FRACTALS AND THEIR DIMENSION

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ABSTRACT. Building on Benoit Mandelbrot's *The Fractal Geometry of Nature*, this expository paper aims to formally define the geometric forms known as fractals. By first defining topological dimension, then detouring through measure theory and dimension theory, this paper will enrich the reader's understanding about the complexity of fractals, while providing basic tools for discussing and quantifying different fractals. Its principal aim is to provide an accessible account of fractals' unique geometric properties.

CONTENTS

1. Overview and Aims

Departing from Benoit Mandelbrot's *The Fractal Geometry of Nature* [3], this paper will describe and motivate the basic concepts needed for an appreciation of the complexity of fractals and their dimension. In particular, we will draw from topology, measure theory, and dimension theory to grasp fractals and their unique properties.

For Mandelbrot, fractals are a response to the perception that geometry is "cold" and "dry" (1). They are an attempt to challenge this notion by appealing to the irregularity and complexity of the geometry of nature. Thus, fractals are a way of understanding geometry's rich possibilities beyond its sterile and purely abstract applications. Examples of complex geometry observable in nature can be seen below in Figure 1.

Among many of Mandelbrot's neologisms, the name "fractal" is rooted in Latin. The word is based on the adjective *fractus* and the corresponding verb *fragere*, which mean to break or to create irregular fragments (Mandelbrot, 4). However, fractals are more than just complex and irregular shapes. Formally defined, they are sets whose **Hausdorff Dimension** D[1] strictly exceeds their Topological Dimension D_T, understood by **Lebesgue Covering Dimension**[2]. Hence, we have our first definition:

> **Definition 1.1.** A set is called a *fractal* if its Hausdorff Dimension D strictly exceeds its topological dimension D_T. Hence, for all fractals, $D > D_T$.

Because it defines what it means to be a fractal, we can refer to Hausdorff Dimension D as a **fractal dimension**. The following sections of this paper are devoted to providing a rich and accessible explanation of fractal and topological dimension. In the end, my hope is that this paper will allow for a richer appreciation of fractals and their complexity, while demonstrating the mathematical equipment required to deal with these forms.

2. Foundational Concepts

This section will briefly review two concepts: first, metric spaces from analysis and geometry, and second, outer measures from measure theory. Despite their origin in different areas of mathematics, both concepts serve as important bases for understanding how sets and subsets can be "measured" or quantified in different

[1] See Definition 5.1
[2] See Definition 1.0

FIGURE 1. Left: complex geometric forms observed in the gills of a mushroom; Right: frozen water forms a unique pattern near the Niagara Falls in Ontario, Canada [6]

ways. Both definitions, and the notion of measurement in the first place, are useful for understanding the work undertaken later in this paper.

2.1. METRIC SPACES

We devote the first part of this paper to a brief summary of metric spaces, an important concept within analysis and geometry.

Definition 2.1. Given a set X, a function $d : X \times X \to \mathbb{R}$ is a *metric* on X if for all $x, y \in X$:

a) $d(x, y) \geqslant 0$ with equality if and only if $x = y$;

b) $d(x, y) = d(y, x)$; and

c) d satisfies the triangle inequality, ie for all $z \in X$, $d(x, z) \leqslant d(x, y) + d(y, z)$

If d is a metric over X, we write (X, d) to denote the *metric space*: a the set equipped with its metric.

Simply put, a metric space is a set X equipped with a "measure" d satisfying the above axioms. With our measure, it is possible to have a notion of "distance" in a set. This concept is invaluable for understanding Hausdorff Dimension, discussed in Section 5. While a simple example is given below, various functions can serve as our measure, including the regular Euclidean distance function. Proving that Euclidean Distance is a metric is fairly easy task and is left to the reader. For the below observation, we notice the one-dimensional formula for Euclidean distance is a metric:

Observation 2.1. We demonstrate a brief observation proving that $d(x, y) := |x - y|$ is a metric over $\mathbb{R}$, hence proving that $(\mathbb{R}, d)$ is a metric space.

Proof. For all $x, y, z \in \mathbb{R}$:

a) $|x - y| \geqslant 0$ and $|x - y| = 0$ if and only if $x = y$;

b) $|x - y| = |y - x|$;

c) By the triangle inequality: $|x - z| \leqslant |x - y| + |y - z|$

Hence, $d(x, y) := |x - y|$ is a metric over $\mathbb{R}$, and $(\mathbb{R}, d)$ is a metric space. □

2.2. Outer Measure

Another useful notion of measurement, this time from measure theory, is that of ‘outer measure’, defined in this section.

Definition 2.2. An *outer measure* ∂ over $\mathbb{R}^n$ is a function assigning a non-negative real number to every subset of $\mathbb{R}^n$ with the following properties:

a) $\partial(\varnothing) = 0$;

b) if $A \subseteq B$ then $\partial(A) \leqslant \partial(B)$; and

c) if $\{A_j\}_{j=1}^{\infty}$ are subsets of X, then $\partial(\bigcup_{j=1}^{\infty} A_j) \leqslant \sum_{j=1}^{\infty} \partial(A_j)$

Essentially, an outer measure satisfies the conditions of a measurement we would in everyday life. The first property simply states that the measure of nothing is zero. For the second, we can think of having a room and measuring a table within it. The measurement of this table should be less than or equal to that of the room. There is no way the table, fully contained within the room, could exceed the room in length. The final property holds that if we have multiple pieces of furniture in the room, the measure of the furniture taken together must not exceed the summed measures of each piece of furniture. This property with furniture is true in general, accounting for situations where furniture is stacked or overlapping.

Further discussion of outer measures, specifically the related concept of a 'metric outer measure' is possible. Such a measure satisfies the properties of an outer measure, while satisfying the following additional property:

Definition 2.3. A *metric outer measure* ∂ over a metric space (X, d) is an outer measure ∂ satisfying the property that for all $A, B \in X$:

$$\partial(A \cup B) = \partial(A) + \partial(B)$$

While it can be shown that Hausdorff measure, defined later in Section 5, is a Metric Outer Measure, the task of proving this property takes us afield in our expository discussion of fractals. For that reason, further elaboration on the properties of Hausdorff measure as a metric outer measure are omitted. However, the reader is encouraged to explore this result, found in Theorem 2.2 in Robinson's textbook [5].

3. Topological Dimension

In this section, we explore the notion of topological dimension, defined formally as Lebesgue Covering Dimension. Informally conceived of as the number of coordinates needed to describe a set, topological dimension is always an integer value. The reason for this will become clear in the discussion below. For a set $X \subseteq \mathbb{R}^{D_T}$, X has topological dimension D_T. Generally, then, individual points have dimension $D_T = 0$, lines have dimension $D_T = 1$, a square has dimension $D_T = 2$, cubes have dimension $D_T = 3$, and so on. Many of the definitions in this section are lifted from James C. Robinson's *Dimensions, Embeddings, and Attractors* [5].

3.1. Coverings and Refinements

We begin by identifying some of the necessary concepts for understanding how the Lebesgue Covering Dimension is defined:

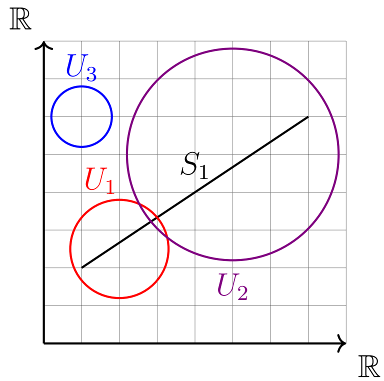
Definition 3.1. Let $S \subseteq R^n$ be a set. A *covering* of S is a collection $\mathbb{U} = \{U_\alpha\}_{\alpha \in A}$ of sets such that:

$$\bigcup_{\alpha \in A} U_\alpha \supseteq S$$

A covering of S consisting of open sets U_α is called an *open cover*.

Simply put, a cover $\mathbb{U}$ of a set S is a collection of sets whose union contains the set S. We apply this definition in the example below:

Example 3.1



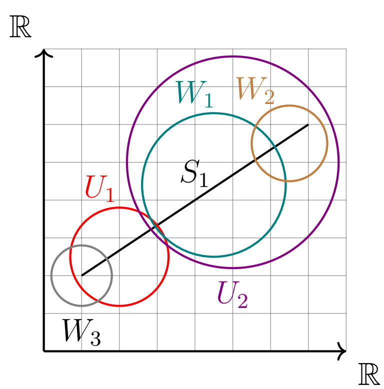
Here, S_1 is a subset of $\mathbb{R}^2$. The collection of sets $\mathbb{U} := \{U_1, U_2, U_3\}$ is a cover of S_1, since $\bigcup_{i=1}^{3} U_i \supseteq S_1$. Notice that $\mathbb{U}_1 := \{U_1, U_2\}$ is also a cover of S_1. However, $\mathbb{U}_2 := \{U_1, U_3\}$ is not a cover of S_1.

Next, we define the refinement of a cover:

Definition 3.2. A *refinement* of a covering $\mathbb{U}$ is a collection $\mathbb{W} = \{W_\lambda\}_{\lambda \in G}$ of sets such that each W_λ is a subset of some U_α.

This definition indicates that each set contained in a collection $\mathbb{W}$ of sets is a refinement of the cover $\mathbb{U}$ only if each set $W_\lambda \in \mathbb{W}$ is contained in at least one set $U_\alpha \in \mathbb{U}$.

Example 3.2



In this figure U_1, U_2, and S_1 are as before. Notice that W_1 and W_2 are completely contained in U_2 and $U_1 \cup W_1 \cup W_2$ is a covering of S_1. Hence, $\mathbb{W} := \{U_1, W_1, W_2\}$ is a refinement of the cover $\mathbb{U}_1$ from the previous example. However, notice that W_3 is not completely contained in either of U_1 or U_2. Hence, any covering of S_1 containing W_3 is not a refinement of $\mathbb{U}_1$.

Finally, we define the order or ply of a covering:

Definition 3.3. The *order* or *ply* of a covering (if finite) is the largest integer n such that there are $n+1$ sets in the covering with a non-empty intersection.

With these definitions understood, we can now properly define Lebesgue covering dimension, which is a formally defined topological dimension. Note that Lebesgue covering dimension is one of many possible topological dimensions. Nevertheless, it is rigorously defined, making it the formal basis for topological dimension for the purposes of our investigation.

3.2. Defining the Lebesgue Covering Dimension

Informally, Lebesgue covering dimension is the smallest number n such that for any open cover $\mathbb{U}$ of a set X, there is a refinement of $\mathbb{W}$ such that every $x \in X$ lies in the intersection of no more than $n+1$ sets in the refinement. A set without a Lebesgue Covering Dimension n is said to be infinite-dimensional.

Example 3.1. From this definition, we can demonstrate that any point $x \in \mathbb{R}$ has the expected dimension $n = 0$:

Proof. Let $\mathbb{U}$ be an arbitrary covering of the set $X := \{x\} \subseteq \mathbb{R}$. We know $\mathbb{W} = \{\{x\}\}$ is a refinement of $\mathbb{U}$. This is because $W = \{\{x\}\}$ is a refinement of $\mathbb{U}$ since if $\mathbb{U}$ covers $\mathbb{X}$, it must contain a set containing x, and so $\{x\} \in W$ is contained in this set. Thus, $\mathbb{U}$ can be refined to $\mathbb{W} := \{x\}$.

Since $\mathbb{W}$ is the singleton $\{x\}$, it cannot be further refined. Consequently, we found that the smallest n such that $X = \{x\}$ lies in the intersection of no more than $n+1$ sets. Namely, X is contained in no more than than 1 set. Hence,

$$n + 1 = 1 \implies n = 0$$

Therefore, the point $x \in \mathbb{R}$ has dimension $n = 0$. □

Formalizing the above definition slightly, we get the following proper definition of Lebesgue covering dimension, from Robinson [5]:

Definition 3.4. Let (X, d) be a metric space. A set $A \subseteq X$ has *Lebesgue covering dimension* $D_T = n$ if every covering $\mathbb{U}$ of X has a refinement of order $\leqslant n$. A set has $D_T = n$ if $D_T \leqslant n$ but it is not true that $D_T \leqslant n - 1$.

From this definition, we can create a function $D_T : X \to \mathbb{Z}$ defined by $D_T(X) =$ the dimension of X. This function takes sets as inputs and their dimension as outputs. Now, we have a definition of topological dimension, as understood by Lebesgue covering dimension. In the following sections, we will motivate and introduce the Hausdorff/Fractal Dimension in contrast to this definition.

4. Motivating Hausdorff Dimension from Lebesgue Covering Dimension

This section will be devoted to a discussion motivating the need for a "fractal dimension" to measure and understand fractal sets. With Lebesgue covering dimension, we have come to appreciate how regular topological dimension is defined. This proves useful in many situations we are used to dealing with in Euclidean geometry, such as points, lines, planes etc. However, our understanding of dimension reaches an impasse, or a point of impossibility, when dealing with fractals and natural phenomena. Consequently, we must move beyond our conventional methods to grapple with these special cases.

Mandelbrot [3] discusses how topology, through the concept of homeomorphism (the existence of a continuous bijection between topological spaces with a continuous inverse) excessively reduces the complexity of coastlines (17). Equivalent to isomorphism in the context of topology, two homeomorphic spaces are considered topologically the same. For the sake of concision, we will avoid further discussion of these ideas in this work. What should be drawn from this discussion, is that topologically speaking, a circle, with a circumference $C = 1$, is "the same" as an island whose coastline measures 1 unit. Hence, complex coastlines become reducible to simple figures, and archipelagos into groups of circles.

FIGURE 2. Measuring the coastline of Britain using increasingly small measurement "yardsticks" from left to right [1]

4.1. CAPTURING THE COMPLEXITY OF COASTLINES

Going beyond topology, Mandelbrot invites us to further consider coastlines when asking "how long is the coast of Britain?" (25). This length must at least be equal to the distance measured in a straight line from the beginning to the end of the coast. Indeed, a typical coastline is irregular, winding, and of a different length than the a straight line drawn from end-to-end on the coastline. Mandelbrot goes on to discuss various ways in which coastlines can be measured, but it proves to be a difficult task.

For example, we know that a line, having topological dimension $D_T = 1$ and length 1 can be measured with yardsticks of a given length ϵ, say $\epsilon = \frac{1}{4}$. Then, dividing my line into segments of length ϵ, I have four segments, corresponding to a length $L(\epsilon) = 4 * \frac{1}{4} = 1$. $L(\epsilon)$, which we can call the *true length*, settles to a single value once a certain threshold of precision is met by my choice of ϵ. The function itself is simply constructed as follows: $L(\epsilon) = N \cdot \epsilon$, where N is the number of lengths needed to 'cover' or measure the line/coastline in question. This function only has natural number outputs, corresponding to the number of lengths ϵ. For example, measuring my unit line with segments of length $\epsilon = 2$, $L(\epsilon) = 0$, since we use 0 ϵ-long segments in measuring the line. However, after a certain point of reducing ϵ, the measurement settles to a well-defined value.

However, this story is more complicated when dealing with real coastlines. As we see in Figure 2, shrinking my yardstick length ϵ increases the amount of detail captured in a given measurement. Consequently, it is not clear whether or not $L(\epsilon)$ will converge to a true length as ϵ shrinks. Indeed, Mandelbrot finds that many natural curves, like coastlines, are *non-rectifiable*. This means that approximate measurements of the curve

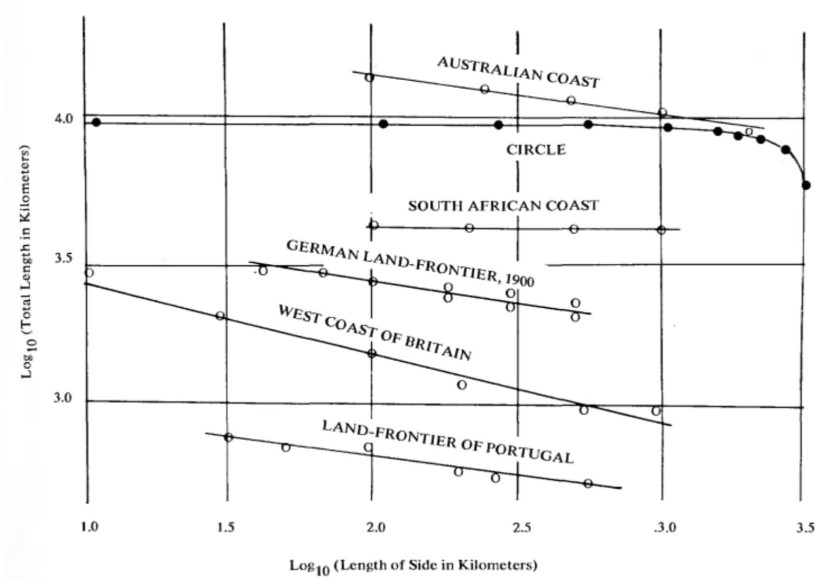


FIGURE 3. Empirical Measurements of Coastlines by Richardson Fry, lifted from [3]

increase as ϵ shrinks, making the value of the measurement diverge as ϵ decreases. Conversely, a *rectifiable* curve, such as the circumference of a circle, quickly converges to its true value. We see this expressed in empirical measurements by Lewis F. Richardson in Figure 3.

Figure 3, from Mandelbrot's textbook (33), plots points along the horizontal axis by the length ϵ of the faces of polygons whose lengths were used to approximate the true length of various coastlines. Meanwhile, the vertical axis marks the log length of the coastline with each measurement. We see that the circumference of a circle quickly converges toward a value as ϵ decreases. However, various coastlines do not converge to a stable value, and appear to diverge.

Hence, techniques for measuring one-dimensional Euclidean curves seem to fail when attempting to gain accurate measurements of naturally occurring curves, like coastlines. This same problem occurs when attempting to measure fractal curves, which have "infinite detail". The issue here lies in the dimension of the measuring tools we are using. Throughout the processes of measurement discussed by Mandelbrot, one-dimensional yard sticks or lengths ϵ are used to determine a true length. However, by changing the dimension of our measuring tool, accurate measurements suddenly become possible.

4.2. The Need for Fractal Dimension

Ultimately, the accurate measurement of fractal curves boils down to using the appropriate measurement tools. For instance, it is clear from early results in Real Analysis that lines cannot meaningfully be measured with points. To exemplify this, we can consider the infinitude of points needed to build a line. For instance, the density of both the rational numbers, $\mathbb{Q}$, and the irrational numbers, $\mathbb{R}\backslash\mathbb{Q}$, on the real line $\mathbb{R}$, demonstrate that an uncountable infinitude of points are contained within $\mathbb{R}$ and any subset of it. Interestingly, an infinite number of points is therefore needed to cover a line of any length.

However, the length of a line of finite length *is* of course measurable, but only by using sets of the same dimension, i.e. other lines. Hence, it does not make sense to measure the length of a line by asking how many points it contains, but how many intervals of a length ϵ it contains. This is reflected in the process of measurement discussed in the previous section. Thus, using a measuring tool of too low of a dimension results in an "infinite measurement", like measuring the length of a line with points. Meanwhile, measuring the length of a set with a tool with too high of a dimension results in a measured length of 0. Indeed, asking about the "surface area" of a line is non-sensical, like asking for the volume of a square. Hence, a tool of the right dimension is required for an accurate measurement of a given set. Hausdorff dimension, which provides unconventional non-integer dimension values, can help us find the right measuring tools for fractals.

5. Hausdorff Dimension

Hausdorff Dimension provides the mathematical equipment required to determine the measuring tool of an appropriate dimension for fractals. In his textbook [3], Mandelbrot presents Hausdorff dimension as a kind of "effective dimension" for fractals. Often a non-integer value for fractals, Hausdorff dimension is often "disconcordant" with other notions of dimension (14). While standard figures in Euclidean geometry, like points, lines, planes, etc. are dimensionally concordant (i.e. have the same value) for different notions of dimension, fractals do not. Indeed, as first discussed in this paper, fractals are defined by having a fractal dimension that strictly exceeds its topological dimension.

Devoted to understanding how Hausdorff dimension is calculated, the following sections will provide the formal definition of Hausdorff dimension. Then, we will see an approximation of Hausdorff dimension from the work of Lewis F. Richardson. A sample calculation, using Richardson's approximation, will also be shown below for the Koch Curve.

5.1. Introduction

Recognizing the need for a notion of dimension which allows for the accurate measurement of fractals and coastlines, we turn to Hausdorff's notion of dimension, as discussed in *Geometric Measure Theory: A Beginner's Guide* by Frank Morgan [4].

We begin with definition of the diameter of a set, an important start for understanding Hausdorff dimension:

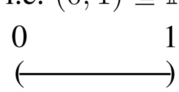
Definition 5.1. For any subset S of $\mathbb{R}^n$, define the *diameter* of S:

$$\text{diam}(S) = \sup\{d(x, y) : x, y \in S\}$$

where d is a metric defined over the set $\mathbb{R}^n$.

For our purposes, we will use the metric d representing Euclidean distance, $\|x - y\|$. Then, we can think of diameter is the longest straight line that can be drawn between two points in the set S. In a circle, this would a straight line passing from one point on the circle's circumference through the centre, or twice the length of the radius. We begin with a simple example:

Example 5.1. Let S be the line from 0 to 1 i.e. $(0, 1) \subseteq \mathbb{R}$. Then,

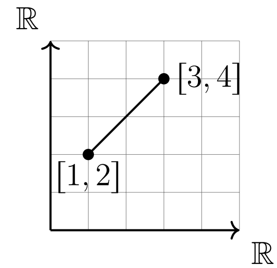


We can visually intuit that:

$$\text{diam}(S) = \sup\{\|x - y\| : x, y \in S\} = \sqrt{(0-1)^2} = \sqrt{1} = 1$$

We show an example for a line segment in a subset of $\mathbb{R}^2$:

Example 5.2. Let L be the line from $(1, 2)$ to $(3, 4)$ over $\mathbb{R}^2$:



Here again, we can intuit that the greatest distance between two points of L is the interval between the two ends, $(1, 2)$ and $(3, 4)$. Here, we use the metric d as the Euclidean distance function in $\mathbb{R}^2$, where x and y are points in $\mathbb{R}^2$. Hence, the calculation of the diameter set is as follows:

$$\text{diam}(L) = \sup\{|x - y| : x, y \in L\} = \sqrt{(1-3)^2 + (2-4)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$$

Indeed, this length is what we would expect by applying the Pythagorean theorem to the right triangle whose hypotenuse is the line L.

Now, we can approach the definition of Hausdorff Dimension by constructing an outer measure:

Definition 5.2. An approximation of the s-dimensional *Hausdorff Measure* of a set $A \subseteq X$, with $\delta > 0$ and $s \geqslant 0$ is defined by:

$$\mathcal{H}_\delta^s := \inf\{\sum_{i \in I} \operatorname{diam}(A_i)^s : X \subset \bigcup_{i \in I} A_i,\ \operatorname{diam}(A_i) < \delta\}$$

Where the infimum is taken over the set of countable covers $\{A_i\}_{i \in I}$ of A

This approximation takes, for a fixed non-negative diameter δ, the infimum of the set of summed diameters of all countable covers A of X. We can demonstrate that this approximation is a an outer measure:

Lemma 5.1. $\mathcal{H}_\delta^s$ is an outer measure on (X, d) for all $\delta > 0$

Proof. Fix $\delta > 0$.

a) Clearly, $\mathcal{H}_\delta^s(\varnothing) = 0$

b) $A \subseteq B \implies \mathcal{H}_\delta^s(A) \leqslant \mathcal{H}_\delta^s(B)$ since we are taking the supremum (by definition of $\operatorname{diam}(S)$) of a set bigger than A.

c) See Lemma 2.1 of [5]

□

Now, we obtain the s-dimensional Hausdorff Measure as follows:

Definition 5.3. The *s-dimensional Hausdorff Measure*, $\mathcal{H}^s$ is defined by $\mathcal{H}_\delta^s$ such that:

$$\mathcal{H}^s = \lim_{\delta \to 0} \mathcal{H}_\delta^s(X)$$

This s-dimensional measure is defined in terms of a limit which does not necessarily exist. Whether or not the limit exists depends on the value of s, which corresponds to the dimension of the measure used. We want to show that as s is decreased, there is a critical value at which the measurement of a given set goes from zero to infinity. We demonstrate this in the following section.

Proposition 5.2. Let A be a subset of (X, d). Take $\tilde{s} > s > 0$: if $\mathcal{H}^s(A) < \infty$ then $\mathcal{H}^{\tilde{s}}(A) = 0$, and if $\mathcal{H}^{\tilde{s}}(A) > 0$, then $\mathcal{H}^s(A) = \infty$.

Proof. Note that the two statements are equivalent, so we prove the first. Assume $\mathcal{H}^s(A) < \infty$. Then, for any $\delta > 0$, there is a cover of A by the sets $\{B_j\}_{j\in J}$ all satisfying $\text{diam}(B_j) \leqslant \delta$ such that for all $\epsilon > 0$:

$$\sum_{j\in J} \text{diam}(B_j)^s \leqslant \mathcal{H}^s(A) + \epsilon$$

Then, for $\tilde{s} > s$,

$$\sum_{j\in J} \text{diam}(B_j)^{\tilde{s}} = \sum_{j\in J} \text{diam}(B_j)^{\tilde{s}-s+s} = \sum_{j\in J} (\text{diam}(B_j)^{\tilde{s}-s} \cdot \text{diam}(B_j)^s)$$

Then, by construction,

$$\text{diam}(B_j)^{\tilde{s}-s} \leqslant \delta^{\tilde{s}-s} \implies \sum_{j\in J} (\text{diam}(B_j)^s \cdot \text{diam}(B_j)^{\tilde{s}-s}) \leqslant \sum_{j\in J} (\text{diam}(B_j)^s \cdot \delta^{\tilde{s}-s}) = \delta^{\tilde{s}-s} \sum_{j\in J} (\text{diam}(B_j)^s$$

Hence,

$$\sum_{j\in J} \text{diam}(B_j)^{\tilde{s}} \leqslant \delta^{\tilde{s}-s} \sum_{j\in J} (\text{diam}(B_j)^s = \delta^{\tilde{s}-s}(\mathcal{H}^s(A) + \epsilon)$$

Finally, we note that δ can get arbitrarily close to 0. Hence, we conclude that $\mathcal{H}^{\tilde{s}}(A) = 0$. □

Now, we can properly define the Hausdorff dimension of a set A using the above proposition:

Definition 5.4. For any subset $A \subseteq (X, d)$, the *Hausdorff Dimension* of A is:

$$\begin{aligned} d_{\mathcal{H}}(A) &= \inf\{0 \leqslant s \leqslant \infty : \mathcal{H}^s(A) = 0\} \\ &= \sup\{0 \leqslant s \leqslant \infty : \mathcal{H}^s = \infty\} \end{aligned}$$

Finally, we have defined Hausdorff dimension, allowing us to distinguish fractal and non-fractal curves by comparing topological dimension (given by Lebesgue covering dimension) with this fractal dimension. However, Hausdorff Dimension, as defined above, is difficult to calculate for a given set. So, we briefly turn to the work of Lewis F. Richardson for his approximation of Hausdorff Dimension for self-similar fractals. We will also provide a sample computation using his approximation.

6. Lewis F. Richardson's Approximation of Hausdorff Dimension

In Mandelbrot's textbook, Richardson's discovery of an approximation of Hausdorff Dimension is described as emerging from his empirical studies (29). Responding to the question of how coastlines can be measured, Richardson argued that the length of a coastline, $L(\epsilon)$, could be approximated as follows:

Definition 6.1. For a given coastline, for some $\epsilon > 0$, the *Richardson Approximation of Length* $L(\epsilon)$ is the following function:

$$L(\epsilon) \approx F\epsilon^{-D} \cdot \epsilon = F\epsilon^{1-D}$$

where $F\epsilon^{-D}$ is the number of yardsticks of length ϵ needed to cover the coastline.

In this approximation, the value of D depends upon the coastline chosen. For our purposes, D is Richardson's approximation of Hausdorff Dimension for coastlines. Interestingly, it can be shown that the value of D is independent of our choice of yardstick length ϵ:

If we take a polygonal approximation of the coastline made up of small intervals of length ϵ, and raise this to the power D, we obtain what Mandelbrot calls an "approximate measure in the dimension D" (30). For Richardson, the number of yardsticks $N = F\epsilon^{-D}$ takes the value $F\epsilon^{-D}\epsilon^{D} = F$. Hence, F appears to approximate $\mathcal{H}^D(A)$ for a given coastline A.

Fractals, heuristically taken as possessing an infinite amount of detail, can be described by Richardson as having a *cascade*. Mandelbrot discusses this notion as the consequence of adding increasingly finer detail to a given drawing/map (34). However, as we will see in the following section, these details and complexity do not necessarily mean unpredictability.

6.1. Self-Similar Fractals and their Generation

A certain class of fractals, called *self-similar*, contain a piece that is geometrically similar to the whole. For such fractals, the shape and the cascade that generate it are self-similar. For fractals in this class, we can use two shapes to generate them iteratively.

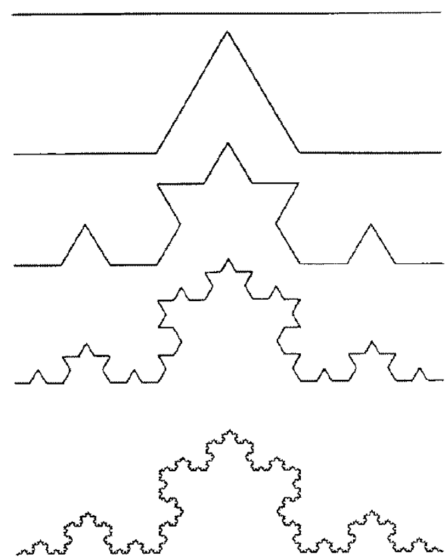


FIGURE 4. Iteratively generating the Koch curve K using the initiator and generator method [2]

First, we begin with an *initiator* and a *generator*. The initiator is our initial shape. The generator is an oriented broken line, made up of N equal sides of length r. With this scheme, we can generate a fractal by replacing each fixed interval along the initiator with a scaled-down copy of the generator. These copies are shrunk and pasted so as the have the same endpoints as each interval of in the initiator.

The initiator-generator construction is demonstrated for the generation of the Koch curve in Figure 4. Here, E_0 is the first initiator, and E_1 is the first generator. To get the first iteration E_1, each straight-line interval of E_0 (i.e. the whole line) is replaced with an appropriately scaled copy of E_1. To attain the next iteration, E_2, we scale down the generator E_1 to $\frac{1}{3}$ its size and paste four copies replacing the straight-line intervals of E_1. To generate E_3, we scale down E_2 to $\frac{1}{3}$ its size, and again paste it along every interval of E_2 that is E_1 (namely, four times). This iterative process can be continued by continuous copying, scaling, and pasting. Taken to the limit, this method generates F, the Koch Curve.

With this construction, Mandelbrot defines an analog of Hausdorff dimension, stemming from Richardson's research. In this construction, the fractal dimension D is defined as follows (30):

$$D = \frac{\log(N)}{log(\frac{1}{r})}$$

In this formula, N is the number of copies of the generator pasted onto the initiator to acquire the next iteration. Meanwhile, r is the scaling factor we apply to the generator so that it becomes an appropriate size. For a curve, this scaling factor is such that the endpoints of the scaled generator map appropriately onto the endpoints of the intervals in the initiator. Of course, this approximation only makes sense for self-similar fractals, which, like K, can be constructed iteratively.

Then, for the Koch curve, we see that each successive iteration involves the scaling of the previous iteration by $r = \frac{1}{3}$, with $N = 4$ copies added to create the next curve. Hence, the dimension of the Koch curve, by this construction, is:

$$D = \frac{\log(N)}{log(\frac{1}{r})} = \frac{\log(4)}{log(3)} \approx 1.2619$$

Now, this value D, which coincides with the Koch Curve's Hausdorff dimension, can be used to measure its length, by using Hausdorff measure. The yardstick of length ϵ acts here like the diameter of each set in a cover, whose added lengths are used to calculate the length of the curve. Indeed, like in Hausdorff Measure, the value of D must be appropriately chosen. Otherwise, an underestimate of length 0 or an overestimate of length ∞ emerges.

By this understanding of dimension and Richardson's approximation of coastline length, we can think of the iterations of the Koch curve as the "level of resolution" of a map of a coastline in the shape of K for a given yardstick length ϵ. For instance, if we attempt to measure a K-shaped coastline using yardsticks of length $\epsilon = \frac{1}{3}$ (assuming the first iteration E_0 has length 1), we find that four yardsticks can be placed along K, which coincides with the measured length of the iteration E_1. Hence, the image of the coastline K using this value $\epsilon = \frac{1}{3}$ is E_1. However, if smaller yardsticks were used, a "higher-resolution" image of K would be possible, with ever-increasing precision. Taken to the limit, we can measure the length of the Koch curve K using a measuring tool of the appropriate dimension, determinable using Richardson's approximation! Suddenly, we can accurately measure coastlines and certain fractals!

7. Conclusion and Acknowledgements

Overall, we have taken Mandelbrot's discussion about fractals to explore how they are defined; namely, shapes whose Hausdorff dimension exceeds their topological dimension. We learned that topological dimension is formally understood as Lebesgue covering dimension, which we then defined. We demonstrated the insufficiency of topological dimension when measuring and discussing natural phenomena and fractals. After motivating Hausdorff dimension, a fractal dimension, we learned how it is defined. Finally, we explored an approximation of Hausdorff dimension for self-similar curves, applying this approximation to calculate the Hausdorff dimension of the Koch curve.

Hopefully, this account of fractals and their dimension has enriched the reader's understanding of how fractals are defined, and the kinds of mathematical tools their analysis requires. Despite the theoretical nature of the definitions and concepts used towards this end, I would like to emphasize that fractals and natural phenomena are found everywhere in concrete reality. Indeed, fractals demonstrate that previous work in mathematics and in the realm of ideality do not even begin to describe the complexity found everywhere in nature.

In light of this development, I do not believe that mathematics, at its very limit, is intended to work purely in the realm of abstraction and ideality. Rather, nature gives us plenty to ponder, often presenting us with incomparable complexity. While mathematical work in the realm of the theoretical remains fascinating, nature, in its everydayness, provides more than enough to push us to the limits of mathematical thought. I hope that through a richer understanding of fractals, we are able to open our hearts and minds to nature's complexity, which we can always find around us.

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