_x \varphi(x, u^{(n)})|_{F=0} = 0,$$

that is, φ is constant on solutions to F.

In a PDE $F(t, x, u^{(n)})$, we more typically denote a conservation law of F as the divergence expression

$$\left(D_t \psi(x, u^{(n)}) + D_x \varphi(x, u^{(n)}) \right) |_{F=0} = 0,$$

to emphasize the time dependence. We call ψ and φ the density and flux of the PDE, respectively.

Conservation laws provide value information about the solutions to a given differential equation. Namely, suppose

$$\Delta(x, u^{(n)}) = 0$$

a differential equation with conserved quantity $\varphi(x, u^{(n)})$. Since

$$D_x \varphi(x, u^{(n)}) = 0 \quad \text{whenever} \quad \Delta = 0$$

we can conclude that solutions must lie on the level set

$$\{(x, u) : \phi(x, u^{(x)}) = c, c \in \mathbb{R}\}.$$

We can then, in a sense, restrict our space of variables of consideration and effectively reduce the number of degrees of freedom in our system.

To elucidate this, we work through the classical example of the Hamiltonian system

$$\Delta = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} - \begin{pmatrix} -y \\ x \end{pmatrix} = 0 \tag{5}$$

where $(x, y) = (x^1, \ldots, x^p, y^1, \ldots, y^p) \in \mathbb{R}^{2p}$ for some positive integer p. Let

$$H(x, y) = \frac{x^2}{2} + \frac{y^2}{2},$$

and notice that

$$\text{Div}(H(x, y)) = x \cdot \dot{x} + y \cdot \dot{y},$$

which vanishes identically on solutions, and is hence a conserved quantity of $\Delta = 0$.

Consider in particular the case $p = 1$, so we are working in the plane $\mathbb{R}^2$. Then, remark that $H(x, y) = \frac{c^2}{2}$ defines a circle in the plane of radius c for various constants c. Hence, solutions to Δ must trace out circles in the plane (in *phase space*).

Definition 4.2 (Characteristic of a Conservation Law)**.** *Let φ be the conservation law of a differential equation $\Delta[u] = 0$. The* characteristic *of φ is a (tuple of) function Q such that*

$$D_x\varphi = Q \cdot \Delta.$$

It turns out that every conservation law (under appropriate assumptions on the domain, etc) can be written in such a way, though such a Q is in general, though, not unique.

A characteristic of eq. (5), for instance, is the vector $Q = (x, y)^T$, as one can readily verify.

Before we can finally state Noether's theorem, we briefly review/introduce the concept of the calculus of variations.

Definition 4.3 (Calculus of Variations)**.** *Let $X = \mathbb{R}^p, U = \mathbb{R}^q$ and let $\Omega \subset X \times U$ be an open, connected subset with smooth boundary. Let $L[u]$ be a smooth function; a* variational problem *asks to find the extrema of the functional*

$$\mathcal{L}[u] = \int_\Omega L[u] \, \mathrm{d}x\,, \tag{6}$$

that is, functions $u = f(x)$ that minimize/extremize $\mathcal{L}$. We call L the Lagrangian *of* *the problem.*

Example 4.4 (Shortest Path)**.** *The classical first example of a variational question asks one to find the shortest path between two points x_1, x_2 in the plane $X \times U = \mathbb{R}^2$. Here, the Lagrangian would thus be the arc length of a planar curve, namely $L[u] = \sqrt{1 + u_x^2}$, yielding the variational problem*

$$\mathcal{L} = \int_{x_1}^{x_2} \sqrt{1 + u_x^2}\, \mathrm{d}x\,.$$

We will affirm the intuitive answer of a straight line shortly.

How does one actually *find* the extrema of such a problem? We won't dive too much into the theory but will introduce some basic necessary concepts.

Definition 4.4 (Euler Operator)**.** *For $1 \leq \alpha \leq q$, the α-th* Euler Operator *is defined*

$$\mathsf{E}_\alpha := \sum_J (-D)_J \frac{\partial}{\partial u_J^\alpha}, \tag{7}$$

where $(-D)_J := (-1)^k D_J$, using the multi-index $J = (j_1, \ldots, j_k)$ for $1 \leq j_\kappa \leq p$ and $k \geq 0$.

We state E in the case $p = q = 1$ with coordinate (x, u) to hopefully elucidate the definition:

$$\mathsf{E} = \frac{\partial}{\partial u} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} - D_x^4 \frac{\partial}{\partial u_{xxx}} + \cdots\,.$$

Theorem 4.1. *Let $u = f(x)$ be an extremal of eq. (6). Then, f is a solution to the system of differential equations given by $E_\alpha(L) = 0, \alpha = 1, \ldots, q$.*

Proof See [Olv86], Proposition 4.2. The proof follows quite naturally (indeed, the Euler operator naturally arises when attempting to find extrema solutions) from the related variational derivative. ■

We call the equations given by

$$E_\alpha(L) = 0$$

the *Euler-Lagrange* or *EL* equations of L.

Example 4.5. *Returning to example 4.4, we find*

$$E(L) = -D_x \frac{u_x}{\sqrt{1+u_x^2}} = -\frac{u_{xx}}{(1+u_x^2)^{3/2}} = 0.$$

This is just a second-order differential equation, with solution $u = mx + b$ (with m, b depending on x_1, x_2), that is, a straight line (alternatively, one can notice that this is the equation for the curvature of an arbitrary curve $u = f(x)$, hence an extremum has 0 curvature ie is a straight line).

Many physical problems arise as the EL equations of a particular Lagrangian. For instance, the wave equation in two spatial dimensions $u_{xx} + u_{yy} = u_{tt}$ arises as the EL of

$$\iiint_\Omega \{\frac{u_x^2}{2} + \frac{u_y^2}{2} - \frac{u_t^2}{2}\} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t \, .$$

We say that a differential equation that arises as the EL of some Lagrangian has "variational structure".

Theorem 4.2 (Variational Symmetry Group)**.** *We say G is a* variational symmetry group *if it is a symmetry group of the variational problem eq.* (6)*. This holds if and only if*

$$\mathbf{pr}^{(n)}\mathbf{v}(L) + L \cdot \mathrm{Div}\xi = 0$$

for all $x, u^{(n)} \in \Omega^{(n)}$ and every infinitesimal generator $\mathbf{v}$ of G, where $\xi = (\xi^1, \ldots, \xi^p)$.

This is nearly identical to our earlier theorem 3.2, with the added divergence term.

Example 4.6 (Symmetry of example 4.4)**.** *Recall the variational problem related to finding the shortest curve between two points. Intuitively, one would expect that a rotation of the curve should not change the length; indeed, we have the infinitesimal generator of $SO(2)$*

$$\mathbf{pr}^{(1)}\mathbf{v} = -u\partial_x + x\partial_u + (1+u_x^2)\partial_{u_x},$$

and

$$\begin{aligned} \mathbf{pr}^{(1)}\mathbf{v}(L) + L \cdot \mathrm{Div}\xi &= \mathbf{pr}^{(1)}\mathbf{v}(\sqrt{1+u_x^2}) + \sqrt{1+u_x^2} \cdot \mathrm{Div}(-u, x) \\ &= \frac{(1+u_x^2) \cdot u_x}{\sqrt{1+u_x^2}} - u_x\sqrt{1+u_x^2} = 0, \end{aligned}$$

as expected.

We finally have the tools and terminology to present Noether's Theorem, which establishes a powerful connection between variational symmetries and conservation laws.

Theorem 4.3 (Noether's Theorem). *Let G be a (local) symmetry group of the variational problem $\mathcal{L} = \int_\Omega L[u]\,\mathrm{d}x$, with infinitesimal generator*

$$\mathbf{v} = \xi^i(x,u)\partial_{x^i} + \phi_\alpha(x,u)\partial_{u^\alpha}.$$

Put

$$Q_\alpha(x,u) = \phi_\alpha - \sum_{i=1}^{p} \xi^i(x,u)\frac{\partial u^\alpha}{\partial x^i}$$

for each $\alpha = 1, \dots, q$, called the characteristic *of* $\mathbf{v}$*. Then, the q-tuple $Q = (Q^1, \dots, Q^q)$ is also the characteristic of a conservation law for the Euler-Lagrange equations $\mathsf{E}(L) = 0$.*

Proof Let $\mathbf{v}_Q = Q_\alpha \partial_{u^\alpha}$ as in the theorem.[13] By theorem 4.2, we have

$$\mathbf{pr}^{(n)}\mathbf{v}(L) + L \cdot \mathrm{Div}(\xi) = 0,$$

and by a simple computation from theorem 3.1, one finds that

$$\mathbf{pr}^{(n)}\mathbf{v}_Q(L) = \mathbf{pr}^{(n)}\mathbf{v}(L) - \sum_{i=1}^{p} \xi^i D_i L.$$

Combining these two expressions we find

$$0 = \mathbf{pr}^{(n)}\mathbf{v}_Q(L) + \underbrace{\sum_{i=1}^{p} \xi^i D_i L + L \sum_{i=1}^{p} D_i \xi^i}_{\circledast} \tag{8}$$

Remark that the expression $\circledast$ is, by the product rule, simply the total divergence of the p-tuple $L\xi := (L\xi^1, \dots, L\xi^p)$.

We can now look closer at the expression $\mathbf{pr}^{(n)}\mathbf{v}_Q(L)$. Expanding according to theorem 3.1, we have

$$\mathbf{pr}^{(n)}\mathbf{v}_Q(L) = \sum_{\alpha,J} (D_J Q_\alpha) \cdot \frac{\partial L}{\partial u^\alpha_J}.$$

[13] We call such a vector field the "evolutionary representative" of $\mathbf{v}$.

From here, we can "integrate by parts" in a sense; namely, remark that by the product rule, we have that

$$D_J(Q_\alpha \cdot \frac{\partial L}{\partial u_J^\alpha}) = (D_J Q_\alpha) \cdot D_{J-1}\frac{\partial L}{\partial u_J^\alpha} + D_{J-1}Q_\alpha \cdot (D_J \frac{\partial L}{\partial u_J^\alpha}).$$

Continuing in this fashion somewhat inductively, we can rewrite

$$\mathbf{pr}^{(n)}\mathbf{v}_Q(L) = \sum_{\alpha,J} Q_\alpha \cdot \underbrace{(-D)_J \frac{\partial L}{\partial u_J^\alpha}}_{\ominus} + \mathrm{Div} A,$$

where A some p-tuple depending on Q, L and their derivatives. Remark that $\ominus$ is just $E_\alpha(L)$. Substituting back into eq. (8), we have thus

$$\sum_{\alpha,J} Q_\alpha E_\alpha(L) + \mathrm{Div} A = \mathrm{Div}(L\xi)$$
$$\implies Q \cdot E(L) = \mathrm{Div}(P),$$

where $P := L\xi - A$, $Q = (Q_1, \ldots, Q_\alpha)$, $E(L) = (E_1(L), \ldots, E_q(L))$. But this final expression is precisely the characteristic form of a conservation law for $E(L)$, hence completing the proof. ∎

Hence, in short, the proof of Noether's theorem reduces to integration by parts thanks to theorem 3.1, theorem 3.2, and theorem 4.2.

Example 4.7 (Kepler's Problem)**.** *In keeping with the "physics flavor" of Noether's Theorem, we consider Kepler's Problem, in which two masses in $\mathbb{R}^3$ experience a central interaction force. We fix our origin at one mass and consider the movement of a second mass m relative to this origin. Let $\mathbf{x} = (x, y, z)(t)$ denote the position of the mass m in space.*

The kinetic energy of this situation is given by the function

$$K = \frac{1}{2} m |\dot{\mathbf{x}}|^2,$$

and the potential energy we define

$$U = \frac{k}{|\mathbf{x}|},$$

ie an inverse-square force, where k a non-zero constant and $|\mathbf{x}| = \sqrt{x^2+y^2+z^2}$ Informally, one should expect the path of the mass to want to minimize the amount of energy it "uses", hence we are dealing with the variational problem with Lagrangian $L = K - U = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - \frac{k}{\sqrt{x^2+y^2+z^2}}$, namely

$$\mathcal{L} = \int_{-\infty}^{\infty} K - U \,\mathrm{d}\mathbf{x}\,.$$

Solving $\mathsf{E}(L) = (\mathsf{E}_x(L), \mathsf{E}_y(L), \mathsf{E}_z(L))$ gives the equations of motion of the mass; for instance, for $\alpha = x$, we have

$$\mathsf{E}_x(L) = \frac{kx}{|\mathbf{x}|^3} - m\ddot{x} = 0.$$

We can summarize succinctly

$$\mathsf{E}(L) = \frac{k\cdot\mathbf{x}}{|\mathbf{x}|^3} - m\ddot{\mathbf{x}} = 0, \tag{9}$$

which matches (thankfully) with the equations of motion one would find using Newton's Laws.

Let $\mathbf{v} = \tau\partial_t + \xi\partial_x + \gamma\partial_y + \zeta\partial_z$ be an arbitrary vector field with each coefficient depending on t, x, y, z. The criterion for $\mathbf{v}$ to be a variational symmetry from theorem 4.2 thus reduces to

$$0 = \mathbf{pr}^{(1)}\mathbf{v}(L) + L\cdot \mathrm{Div}\tau = \mathbf{pr}^{(1)}\mathbf{v}(L) + L\cdot(\tau_t + \dot{x}\tau_x + \dot{y}\tau_y + \dot{z}\tau_z). \tag{10}$$

Consider the vector field $\mathbf{v}_t = \partial_t$ corresponding to translation in time, ie with $\tau = 1, \xi = \gamma = \zeta = 0$. According to theorem 3.1, $\mathbf{pr}^{(1)}\mathbf{v}_t = \mathbf{v}_t$. L has no explicit time dependence, and $\tau_t = \tau_x = \tau_y = \tau_z = 0$, and hence it's clear that the symmetry condition is satisfied. This vector field has characteristic

$$Q = (Q_x, Q_y, Q_z) = (-\dot{x}, -\dot{y}, -\dot{z}).$$

Noether's Theorem promises that this is also the characteristic of a conservation law of *the equations of motion; indeed, we have*

$$\begin{aligned}
Q \cdot \mathsf{E}(L) &= m\dot{x}\ddot{x} - \frac{kx\dot{x}}{|\mathbf{x}|^3} + \text{(same with } y, z\text{)} \\
&= D_t\left[\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) + \frac{k}{\sqrt{x^2+y^2+z^2}}\right] \\
&= D_t(K+U),
\end{aligned}$$

that is, $\mathbf{v}_t$ *corresponds to the conservation of energy in the system.*

Let us consider now the family of vector fields corresponding to rotations about fixed axes. In particular, let $\mathbf{v}_z = -y\partial_x + x\partial_y$ *be the infinitesimal generator of rotations about the z-axis, with prolongation* $\mathbf{pr}^{(1)}\mathbf{v}_z = \mathbf{v}_z - \dot{y}\partial_{\dot{x}} + \dot{x}\partial_{\dot{y}}$*. Plugging into eq. (10), we find*

$$\frac{-kx\cancel{y}}{\cancel{|\mathbf{x}|^3}} + \frac{ky\cancel{x}}{\cancel{|\mathbf{x}|^3}} + m\left[\cancel{-\dot{x}\dot{y}} + \cancel{\dot{y}\dot{x}}\right] = 0,$$

hence $\mathbf{v}_z$ *indeed generates a variational symmetry group. It has characteristic*

$$P = (-y, x, 0),$$

and again appealing to Noether's Theorem, we find

$$\begin{aligned}
P \cdot \mathsf{E}(L) &= \frac{-kx\cancel{y}}{\cancel{|\mathbf{x}|^3}} + m\dot{y}\ddot{x} + \frac{kx\cancel{y}}{\cancel{|\mathbf{x}|^3}} - m\dot{x}\ddot{y} \\
&= D_t(my\dot{x} - mx\dot{y}),
\end{aligned}$$

corresponding to the conservation of angular momentum about the z-axis.

One naturally may ask when precisely a given differential equation actually arises from a variational problem, for, if one does, Noether's theorem can be applied. This question is called the "inverse problem to the calculus of variations", and requires quite a lot of background to solve, which we won't do here (see [Olv86], chapter 5.4 for a simplified version or [And89] for a much more rigorous treatment). In short, a differential equation $P[u]$ is the EL for some Lagrangian if and only if its "*Fréchet derivative*" is self-adjoint.

4.4 Generalized Symmetries

Recall that the types of symmetries we have considered all have infinitesimal generators of the general form

$$\mathbf{v} = \xi^i(x,u)\partial_{x^i} + \phi^\alpha(x,u)\partial_{u^\alpha}, \tag{11}$$

namely, the coefficient functions only depended on the variables x, u. In practice, we call such symmetries "geometric" or "point" symmetries, since they literally transform the geometry of the underlying space, in turn transforming the differential equation defined on the space. More generally, we can consider symmetries of the form

$$\mathbf{v} = \xi^i[u]\partial_{x^i} + \phi^\alpha[u]\partial_{u^\alpha},$$

called *generalized symmetries*; that is, the coefficient functions can rely on x, u, and derivatives of u. While we won't get into the theory of such vector fields, we remark the following.

- Is this even well-defined? Our original construction of infinitesimal generators relied on defining vector fields on the underlying space $X \times U$, then prolonging. This had a clear, geometric interpretation, as we particularly can see in applications. To formally define such generalized symmetries, one needs to be more explicit than we have been with regards to the jet space, and thus define these vector fields over the space $X \times U^{(n)}$.
- In the same vein as the previous point, another issue that may arise is in the exponentiation phase of computations; do these generalized vector fields necessarily exponentiate to some well-defined group?
- Simply "allowing" generalized symmetries makes working with point symmetries often more convenient, as then we can always rewrite our symmetries in "evolutionary form", where all ξ^i are identically zero. Namely, given a point symmetry eq. (11), and taking

  $$Q_\alpha := \phi^\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha$$

  for each $\alpha = 1, \ldots, q$, then we have a corresponding *evolutionary representative*

  $$\mathbf{v}_Q = Q_\alpha \partial_{u^\alpha},$$

(noting that each Q_α may now depend explicitly on derivatives of u and is hence generalized) and one can show that a function is invariant under $\mathbf{v}$ if and only if it is invariant under $\mathbf{v}_Q$, hence the two are essentially equivalent.

- Theorem 3.2 extends naturally to generalized symmetries; however, one must fix before computation the degree of derivatives of u that the coefficient functions depend on.

One of the more remarkable results one has upon consideration of generalized symmetries is a natural extension of Noether's Theorem; indeed, under appropriate assumptions of domain, non-degeneracy, etc, the theorem becomes a *one-to-one correspondence* between conservation laws[14] and symmetries. Given the relative algebraic ease that computing symmetry groups takes compared to conservation laws, this is quite a powerful result. See [Noe18], [Olv86].

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Mathematica code used in this project can be found here.

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