Non-Archimedean Valuations and Monsky's Theorem

Rob Li
Mentor: Aaron Shalev
DRP Winter 2024, McGill University Mathematics & Statistics

June 2024

1 Introduction

This semester we explored non-Archimedean analysis. When learning real analysis, fundamental properties such as the Archimedean property and triangle inequality quickly become second nature. Analysis without these assumptions, we discover, is just as natural. Moreover, exploration of this topic goes beyond theoretical curiosity with applications to the p-adic numbers and the Levi-Civita field among other important and exciting areas of mathematics.

In this write-up, we seek to provide an introduction to the concept of non-Archimedean valuations upon which non-Archimedean analysis is based on. We will then demonstrate one of the suprising applications of non-Archimedean analysis and the ultramatric inequality in the p-adic absolute value, and the proof of Monsky's Theorem.

1.1 Monsky's Theorem If a square is cut into n triangles of equivalent area, then n must be even.

For any even number $n = 2m$, one can quite easily cut a square into n triangles of equal area by perhaps dividing the square into m rectangular strips along the bottom, and then dividing each of these diagonally. However, the difficulty (and impossibility) for n odd will be explained with an elegant proof applying the non-Archimedean properties of the 2-adic valuation in the latter half of this write up.

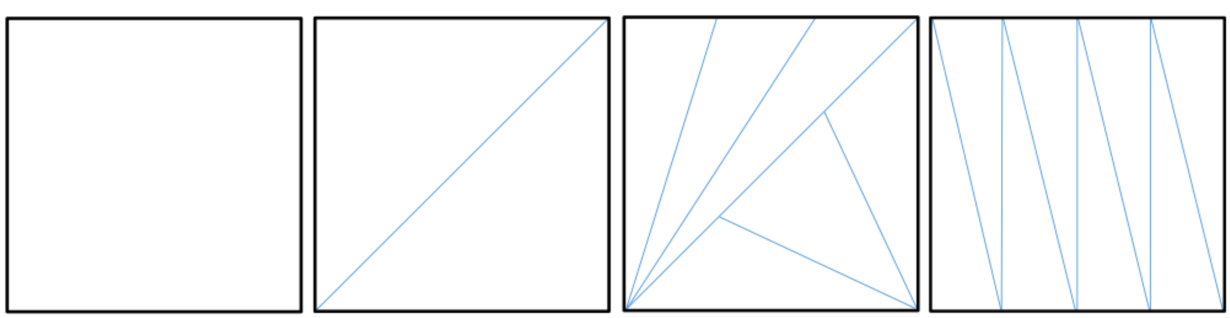

Figure 1: Four squares cut into, from left to right, 0, 2, 6, 8 triangles of equal area respectively

2 Valuations

First, we will generalize the concept of the usual absolute value on $\mathbb{R}$ as *valuations*. We will discuss the preliminary properties, the concept of *non-Archimedian* valuations, and the *ultrametric* inequality which will be foundational for both non-Archimedian analysis and the proof of Monsky's theorem.

2.1 Definition: Let $\mathbb{F}$ be a field. A **valuation** on $\mathbb{F}$ is a map $|\cdot| : \mathbb{F} \to \mathbb{R}$ such that for some real number $C \geq 1$, the following axioms hold:

(1) $\forall\ x \in \mathbb{F}$, $|x| \geq 0$ with $|x| = 0 \iff x = 0$

(2) $\forall\ x, y \in \mathbb{F}$, $|xy| = |x||y|$

(3) $\forall\ x \in \mathbb{F}$ such that $|x| \leq 1$, $|x+1| \leq C$

2.2 Proposition: For a valuation $|\cdot| : \mathbb{F} \to \mathbb{R}$

2.2.1 $|1| = 1$

Proof: By 1.1 (2), $\forall\ x \in \mathbb{F}$ $|x| = |x \cdot 1| = |x||1| \Rightarrow |1| = 1$

2.2.2 $\forall\ x \in \mathbb{F}$ $|x^n| = 1 \Rightarrow |x| = 1$

Proof: By 2.1 (2), $1 = |x^n| = |x|^n \Rightarrow |x| = 1$

2.2.3 $|-1| = 1$

Proof: Apply 2.2.2 with $n = 2$

2.2.4 $|-x| = |x|$

Proof: Corollary of 2.2.3

2.2.5 $|x^{-1}| = |x|^{-1}$

Proof: $|x||x^{-1}| = |1| = 1 \Rightarrow |x^{-1}| = |x|^{-1}$

2.3 Example: The trivial valuation, $|\cdot| : \mathbb{F} \to \mathbb{R}$ is defined by $\forall\ x \neq 0$ $|x| = 1$ and $|0| = 0$

Proof: All (1), (2), and (3) follow clearly with $C = 1$.

2.4 Proposition: If $|\cdot| : \mathbb{F} \to \mathbb{R}$ is a valuation on $\mathbb{F}$, and $\lambda \in \mathbb{R}^+$ then, $|\cdot|_\lambda := |\cdot|^\lambda$ is also a valuation.

Proof: Properties (1) and (2) clearly follow.

(3) follows from: $|x|_\lambda = |x|^\lambda \leq 1 \Rightarrow |x| \leq 1$

so by (3) for $|\cdot|$, $|x+1| \leq C \Rightarrow |x+1|_\lambda = |x+1|^\lambda \leq C^\lambda$.

$\therefore |\cdot|$ is a valuation on $\mathbb{F}$ with constant C^λ

2.5 Definition: Two valuations $|\cdot|_1$ and $|\cdot|_2$ are equivalent if $\exists \lambda \in \mathbb{R}^+$ such that $|\cdot|_2 = |\cdot|_1^\lambda$.

2.6 Definition: A valuation $|\cdot|$ on field $\mathbb{F}$ satisfies the **triangle inequality** if $\forall x, y \in \mathbb{F}$, $|x+y| \leq |x| + |y|$.

2.7 Proposition: $|\cdot|$ satisfies the triangle inequality if and only if one can take $C = 2$ in the definition 2.1 (3).

This is a very useful characterization, but we will not use it beyond 2.10. The proof for this can be found in [2].

2.8 Definition: We will call a valuation $|\cdot|$ on field $\mathbb{F}$ that satisfies the triangle inequality an **absolute value**.

2.9 Example: We are familiar with the absolute value on real numbers defined by $x \mapsto \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

2.10 Proposition: Every valuation is equivalent to one that satisfies the triangle inequality.

Proof: Let $|\cdot|$ be a valuation and let $C \geq 1$ be the associated constant.

Consider $|\cdot|' := |\cdot|^{\log_C 2}$.

$|x+1|' = |x+1|^{\log_C 2} \leq C^{\log_C 2} = 2$ and use 2.7.

2.11 Proposition: By 2.10 we may assume that a valuation $|\cdot|$ observes that triangle inequality. As a consequence $|\cdot|$ induces a natural distance function and metric space. In particular, the distance function in question is $d : \mathbb{F}^2 \to \mathbb{R}$ defined by $d(x, y) = |x - y|$, and the axioms for metric follow from the axioms for absolute values.

2.12 Definition: A valuation $|\cdot|$ on field $\mathbb{F}$ satisfies the **ultrametric inequality** (a.k.a. the **strong triangle inequality**) if $\forall x, y \in \mathbb{F}$, $|x + y| \leq \max(|x|, |y|)$.

2.13 Proposition: $|\cdot|$ satisfies the ultrametric inequality if and only if $C = 1$ in the definition 2.1(3).

Proof of ($\Rightarrow$): Suppose the ultrametric inequality holds for our valuation $|\cdot|$, and let $x \in \mathbb{F}$ be such that $|x| \leq 1$

$$|x+1| \leq \max\left(|x|, |1|\right) \leq 1 =: C$$

Proof of ($\Leftarrow$): Suppose $C = 1$ and let $x, y \in \mathbb{F}$. The cases where either x or y are 0 follow easily so suppose that neither are 0. Without loss of generality, let $|x| \leq |y|$:

$$\left|\frac{x}{y}\right| \leq 1 \Rightarrow \left|\frac{x}{y} + 1\right| \leq 1 \Rightarrow |x + y| \leq |y| = \max\left(|x|, |y|\right)$$

2.14 Proposition: For $|\cdot|$ non-Archimedian and $|x| < |y|$, $|x + y| = |y|$

Proof: By the ultrametric inequality,

$|x + y| \leq \max(|x|, |y|) = |y| = |x - x + y| \leq \max(|-x|, |x + y|) = \max(|x|, |x + y|)$

$\max(|x|, |x + y|) \neq |x|$ as otherwise, $|y| \leq |x| < |y|$

So $\max(|x|, |x + y|) = |x + y|$ and $|x + y| = |y|$.

2.15 Definition: A valuation $|\cdot|$ on field $\mathbb{F}$ is **non-Archimedean** if it satisfies the ultrametric inequality. Additionally the metric space induced by a non-Archimedean valuation is an **ultrametric** space

The subject of non-Archimedean analysis takes off from here. The consequences of the ultrametric inequality shake up previous intuitions of analysis. We discover that in an ultrametric space, nondisjoint balls are necessarily concentric, infinitely small elements exist, and differing conceptions of *completeness* follow. However, these results will not be discussed here, as they do not apply to the proof of Monsky's theorem.

3 p-Adic Absolute Value and the order function

We will now give a concrete example of a non-Archimedean valuation with the p-adic absolute value. The p-adic absolute value defines magnitude and distance using the multiplicity of primes in the factorization of rational numbers. It can be later extended to the real or imaginary numbers by a consequence of a theorem by Chevalley 3.6. Alternatively, the completion of $\mathbb{Q}$ with respect to the ultrametric space and distance function induced by the p-adic absolute value gives rise to the p-adic numbers $\mathbb{Q}_p$. The p-adic numbers are an interesting number system that proves helpful in a vast array of problems including in the proof of Fermat's Last Theorem. For the purposes of Monsky's theorem, we are mostly interested in the p-adic absolute value and its properties.

3.1 Definition: Let p be a prime number. $\forall q \in \mathbb{Q}\backslash\{0\}$ $\exists$ unique $a, b, n \in \mathbb{Z}\backslash\{0\}$, with $\gcd(p, ab) = 1$ such that $q = \frac{a}{b}p^n$. The **p-adic Valuation** on $\mathbb{Q}$, $\nu_p(\cdot)$, is defined by $\nu_p(0) = \infty$ and $\forall q \in \mathbb{Q}\backslash\{0\}$, $\nu_p(q) = n$.

It should be worth noting that the p-Adic valuation is not a valuation as outlined by the definition in 2.1. In particular, 2.1(1) fails. Rather, this is an example of an order function. Order functions are a different approach to valuation theory, and in the case of the p-adic absolute value, the order function above, *the p-adic valuation*, is used in the definition. Some authors will use the term *valuation* to refer to order functions and use the term absolute value to only refer to the functions mentioned in 2.1 that also abide by the triangle inequality. In this write up, we will use *valuation* for functions $|\cdot|$ that fit definition 2.1, and we will notate order functions with $\nu(\cdot)$.

3.2 Definition: A function $\nu : \mathbb{F} \to \mathbb{Z}\bigcup\{\infty\}$ is an **order function** if it satisfies:

(1) $\nu(0) = \infty$

(2) $\forall x, y \neq 0$, $\nu(xy) = \nu(x) + \nu(y)$

(3) $\forall x, y \neq 0$, $\nu(x + y) \geq \min(\nu(x), \nu(y))$

3.3 Proposition: For an order function $\nu : \mathbb{F} \to \mathbb{Z}\bigcup\{\infty\}$

3.3.1 $\nu(1) = 0$

Proof: By 3.2(2), $\nu(1) = \nu(1) + \nu(1) \Rightarrow \nu(1) = 0$.

3.3.2 $\nu(x^{-1}) = -\nu(x)$

Proof: By 3.2(2) and 3.3.1, $\nu(x) + \nu(x^{-1}) = \nu(xx^{-1}) = \nu(1) = 0 \Rightarrow \nu(x^{-1}) = -\nu(x)$.

3.3.2 $\nu(-x) = \nu(x)$

Proof: By 3.2(2), $\nu(1) = \nu(-1) + \nu(-1) \Rightarrow \nu(-1) = 0 \Rightarrow \nu(-x) = 0 + \nu(x)$.

3.4 Proposition: The p-adic valuation $\nu_p(\cdot)$ is an order function.

Proof: We check the definition

(1) $\nu(0) := 0$

(2) $\forall q, r \neq 0$, $\nu_p(qr) = \nu_p(q) + \nu_p(r)$

The cases where either or q and r are zero follow easily. So we assume neither are zero.

$\exists a, b, c, d, n, m \in \mathbb{Z}$ such that $q = \frac{a}{b}p^n$ and $r = \frac{c}{d}p^m$. $\nu_p(q) = n, \nu_p(r) = m$

$qr = \frac{ac}{bd}p^{n+m} = \frac{\alpha}{\beta}p^{n+m}$ with $\gcd(p, \alpha\beta) = 1$.

$\therefore \nu_p(qr) = \nu_p(q) + \nu_p(r)$.

(3) $\forall q, r \neq 0$, $\nu_p(q + r) \geq \min(\nu_p(q), \nu_p(r))$

The cases where $q = 0$ or $r = 0$ are clear, so we suppose neither are zero.
$\exists a, b, c, d, n, m \in \mathbb{Z}$ such that $q = \frac{a}{b}p^n$ and $r = \frac{c}{d}p^m$. $\nu_p(q) = n, \nu_p(r) = m$.
Without loss of generality, let $\nu_p(q) = n \leq m = \nu_p(r)$.
Then, $q + r = \frac{a}{b}p^n + \frac{c}{d}p^m = \frac{ap^n b + cp^m d}{bd} = \frac{ab + cp^{m-n}d}{bd}p^n$ which has an order at least $\nu_p(q) = n$

3.5 Definition: The **p-adic absolute value** for prime number p, $|\cdot|_p$, is defined by $\forall q \in \mathbb{Q}\backslash\{0\}$, $|q|_p = p^{-v_p(q)}$, and $|0|_p = 0$.

3.6 Proposition: For any base $b > 1$ and an order function $\nu : \mathbb{F} \to \mathbb{Z}\bigcup\{\infty\}$, $|\cdot| : \mathbb{F} \to \mathbb{R}$ defined by $\forall x \neq 0, |x| := b^{-\nu(x)}$ and $|0| := 0$ is a non-Archimedean valuation.
Proof: We check the definition

(1) $\forall\ x \in \mathbb{F}$, $|x| \geq 0$ with $|x| = 0 \iff x = 0$
$|x| \geq 0$ clearly.
$|0| := 0$ so $x = 0 \Rightarrow |x| = 0$. $x \neq 0 \Rightarrow \nu(x) < \infty \Rightarrow |x| = b^{-\nu(x)} > 0$.

(2) $\forall\ x,\ y \in \mathbb{F}$, $|xy| = |x||y|$
The cases where either or x and y are zero follow easily, so we assume neither are zero.
In this case, $|xy| = b^{-\nu(xy)} = b^{\nu(x)\nu(y)} = b^{-\nu(x)}b^{-\nu(y)} = |x||y|$.

(3) $|\cdot|$ satisfies the ultrametric inequality
$\nu(x + y) \geq \min(\nu(x), \nu(y))$
$\Rightarrow b^{-\nu(x+y)} \leq b^{\max(-\nu(x), -\nu(y))} = \max(b^{-\nu(x)}, b^{-\nu(y)})$
$\Rightarrow |x + y| \leq \max(|x|, |y|)$.

As a result, the p-adic absolute value $|\cdot|_p$ is a non-Archimedian valuation on $\mathbb{Q}$.

3.7 Examples:

$$|19|_{17} = \left|\frac{19}{1}17^0\right|_{17} = 17^{-0} = 1$$

$$\left|\frac{1}{135}\right|_3 = \left|\frac{1}{5}3^{-3}\right|_3 = 3^3 = 27$$

$$|-64|_2 = \left|\frac{-1}{1}2^4\right|_2 = 2^{-4} = \frac{1}{64}$$

$$\left|\frac{15}{12}\right|_2 = \left|\frac{5}{1}2^{-2}\right|_2 = 2^2 = 4$$

The following proposition will be important in the proof of Monsky's Theorem.
3.8 Proposition: For $n \in \mathbb{Z}$, $|n|_2 < 1 \iff n$ is even.
Proof of ($\Rightarrow$): Suppose $n \in \mathbb{Z}$ and $|n|_2 < 1$, we then have $n = \frac{a}{b}2^k$; $a, b, k \in \mathbb{Z}$ and $\gcd(k, ab) = 1$ with
$|n|_2 = 2^{-k} < 1 \Rightarrow k > 1$ and $\gcd(k, ab) = 1 \Rightarrow b = 1$. So $\therefore n = 2^k a$ is even.
Proof of ($\Leftarrow$): Suppose $n = a2^k$ with $a, k \in \mathbb{Z}$ and $k \geq 1$. $|n|_2 = 2^{-k} < 1$

3.9 Definition: Let $G_\nu \subset \mathbb{F}$ be a subring. G_ν is a **valuation ring** if $\forall x \in \mathbb{F}\backslash\{0\}$ either $x \in G_\nu$ or $x^{-1} \in G_\nu$

With this definition, we can make the following observations:
If $\nu(\cdot)$ is an order function, $b > 1$ is some base, and $\forall x \neq 0$ $|x| := b^{-\nu(x)}$ and $|0| := 0$ is a non-Archimedean valuation then,
$G_\nu := \{x \in \mathbb{F} : |x| \leq 1\} = \{x \in \mathbb{F} : \nu(x) \geq 1\}$ is a valuation ring, and
$M := \{x \in G_\nu : |x| < 1\} = \{x \in G_\nu : \nu(x) > 1\}$ is the unique maximal ideal of G_ν.
In fact, the existence of a valuation ring on $\mathbb{F}$ and the existence of an order function on $\mathbb{F}$ are interchangeable. An elaboration on these observations can be found in both [2] and [3]. They provide background for the following theorem by Chevalley:

3.10 Chevalley's Theorem: Let $\mathbb{F}$ be a field, $A \subset \mathbb{F}$ is a subring, and $P \subset A$ is a prime ideal of A. Then there exists a valuation ring $G \subset \mathbb{F}$ such that $A \subset G$ with maximal ideal M such that $M \bigcap A = P$.

The proof of this theorem makes use of ring localizations and Zorn's lemma. It can be found in [3]. This theorem is necessary for us to extend the p-adic valuation to $\mathbb{R}$ for our proof of Monsky's theorem.

3.11 Proposition: Let $\mathbb{F}_1 \subset \mathbb{F}_2$ be a subfield. Let $G_1 \subset \mathbb{F}_1$ be a valuation ring with maximal ideal M_1. There exists an a valuation ring on $\mathbb{F}_2$, G_2 with $G_2 \bigcap \mathbb{F}_1 = G_1$. We will call G_2 an extension of G_1.
Proof: G_1 is a subring of $\mathbb{F}_2$ with maximal ideal say M_1, so by Chevalley's theorem 3.10, $\exists$ a valuation ring G_2 of $\mathbb{F}_2$ with maximal ideal M_2 such that $G_1 \subset G_2$ and $M_2 \bigcap G_1 = M_1$.

Since both G_1, G_2 are valuation rings with respective maximal ideals M_1, M_2,

$$M_2 \bigcap \mathbb{F}_1 = M_2 \bigcap G_1 = M_1$$

$$G_2\backslash\{0\} \bigcap \mathbb{F}_1 = G_2\backslash\{0\} \bigcap G_1 = G_1\backslash\{0\}$$

So $G_2 \bigcap \mathbb{F}_1 = G_1$

Using 3.11 and the fact that the existence of a valuation ring a the existence of an order function are interchangeable, we can extend the p-adic valuation $\nu_p : \mathbb{Q} \to \mathbb{Z}\backslash\{0\}$ to $\nu_p' : \mathbb{R} \to \mathbb{Z}\backslash\{0\}$. With this extended p-adic valuation, we naturally also have an extended absolute value $|\cdot|_p' : \mathbb{R} \to \mathbb{R}$. We will henceforth denote this extended p-adic absolute value function with $|\cdot|_p$ for brevity in our proof of Monsky's theorem.

4 Sperner's Lemma and Monsky's Theorem

In this section we apply the 2-adic valuation and its ultrametric properties to prove Monsky's theorem. This proof draws upon Sperner's lemma which is a property of graph colouring. We will begin by describing how a polygon can be "cut".

Let $R \subset \mathbb{R}^2$ be a polygon.

4.1 Definition: A **dissection** of $R \subset \mathbb{R}^2$ is a decomposition of R into a finite number of triangles $\{T_n\}_{n=1}^{N}$. Each triangle is characterized by being a compact set in $\mathbb{R}^2$ with a boundary consisting of three line segments. The intersection of these triangles is the boundaries of the triangles, and their union is R.

4.2 Definition: An **equidissection** is a dissection of polygon R such that each triangle has the same area.

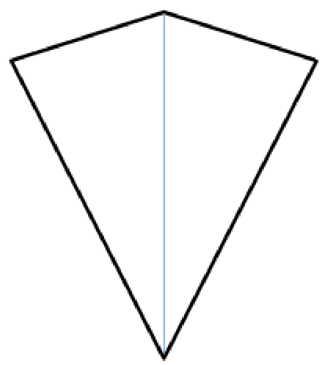

Figure 2: An equidissection of a kite into two triangles

4.3 Definition: A **triangulation** of polygon R is dissection such that no edge of a triangle is adjacent to more than two triangles.

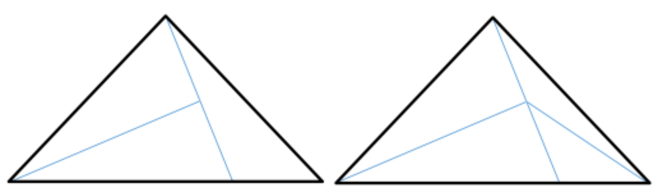

Figure 3: A dissection (left) differs from a triangulation (right) in that the edges of a triangulation are end to end adjacent. Each triangle edge on the right is the boundary of at most two triangles.

We will now initialize some terms for our colouring of the graph to set up Sperner's Lemma.

For a polygon R and a triangulation of R, we may colour each vertex (x_k, y_k) **red**, **blue**, or **yellow**.

An edge between a red vertex and a blue vertex is coloured **purple**.

Lastly, any triangle with one red, one blue, and one yellow vertex is coloured **rainbow**.

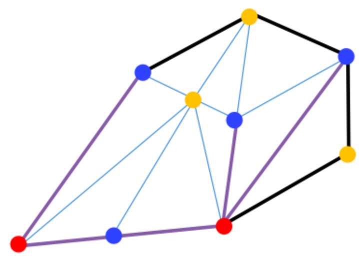

Figure 4: A colouring of polygon R. There are exactly 5 rainbow triangles.

4.4 Sperner's Lemma: Given a red-blue-yellow coloured triangulation of polygon R, the number of purple edges on the boundary of R and the number of rainbow triangles inside R have the same parity.

Proof: By Double Counting

We will draw one dot on each side of all the purple edges.

Now we count $d \in \mathbb{N}$, the number of dots in the interior of R.

Each purple edge in the interior contributes two dots to d.

Purple edges on the boundary of R contribute one dot to d.

Non-purple edges contribute zero dots to d.

So the parity of the number of purple edges on the boundary is the same as the parity of d.

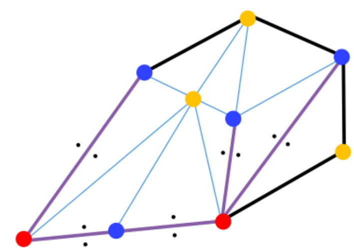

Figure 5: Sperner's Lemma proof: The double counting proof applied to the coloured polygon from the previous figure. See that an odd number of dots (3) are outside the shape and there are an odd number of rainbow triangles (5) accordingly.

Another way of counting d is to count all the dots inside all triangles of the triangulation.

See that a triangle cannot have three dots.

Otherwise, this would imply its three edges are purple which is impossible as one purple edge implies one of the vertices is red and another is blue, but the two other purple edges would imply that the third is both red and blue.

Rainbow triangles are characterized by having only one dot in the interior.

One interior dot means exactly one purple edge, one red vertex, one blue vertex with the third vertex necessarily being yellow.

All other triangles contribute an even number of dots.

So the parity of the total number of dots in all triangles (same as d) and the number of rainbow triangles is the same.

Therefore, the parity of the number of boundary purple edges and the number of rainbow triangles is the same.

With this lemma in mind, we now construct the proof of Monsky's Theorem. In order to describe the dissections of a square, we will provide a set of colouring rules to every point in the square. The rules are based on our 2-adic valuation extended to $\mathbb{R}$ as described in 3.11.

Consider the square $S := \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\}$.
We will now colour the square based on the following rules.

$\{(x, y) \in S : |x|_2 < 1, |y|_2 < 1\}$ are coloured Blue.
$\{(x, y) \in S : |y|_2 \geq 1, |y|_2 > |x|_2\}$ are coloured Red.
$\{(x, y) \in S : |x|_2 \geq 1, |x|_2 \geq |y|_2\}$ are coloured Yellow.

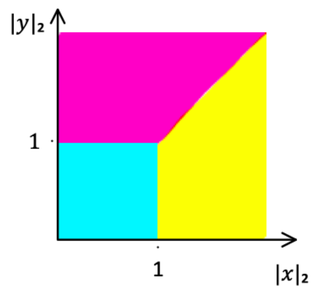

Figure 6: The colouring rules visualized.

4.5 Lemma: The colouring of the points are translation invariant with respect to the blue points; ie: if $(x_b, y_b), (x, y)$ are Blue, (x_r, y_r) is Red, and (x_y, y_y) is Yellow then, $(x_b - x, y_b - y)$ is Blue, $(x_r - x, y_r - y)$ is Red, and $(x_y - x, y_y - y)$ is Yellow.

Proof: Following from the ultrametric inequality,
$|x|_2, |y|_2, |x_b|_2, |y_b|_2 < 1 \Rightarrow |x_b - x|_2 \leq \max(|x_b|_2, |x|_2) < 1$ and $|y_b - y|_2 \leq \max(|y_b|_2, |y|_2) < 1$
So, $(x_b - x, y_b - y)$ is Blue.
$|-y|_2 = |y|_2 < 1 \leq |y_r|_2 \Rightarrow |y_r - y|_2 = |y_r|_2 \geq 1$ by 2.14.
$|x_r|_2 < |y_r|_2$ and $|x|_2 < 1 \leq |y_r|_2 \Rightarrow |x_r - x|_2 \leq \max(|x_r|_2, 1) < |y_r|_2 = |y_r - y|_2$
So $(x_r - x, y_r - y)$ is Red.
$|-x|_2 = |x|_2 < 1 \leq |x_y|_2 \Rightarrow |x_y - x|_2 = |x_y|_2 \geq 1$ by 2.14.
$|y_y|_2 \leq |x_y|_2$ and $|y|_2 < 1 \leq |x_y|_2 \Rightarrow |y_y - y|_2 \leq \max(|y_y|_2, 1) \leq |x_y|_2 = |x_y - x|_2$
So $(x_y - x, y_y - y)$ is Yellow.

4.6 Lemma: Let $T \subset S$ be a rainbow triangle then, $|\text{area}(T)|_2 > 1$.

Proof: Let T have vertices $(x_b, y_b), (x_r, y_r)$, and (x_y, y_y) coloured blue, red, and yellow respectively.
Due to 4.5 the colouring of T is invariant with respect to blue points.
So, $(0, 0), (x_r - x_b, y_r - y_b)$, and $(x_y - x_b, y_y - y_b)$ are also blue, red, and yellow respectively.
Let $(x'_r, y'_r) = (x_r - x_b, y_r - y_b)$ and $(x'_y, y'_y) = (x_y - x_b, y_y - y_b)$
The triangle described by the translated points is denoted as T' and has the same area as T.

$$\text{area}(T) = \text{area}(T') = \frac{1}{2}\det\left(\begin{bmatrix} x'_r & x'_y \\ y'_r & y'_y \end{bmatrix}\right) = \frac{1}{2}(x'_r y'_y - x'_y y'_r)$$

By the colouring, $|x'_r|_2 < |y'_r|_2$ and $|y'_y|_2 \leq |x'_y|_2$. $\Rightarrow |x'_r y'_y|_2 < |x'_y y'_r|_2$

So by 2.14 and $|x'_y|_2, |y'_r|_2 \geq 1$

$$|\text{area}(T)|_2 = \left|\frac{1}{2}\right|_2 \left|x'_r y'_y - x'_y y'_r\right|_2 = 2\max\left(|x'_r y'_y|_2, |x'_y y'_r|_2\right) = 2|x'_y y'_r|_2 = 2|x'_y|_2|y'_r|_2 \geq 2 > 1$$

4.7 Lemma: No line segment in S can have blue, red, and yellow points.

Proof: By lemma 4.6, the area of any rainbow triangle is non zero. Therefore three different coloured points cannot exist on the same line.

As it turns out, Sperner's Lemma alone is not enough to prove Monsky's Theorem because it applies to triangulations on S rather than dissections. However, 4.7 allows us to use an extended version of Sperner's lemma for S.

4.8 Lemma: In a dissection of S, the number of purple edges on the boundary of S and the number of rainbow triangles inside S have the same parity.

Proof: The proof for this is almost identical to that of Sperner's lemma 4.4. We only need to adjust how we count dots within the triangles. Instead of rainbow triangles being characterized by having one dot, they are characterized by having an odd number of dots. Rainbow triangles have one line segment bounded by a red and blue vertex. This line may or may not be divided into smaller edges by neighboring lines. We know that that the vertices along this line can only be red or blue by 4.7. Since the vertex colour changes from blue to red over the entire segment, vertex colour can change an odd number of times along the line segment. Therefore, there are an odd number of purple edges, and an odd number of dots in the rainbow triangle.

All other triangles have zero or two line segment boundaries that end in red and blue. Each of these segments contribute an odd number of purple edges and dots to the triangle. Since we have an even number of these boundary segments, non-rainbow triangles are characterized by an even number of dots.

These adjustments maintain the conclusions about parity for the proof.

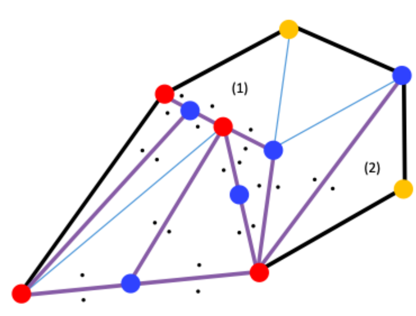

Figure 7: Extended Sperner's Lemma: The adjusted double counting proof and a dissected shape. Notice that both rainbow triangles (1) and (2) have an odd number of dots. The existence of and even number of rainbow triangles coincides accordingly with an odd number of purple edges on the boundary (or dots outside the shape).

4.9 Lemma: In any triangulation of the square S, there is an odd number of purple edges on the boundary.

Proof: By our colouring, the point $(0, 0)$ is blue, $(0, 1)$ is red, and $(1, 0), (1, 1)$ are yellow.

From our conclusion in 4.7 we see that the only boundary line segment that contains purple edges is that from $(0,0)$ to $(0,1)$

Furthermore, this line segment must have an odd number of purple edges since it begins with a blue vertex and ends with a red vertex.

4.10 Monsky's Theorem: In any equidissection of a square into n triangles, n is even.

Proof: Without loss of generality, we can transform any square into S defined above.

The total area of S is 1, so each triangle has an area $\frac{1}{n}$

By Lemma 4.9, we have an odd number of purple edges on the boundary.

By the extended Sperner's lemma 4.8, there must be an odd number of rainbow triangles.

By Lemma 4.5, choosing one of these rainbow triangles, the area satisfies: $|\text{area}(T)|_2 > 1$

$|\frac{1}{n}|_2 > 1 \Rightarrow |n|_2^{-1} > 1 \Rightarrow |n|_2 < 1$

Finally, by Proposition 3.8, n is even.

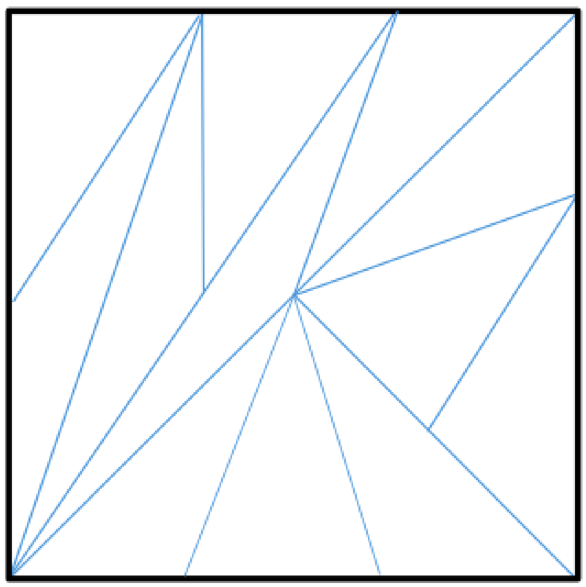

Figure 8: A square dissected into twelve triangles of equal area

5 Further Explorations

It turns out that the proof above is the only known proof of Monsky's theorem [3]. Drawing on seemingly unrelated areas of mathematics, Monsky's theorem offers a surprising application of non-Archimedean. One might begin with the initial question of dividing a square into triangles of equal area and contemplate its adaptation to different shapes. Indeed, the above proof inspires a generalization to other regular polygons, as stated in the following theorem:

5.1 Theorem: For $n > 4$, a regular n-gon can be dissected into m triangles of equal area if and only if m is a multiple of n.

The proof for this can be found in [5].

Listed below, other problems of a similar spirit can be found detailed in [3]

(1) A n-dimensional hypercube can be equidissected into m simplicies if and only if m is a multiple of $n!$

(2) Some polygons cannot be equidissected.

References

[1] W. Sablan, "An Exposition of Monsky's Theorem," University of Chicago Mathematics REU 2019, Chicago, IL, USA, 2019. [Online]. Available: https://math.uchicago.edu/ may/REU2019/REUPapers/Sablan.pdf

[2] T. Diagana and F. Ramaroson, *Non-Archimedean Operator Theory*. Cham, Switzerland: Springer, 2016.

[3] A. Ferré Moragues, "What is... Monsky's theorem?" Department of Mathematics, Ohio State University, Columbus, OH, USA, 2016. [Online]. Available: https://math.osu.edu/sites/math.osu.edu/files/ferre_Monsky_Thm2016.pdf

[4] P. Monsky, "On Dividing a Square Into Triangles," *The American Mathematical Monthly*, vol. 77, no. 2, pp. 161-164, Feb. 1970.

[5] R. Fan, "Monsky's Theorem," [Online]. Available: http://simonrs.com/eulercircle/pftb2/roger-monsky.pdf