# MATH 248: ASSIGNMENT 6: SOLUTIONS

**Remark.** This document contains solutions to the problems in the sixth assignment. I have also sprinkled in a few exercises, which I hope you will find engaging.

The problems in this assignment use the following three theorems

**Green's Theorem.** Let $D$ be a simple region and let $C$ be its boundary. Suppose $P : D \to \mathbb{R}$ and $Q : D \to \mathbb{R}$ are of class $C^1$. Then

$$\int_{C^+} P\,dx + Q\,dy = \int_D Q_x - P_y\,dx\,dy$$

Where we integrate around the boundary curve with *positive orientation.*

In the setting of Green's theorem, any *counter-clockwise* parametrization of the boundary curve is positively oriented.

**Stokes' Theorem.** Let $S$ be an oriented surface defined by a one-to-one parametrization $\Phi : D \subset \mathbb{R}^2 \to S$, where $D$ is a region to which Green's theorem applies. Let $\partial S$ denote the oriented boundary of $S$ and let $\mathbf{F}$ be a $C^1$ vector field on $S$. Then

$$\int_S \operatorname{curl}(\mathbf{F}) \cdot dS = \int_{\partial S} \mathbf{F} \cdot ds$$

**Gauss' Divergence Theorem.** Suppose that $W$ be a closed and bounded region in $\mathbb{R}^3$. Let $\partial W$ be its oriented boundary, and let $\mathbf{F}$ be a smooth vector field defined on all of $W$. Then

$$\int_W \operatorname{div}(\mathbf{F})\,dV = \int_{\partial W} (\mathbf{F} \cdot n)\,dS.$$

Where $n$ is the outer normal to the boundary of $W$.

From the statement of the Stokes' theorem it is clear that whenever we wish to apply Stokes' theorem to a surface $S$, we will need to have some parametrization of $S$ ready. Additionally, in the applications of Gauss's divergence theorem we will need a parametrization of the boundary of the region $W$.

**Problem 1.** Let $D$ be a region for which Green's Theorem holds. Suppose $u$ is harmonic; that is,

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \qquad \text{for all } (x,y) \in D \tag{1}$$

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Prove that

$$\int_{\partial D} u_y dx - u_x dy = 0$$

*Solution.* From the statement of the problem, we know that we can apply Green's Theorem to $\int_{\partial D} u_y dx - u_x dy$. We set $P = u_y$ and $Q = u_x$. We obtain

$$\int_{\partial D} u_y dx - u_x dy = \int_D -u_{xx} - u_{yy}\, dx\, dy = -\int_D u_{xx} + u_{yy}\, dx\, dy = 0$$

Where we have set the integral equal to zero since the integrand is zero everywhere in the disk. This completes the solution. ⌟

**Problem 2.** Suppose $p \in D$ is such that $\overline{B}_R(p) \subset D$, and suppose that $u$ is continuous in $D$ and satisfies Laplace's equation (1) on $D - \{p\}$. Assume that

$$\int_{\partial D} u_y\, dx - u_x\, dy = 0$$

and prove that

$$u(p) = \frac{1}{2\pi R}\int_{\partial B_R(p)} u\, ds.$$

Hint: consider $I(\rho) = \frac{1}{\rho}\int_{\partial B_\rho(p)} u\, ds$, for $0 < \rho \leq R$, using Green's Theorem to deduce that $\frac{d}{d\rho} I(\rho) = 0$, and then calculate $\lim_{\rho\to 0} I(\rho) = 2\pi u(\rho)$.

*Solution.* First we define

$$I(\rho) = \frac{1}{2\pi\rho}\int_{\partial B_\rho(p)} u\, ds$$

We should think of the quantity $I(\rho)$ as the average value of $u$ over the circle $\partial B_\rho(p)$. Since we are integrating over a circle, we can write $I(\rho)$ in terms of an iterated integral using polar coordinates $(x, y) = (p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)$, where $p = (p_1, p_2)$.

$$(2)\quad I(\rho) = \frac{1}{2\pi\rho}\int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\rho\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\, d\theta$$

**Intuition.** Before we continue with the solution, let's sketch out our strategy so that we know where we are going. The goal of the problem is to show that $I(\rho) = u(p)$, *for all* $p$. One way to show that a differentiable function $f : [0, R] \to \mathbb{R}$ is identically a constant $c$ is to prove that $f'$ is identically zero on $(0, R)$ (so that $f$ is constant) and that $f(0) = c$.
To prove that $I(\rho) = u(p)$, we will first prove that $\rho \mapsto I(\rho)$ is a differentiable function, that $I'(\rho) = 0$ for all $\rho > 0$, and that $I(\rho)$ is continuous up to $\rho = 0$ where it satisfies $I(0) = u(p)$.
It turns out that Green's function and the fact that

$$\int_{\partial D} u_y\, dx - u_x\, dy = 0$$

will play a key role in proving that $I'(\rho) = 0$.

The fact that $I(\rho)$ is differentiable follows from the following theorem, which gives us a sufficient condition to "bring derivatives" into the integral sign:

**Theorem.** Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a differentiable function such that $\frac{\partial}{\partial y} f(x,y)$ is integrable over $[a,b] \times [c,d]$. Then

$$\frac{d}{dy} \int_a^b f(x,y)\,dx = \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx$$

**Proof.** First we remark that the fundamental theorem of calculus tells us that for each pair $y_1 < y_2$ in $[c,d]$, we have

$$f(x,y_2) - f(x,y_1) = \int_{y_1}^{y_2} \frac{\partial}{\partial y} f(x,y)\,dy$$

and integrating the above equation over $[a,b]$, we obtain

$$\int_a^b f(x,y_2)\,dx - \int_a^b f(x,y_1)\,dx = \int_a^b \int_{y_1}^{y_2} \frac{\partial}{\partial y} f(x,y)\,dy\,dx$$

But now, since we assume that $\frac{\partial}{\partial y} f(x,y)$ is integrable over $[a,b]\times[c,d]$, it is also integrable over $[a,b] \times [y_1,y_2]$, and so Fubini's theorem applies and we may switch the order of integration

$$\int_a^b f(x,y_2)\,dx - \int_a^b f(x,y_1)\,dx = \int_{y_1}^{y_2} \int_a^b \frac{\partial}{\partial y} f(x,y)\,dx\,dy$$

Fixing $y_1$ and differentiating with respect to $y_2$, another application of the fundamental theorem of calculus implies that

$$\frac{d}{dy_2} \int_a^b f(x,y_2)\,dx = \int_a^b \frac{\partial}{\partial y} f(x,y_2)\,dx$$

which (after setting $y_2 = y$) is what we wanted to show. □

Applying this theorem to $I(\rho)$ given by (2), we deduce that

$$(3)\quad I'(\rho) = \frac{1}{2\pi} \int_0^{2\pi} u_x(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\cos\theta + u_y(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\sin\theta\,d\theta$$

Now we recall the definition of a line integral: if $\gamma : [a,b] \to \mathbb{R}^2$ is a parametrized curve with components $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then

$$\int_\gamma P(x,y)\,dx + Q(x,y)\,dy = \int_a^b P(\gamma(t))\gamma_1'(t) + Q(\gamma(t))\gamma_2'(t)\,dt$$

Since the mapping $\theta \mapsto (p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)$ parametrizes $\partial B_\rho(p)$, we can write

$$\int_{\partial B_\rho} P\,dx + Q\,dy = \int_0^{2\pi} Q(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\cos\theta - P(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\sin\theta\,d\theta$$

Applying this to (3), we see that

$$I'(\rho) = \frac{1}{2\pi} \int_{\partial B_\rho(p)} u_x \, dy - u_y \, dx$$

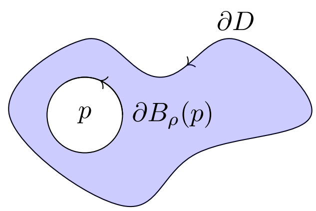
**Remark.** The key part of this argument is to recognize that $u_x \, dy - u_y \, dx$ is somehow hidden inside of

$$u_x(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\cos\theta + u_y(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\sin\theta \, d\theta$$

Perhaps this can be made more obvious if we write the preceeding expression as

$$u_x(\cdots) \underbrace{\cos\theta d\theta}_{dy} - u_y(\cdots) \underbrace{(-\sin\theta \, d\theta)}_{dx}$$

The idea for the rest of the solution can be summarized nicely in a picture:




In the above figure, $\partial D$ is the (oriented boundary) of the entire region $D$, $\partial B_\rho(p)$ is the oriented boundary of the disk $B_\rho(p)$ of radius $\rho$ centered at $p$. Let $\Omega$ be the shaded region in the above figure; that is $\Omega = D - B_\rho(p)$. We want to use Green's theorem to conclude that $\int_{\partial B_\rho(p)} u_x \, dy - u_y \, dx = 0$. To do this, we note that $u_x$ and $u_y$ are differentiable inside of the shaded region $\Omega$ (since $\Omega$ doesn't contain the point $p$, $u$ is harmonic on $\Omega$), and so we may apply Green's theorem (with $-u_y = P$ and $u_x = Q$). However, note that we need to have $B_\rho(p) \subset \overline{D}$, or else $\partial\Omega$ will not be $\partial D - \partial B_\rho(p)$.

$$\int_\Omega u_{yy} + u_{xx} \, dx \, dy = \int_{\partial\Omega} u_x \, dy - u_y \, dx = \int_{\partial D} u_x \, dy - u_y \, dx - \int_{\partial B_\rho(p)} u_x \, dy - u_y \, dx$$

Here we have used the fact that $\partial\Omega = \partial D - \partial B_\rho(p)$, where here the minus sign in front of $\partial B_\rho(p)$ denotes the reverse orientation. Since $u_{xx} + u_{yy} = 0$ inside of $\Omega$, we deduce that

$$\int_{\partial D} u_x \, dy - u_y \, dx - \int_{\partial B_\rho(p)} u_x \, dy - u_y \, dx = 0 \implies I'(\rho) = \int_{\partial B_\rho(p)} u_x \, dy - u_y \, dx = 0$$

since we assume that $\int_{\partial D} u_x \, dy - u_y \, dx = 0$. Thus we have shown that $I(\rho) =$ constant, as long as $\rho$ is small enough that $B_\rho(p) \subset \overline{D}$.

It remains to prove that $\lim_{\rho\to 0} I(\rho) = u(p)$, and (by the remarks in the box labelled "Intuition") it will follow that $I(\rho) = u(p)$ for all $\rho$. We recall equation (2):

$$I(\rho) = \frac{1}{2\pi}\int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta)\, d\theta$$

To show that $I(\rho) \to u(p)$ as $\rho \to 0$, we need to pick some $\epsilon > 0$, and then find some $\delta > 0$ such that $\rho < \delta$ implies that $|I(\rho) - u(p)| < \epsilon$. Thus, let us fix some $\epsilon > 0$.

Since $u$ is continuous at $p$, there exists a $\delta$ such that

$$|u(p + x) - u(p)| < \epsilon \quad \text{whenever} \quad \|x\| < \delta \tag{4}$$

Now choose $\rho < \delta$. Then write

$$I(\rho) - u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta) - u(p_1, p_2)\, d\theta$$

where we have used the fact that $u(p_1, p_2) = \frac{1}{2\pi}\int_0^{2\pi} u(p_1, p_2)\, d\theta$. Now we take absolute values of the above equation and use the fact that $\left|\int f\, d\theta\right| \le \int |f|\, d\theta$ to set up an inequality:

$$|I(\rho) - u(p)| =\le \frac{1}{2\pi}\int_0^{2\pi} |u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta) - u(p_1, p_2)|\, d\theta$$

Now since $\|(\rho\cos\theta, \rho\sin\theta)\| = \rho < \delta$, we can deduce from (4) that

$$\frac{1}{2\pi}\int_0^{2\pi} |u(p_1 + \rho\cos\theta, p_2 + \rho\sin\theta) - u(p_1, p_2)|\, d\theta < \frac{1}{2\pi}\int_0^{2\pi} \epsilon\, d\theta = \epsilon$$

and so

$$|I(\rho) - u(p)| < \epsilon \quad \text{whenever } \rho < \delta.$$

This proves that $\lim_{\rho\to 0} I(\rho) \to u(p)$, and since we proved that $I'(\rho)$ is identically 0 (for $B_\rho(p) \subset \overline{D}$), we deduce that $I(\rho) = u(p)$ for all $\rho$ small enough that $B_\rho(p) \subset \overline{D}$. This completes the solution. ⌟

**Problem 3.** Define $B = \left\{(x, y) : x^2 + y^2 \le 1\right\}$, and for all $\delta > 0$, define

$$B_\delta = \left\{(x, y) : x^2 + y^2 \le \delta\right\}.$$

Suppose $f$ is a continuous function and $\|\nabla f\| \le 1$ on $B$. Suppose that

$$f_{xx} + f_{yy} = e^{x^2+y^2}, \quad \text{in } B - \{0\}.$$

Use the boundedness of $\nabla f$ to show that

$$\lim_{\delta\to 0}\int_{\partial B_\delta} f_y\, dx - f_x\, dy = 0.$$

Use this fact and Green's theorem to evaluate

$$\int_{\partial B} f_y\, dx - f_x\, dy.$$

*Solution.* This problem is similar to Problem 10. I give a thorough solution to Problem 10, and so my solution to this problem will be more concise.

Pick some $\delta \leq 1$; we can rewrite

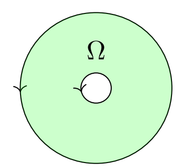
$$\int_{\partial B_\delta} f_y dx - f_x dy = \int_{\partial B_\delta} \nabla f \cdot (-dy, dx)$$

Now we use the Cauchy-Schwarz inequality, and conclude

$$\left| \int_{\partial B_\delta} \nabla f \cdot (-dy, dx) \right| \leq \int_{\partial B_\delta} \|\nabla f\| \, ds \leq \int_{\partial B_\delta} ds = 2\pi\delta$$

Where $ds = \sqrt{dx^2 + dy^2}$, and where we have used the boundedness of $\nabla f$. Thus we see that in the limit $\delta \to 0$ we have $\int_{\partial B_\delta} f_y dx - f_x dy \to 0$.

Now pick some $0 < \delta < 1$, and let $\Omega$ denote the region in between $B_\delta$ and $B$:




We use Green's Theorem on $\Omega$, and conclude that

$$\int_{\partial B} f_y dx - f_x dy - \int_{\partial B_\delta} f_y dx - f_x dy = \int_{\partial \Omega} f_y dx - f_x dy = -\iint_\Omega f_{xx} + f_{yy} \; dxdy$$

Where we have chosen the orientation of both $\partial B$ and $\partial B_\delta$ to be counterclockwise. Now we use the fact that we know $f_{xx} + f_{yy} = e^{x^2+y^2}$, and we obtain

$$\int_{\partial B} f_y dx - f_x dy = \int_{\partial B_\delta} f_y dx - f_x dy - \iint_\Omega e^{x^2+y^2} dxdy$$

Using polar coordinates, this becomes

$$\int_{\partial B} f_y dx - f_x dy = \int_{\partial B_\delta} f_y dx - f_x dy - \int_0^{2\pi} \int_\delta^1 e^{r^2} r \, dr$$

The integral on the right can be explicitly evaluated

$$\int_0^{2\pi} \int_\delta^1 \exp\left(r^2\right) r \, dr = \pi \int_\delta^1 2re^{r^2} \, dr = \pi(e - e^{\delta^2})$$

and so we obtain

$$\int_{\partial B} f_y dx - f_x dy = \int_{\partial B_\delta} f_y dx - f_x dy - \pi(e - e^{\delta^2})$$

Now this is true for all $\delta$, so we can let $\delta \to 0$, and then the first term on the right hand side vanishes, and so we obtain

$$\int_{\partial B} f_y dx - f_x dy = -\pi(e-1)$$

This completes the solution. ⌟

**Problem 4.** Evaluate the integral $\int_S (\nabla \times \mathbf{F}) \cdot dS$, where $S$ is the portion of a sphere defined by $x^2 + y^2 + z^2 = 1$, and $x + y + z \geq 1$, and where $\mathbf{F} = \mathbf{r} \times (\mathbf{i} + \mathbf{j} + \mathbf{k})$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, by observing that

$$\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{F}\, d\mathbf{r}$$

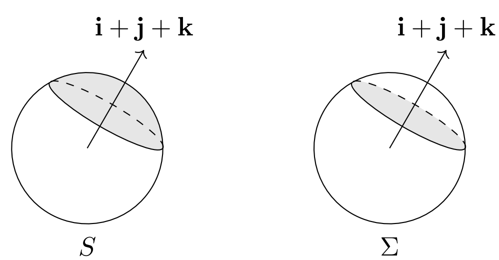
for any other surface $\Sigma$ with the same boundary as $S$. By picking $\Sigma$ appropriately, the new surface integral $\int_\Sigma (\nabla \times \mathbf{F}) \cdot dS$ may be easy to compute. Show that this is the case if $\Sigma$ is taken to be the portion of the plane $x + y + z = 1$ inside the circle $\partial S$.

*Solution.* First we observe that $\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{r}$ for any $\Sigma$ which shares the same boundary as $S$. This is obvious because if $\Sigma$ and $S$ share their boundary, then $\partial S$ and $\partial \Sigma$ are equal as curves, and so $\int_{\partial S} \mathbf{g} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{g} \cdot d\mathbf{r}$, for any choice of $\mathbf{g}$.

Next, we apply Stokes' theorem to the two surface integrals $\int_S (\nabla \times \mathbf{F}) \cdot dS$ and $\int_\Sigma (\nabla \times \mathbf{F}) \cdot dS$ to conclude that

$$\int_S (\nabla \times \mathbf{F}) \cdot dS = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{r} = \int_\Sigma (\nabla \times \mathbf{F}) \cdot dS \tag{5}$$

We have used Stokes' theorem in the first and third equality. Equation (5) gives us some flexibility in computing $\int_S (\nabla \times \mathbf{F}) \cdot dS$. We want to pick a new surface $\Sigma$ with a simple parametrization and with the property that $\nabla \times \mathbf{F}$ can be expressed nicely in terms of the chosen parametrization. Let us consider the following figure representing the region $S$; we show next to $S$ a natural choice for $\Sigma$.




It is clear from the above figure that if we choose $\Sigma$ to have the orientation so that its outer normal points in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$, then $\Sigma$ and $S$ will the same boundary. In words, we can describe $\Sigma$ as the region inside the plane $x + y + z = 1$ contained inside the boundary circle of $S$. The normal vector to the surface $\Sigma$ is a constant vector field which is $\mathbf{n} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ at every point on the surface.

The rest of the solution is fairly straightforward: we will simply compute $\int_\Sigma (\nabla \times F) \cdot dS$.

We now compute $\nabla \times \mathbf{F}$. Note that $\mathbf{F} = \mathbf{r} \times (\mathbf{i}+\mathbf{j}+\mathbf{k})$, which is

$$(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \times (\mathbf{i}+\mathbf{j}+\mathbf{k}) = (y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}$$

Then we compute the curl of this vector field (show this)

$$\left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right] \times [(y-z)\mathbf{i} + (z-x)\mathbf{j} + (x-y)\mathbf{k}] = -2(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

Since $\int_\Sigma \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int_\Sigma (\nabla \times F) \cdot \mathbf{n}\, dS$, and $\mathbf{n} = (\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3}$, we deduce that

$$\int_\Sigma (\nabla \times F) \cdot \mathbf{n}\, dS = \int_\Sigma \frac{-2}{\sqrt{3}}(\mathbf{i}+\mathbf{j}+\mathbf{k}) \cdot (\mathbf{i}+\mathbf{j}+\mathbf{k})\, dS = -2\sqrt{3}\int_\Sigma dS = -2\sqrt{3}(\text{area of } \Sigma)$$

Thus our solution will be complete if we can compute the area of $\Sigma$. Note that in the above calculations all we need to know about $\Sigma$ was that it had a constant outward normal vector field. There are a few ways we could proceed to compute the area of $\Sigma$.

(i) Parametrize $\Sigma$ in terms of 2-dimensional region $S$ by a smooth function $\Phi : D \to \Sigma$. Then write
$$\int_\Sigma dS = \int_D |\Phi_u \times \Phi_v|\, dS$$
Where $\Phi_u$ and $\Phi_v$ are the first order derivatives of $\Phi$. In this problem, we will first parametrize the plane $x+y+z=1$ by means of a map $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$, and then we will find which points in the domain get mapped to $\Sigma$.

(ii) Argue that $\Sigma$ is really a disk, and use the fact that the area of a disk is $\pi r^2$, where $r$ is its radius. This strategy only requires us to find the radius of $\partial\Sigma$, which shouldn't be too hard.

I prefer the second method, but it is pretty special since it uses the fact that $\partial\Sigma$ is a circle. In other problems the second method would not work.

For the first method, consider the linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ defined by the matrix

$$(u,v,w) \mapsto \frac{1}{\sqrt{6}}(u,v,w)\begin{pmatrix} 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \\ \sqrt{2} & \sqrt{2} & \sqrt{2} \end{pmatrix}$$

Since the rows of this matrix form an orthonormal basis, this matrix is orthogonal, and so it preserves the inner product on $\mathbb{R}^3$. Note that it maps the $u$-$v$ plane to the plane orthogonal to $\mathbf{i}+\mathbf{j}+\mathbf{k}$ passing through the origin. It follows that the map

$$(u,v) \mapsto \frac{1}{\sqrt{6}}(u,v)\begin{pmatrix} 1 & 1 & -2 \\ \sqrt{3} & -\sqrt{3} & 0 \end{pmatrix} + \mathbf{p}$$

parametrizes the plane orthogonal to $\mathbf{i}+\mathbf{j}+\mathbf{k}$ passing through the point $\mathbf{p}$. Since we know that the plane $x+y+z=1$ passes through the point $\mathbf{p}=(1/3,1/3,1/3)$, we deduce that

$$(u,v)\mapsto \frac{1}{\sqrt{6}}(u,v)\begin{pmatrix}1 & 1 & -2\\ \sqrt{3} & -\sqrt{3} & 0\end{pmatrix}+(1/3,1/3,1/3) \tag{6}$$

Call this map $\Phi:\mathbb{R}^2\to\mathbb{R}^3$. Then $\Phi$ parametrizes the plane containing $\Sigma$. Now we need to find the region $D$ such that the map defined by (6) maps $D$ onto $\Sigma$. We know that $\Sigma$ is defined as the set of all points satisfying $x+y+z=1$ and $x^2+y^2+z^2\leq 1$. Since every point $(x,y,z)$ in the image of the map defined by (6) satisfies $x+y+z=1$, we only need to find those points which satisfy $x^2+y^2+z^2\leq 1$. We set up the equation

$$\left[\frac{1}{\sqrt{6}}(u,v)\begin{pmatrix}1 & 1 & -2\\ \sqrt{3} & -\sqrt{3} & 0\end{pmatrix}+(1/3,1/3,1/3)\right]\cdot\left[\frac{1}{\sqrt{6}}\begin{pmatrix}1 & \sqrt{3}\\ 1 & -\sqrt{3}\\ -2 & 0\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}+\begin{pmatrix}1/3\\ 1/3\\ 1/3\end{pmatrix}\right]\leq 1$$

Simplifying, we obtain

$$(u,v)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}+1/3\leq 1 \implies u^2+v^2\leq\frac{2}{3}$$

Thus we see that the disk $D=\{(x,y): u^2+v^2\leq 2/3\}$ gets mapped (in a one-to-one fashion) to $\Sigma$. Furthermore, since the map we used to parametrize $\Sigma$ is an orthogonal linear transformation composed with a translation, it preserves areas (since translations and orthgonal transformations preserve areas), and so we conclude that the area of $\Sigma$ is the area of $D$, which is just $\pi r^2 = 2\pi/3$. For an alternate argument that the area of $\Sigma$ is the area of $D$, we conclude from our parametrization $\Phi: D\to\Sigma$ that

$$\int_\Sigma dS=\int_D|\Phi_u\times\Phi_v|\,dS$$

Since $\Phi_u=(1/\sqrt{6},1/\sqrt{6},-2/\sqrt{6})$ and $\Phi_v=(1/\sqrt{2},-1/\sqrt{2},0)$ at every point, and they are orthogonal unit vectors, we deduce that $|\Phi_u\times\Phi_v|$ is identically 1. Thus

$$\int_\Sigma dS=\int_D dS=2\pi/3$$

Now we return to the equation on a previous page to conclude that

$$\int_S(\nabla\times F)\cdot\mathbf{n}\,dS=\int_\Sigma(\nabla\times F)\cdot\mathbf{n}\,dS=-2\sqrt{3}\left[\frac{2\pi}{3}\right]=\frac{-4\pi}{\sqrt{3}}$$

Which is our final answer. I leave the alternate strategy (ii) I proposed as an exercise.

**Exercise.** Prove that $\Sigma$ is indeed a circle by showing that every plane intersects a sphere in a disk. Prove this first in the case where the plane is normal to the vector $\mathbf{k}$, and then use a rotation matrix to prove the general case. Find the radius of $\Sigma$ by finding the midpoint of $\Sigma$. Hint: the midpoint must lie on the line parallel to $\mathbf{i}+\mathbf{j}+\mathbf{k}$, because both the sphere and the plane are invariant under rotations about $\mathbf{i}+\mathbf{j}+\mathbf{k}$.

I include a few more exercises that you can do if you want to practice some of the techniques used in this solution.

**Exercise.**

(1) Find a parametrization $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ for the plane $x - 2z + y = 3$. Choose $\Phi$ so that its first order derivatives $\Phi_u(u, v)$ and $\Phi_v(u, v)$ form an orthonormal basis of the plane $x - 2z + y = 3$ at each point $(u, v)$. Hint: Let $\Phi$ be a map of the form $(u, v) \mapsto (u, v)A + t$, where $A$ is a matrix and $t$ is a fixed vector. What can you deduce about the tangent vectors $\Phi_u$ and $\Phi_v$ in terms the matrix $A$?

(2) Let $\Omega$ be a closed and bounded domain in $\mathbb{R}^2$, and suppose that the integral $\int_\Omega dS$ exists (so that we can define an area for $\Omega$). What is the area of $\Phi(\Omega)$ in terms of $\Omega$?

**Exercise.** Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be a map of the form $(u, v) \mapsto (u, v)A + t$, where $A$ is a $2 \times 3$ matrix and $t$ is a fixed vector in $\mathbb{R}^3$. Let $\Omega$ be a closed and bounded domain in $\mathbb{R}^2$, and suppose that the integral $\int_\Omega dS$ exists. What is the area of $\Phi(\Omega)$ in terms of $\Omega$? Under what conditions on $\Omega$ and $\Phi$ is the area of $\Phi(\Omega)$ equal to 0?

⌟

**Problem 5.** For a surface $S$ and a fixed vector $\mathbf{v}$, prove that

$$2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s}$$

where $\mathbf{r}(x, y, z) = (x, y, z)$.

*Solution.* The strategy behind this problem is very simple. First it is clear that the vector field $\mathbf{v} \times \mathbf{r}$ is very smooth (it component functions are linear combinations of $x$,$y$, and $z$), and so we may apply Stokes' theorem to write

$$\int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s} = \iint_S [\nabla \times (\mathbf{v} \times \mathbf{r})] \cdot \mathbf{n} \, dS.$$

The rest of the solution will show that

$$[\nabla \times (\mathbf{v} \times \mathbf{r})] = 2\mathbf{v}$$

and so we will obtain

$$\int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s} = 2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS.$$

To show that $[\nabla \times (\mathbf{v} \times \mathbf{r})] = 2\mathbf{v}$ is not too hard. We begin with

$$\mathbf{v} \times \mathbf{r} = (v_2 z - v_3 y, v_3 x - v_1 z, v_1 y - v_2 x)$$

Then

$$[\nabla \times (\mathbf{v} \times \mathbf{r})]_1 = \frac{\partial}{\partial y}(v_1 y - v_2 x) - \frac{\partial}{\partial z}(v_3 x - v_1 z) = 2v_1$$

similar computations show that $[\nabla \times (\mathbf{v} \times \mathbf{r})]_2 = 2v_2$ and $[\nabla \times (\mathbf{v} \times \mathbf{r})]_3 = 2v_3$, so that we have

$$[\nabla \times (\mathbf{v} \times \mathbf{r})] = 2\mathbf{v}$$

and using the remarks at the beginning of the solution, we conclude that

$$\int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot d\mathbf{s} = 2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS.$$

⌟

**Problem 6.**

(a) Show that $\mathbf{F} = -\mathbf{r}/\|\mathbf{r}\|^3$ is the gradient of $f(x, y, z) = 1/r$, where $r = \sqrt{x^2 + y^2 + z^2}$.

(b) What is the work done by the force $\mathbf{F} = -\mathbf{r}/\|\mathbf{r}\|^3$ in moving a particle from a point $\mathbf{r}_0 \in \mathbb{R}^3$ to "$\infty$", where $\mathbf{r}(x, y, z) = (x, y, z)$.

*Solution.* We want to show that $\nabla(1/r) = \mathbf{F}$. This is a straightforward computation: we know that

$$\nabla(1/r) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\mathbf{i} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\mathbf{j} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\mathbf{k}$$

And now the chain rule for differentiation implies that

$$\nabla(1/r) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}\mathbf{i} + \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial y}\mathbf{j} + \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial z}\mathbf{k}$$

We compute

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{1}{r^2} \qquad \frac{\partial r}{\partial x} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{r}$$

and so

$$\nabla(1/r) = \frac{-\mathbf{r}}{\|\mathbf{r}\|^3} \qquad \text{since } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } \|\mathbf{r}\| = r.$$

We note one subtlety: $\mathbf{F}(x, y, z) = \nabla(1/r)$ for all $x, y, z \in \mathbb{R}^3 - \{0\}$, and $\mathbf{F}(x, y, z)$ is undefined at $(x, y, z) = 0$. In fact, it is easy to see that $\lim_{(x,y,z)\to 0} \mathbf{F}(x, y, z)$ diverges, so that the vector field $\mathbf{F}(x, y, z)$ cannot be extended to a vector field on all of $\mathbb{R}^3$. Now we move on to part (b) of the problem. We use the following definition of work of a vector field $\mathbf{F}$ along a curve $\mathbf{c}$:

$$\text{work of } \mathbf{F} \text{ along } \mathbf{c} = \int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}$$

Our strategy is the following: We will prove that work $W(\mathbf{r}_0, \mathbf{r}_1)$ done in travelling from $\mathbf{r}_0$ to $\mathbf{r}_1$ is independent of the path $\mathbf{c}$ used to travel from $\mathbf{r}_0$ to $\mathbf{r}_1$. Then we will compute

$W(\mathbf{r}_0, \mathbf{r}_1)$ as a function of $\mathbf{r}_1$, and show that the limit $\lim_{\mathbf{r}_1 \to \infty} W(\mathbf{r}_0, \mathbf{r}_1)$ converges. This will be our answer for part (b).

Let $\mathbf{c} : [0, 1] \to \mathbb{R}^3 - \{(0, 0, 0)\}$ and be a curve which has $\mathbf{c}(0) = \mathbf{r}_0$ and $\mathbf{c}(0) = \mathbf{r}_1$. Then

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

Now we write $\mathbf{F}$ as the gradient of $f(x, y, z) = 1/r(x, y, z)$.

$$\int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt = \int_0^1 \nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt$$

Then the chain rule tells us that $\nabla f(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \frac{d}{dt} f(\mathbf{c}(t))$, and so

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \frac{d}{dt} f(\mathbf{c}(t))\, dt$$

The fundamental theorem of calculus kicks in, and allows us to conclude that

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \frac{d}{dt} f(\mathbf{c}(t))\, dt = \frac{1}{\|\mathbf{r}_1\|} - \frac{1}{\|\mathbf{r}_0\|}$$

This proves that $W(\mathbf{r}_0, \mathbf{r}_1)$ is independent of the curve $\mathbf{c}$. Furthermore, it is clear that as $\|\mathbf{r}_1\| \to \infty$, we have

$$\lim_{\mathbf{r}_1 \to \infty} W(\mathbf{r}_0, r_1) \to -\frac{1}{\|\mathbf{r}_0\|}$$

Thus the work done by $\mathbf{F}$ in moving a particle from $\mathbf{r}_0$ to $\infty$ is $-1/\|\mathbf{r}_0\|$. Note that this work is *negative*, which can be understood physically as the fact that $\mathbf{F}$ is attractive (so that it *requires* energy to go to $\infty$). ⌟

**Problem 7.** Let

$$\mathbf{F} = \frac{-GmM\mathbf{r}}{\|r\|^3}$$

be the gravitational force field defined in $\mathbb{R}^3 - \{0\}$.

(a) Show that $\mathrm{div}(\mathbf{F}) = 0$.

(b) Show that $\mathbf{F} \neq \mathrm{curl}(\mathbf{F})$ for any $C^1$ vector field $\mathbf{G}$ on $\mathbb{R}^3 - \{0\}$.

*Solution.* For simplicity, set $GmM = 1$ (this does not affect the important parts of this problem in any way). Let $\mathbf{F} = (F_1, F_2, F_3)$. Then

$$\frac{\partial}{\partial x} F_1(x, y, z) = \frac{\partial}{\partial x}\left(\frac{x}{(x^2 + y^2 + z^2)^{3/2}}\right) = \frac{1}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}}$$

Similar results hold for the $y$ and $z$ derivatives (simply replace $x$ by $y$ or $z$ in the above equation), and we conclude that

$$\frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3 = \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} = 0$$

Which is what we wanted to show.

Part (b) requires a little thought. How can we show that $\mathbf{F} \neq \text{curl}(\mathbf{G})$, for all possible vector fields $\mathbf{G}$? One approach is to find a property that every vector field of the form $\text{curl}(\mathbf{G})$ shares, and the show that $\mathbf{F}$ *doesn't* have this property. One basic property of vector fields of the form $\text{curl}(\mathbf{G})$ is that they can be used in Stokes' theorem: namely, if $S$ is an oriented surface with oriented boundary $\partial S$, then

$$\iint_S \text{curl}(\mathbf{G}) \cdot \mathbf{n}\, dA = \int_{\partial S} \mathbf{G} \cdot d\mathbf{s}$$

In particular, if $S$ is a surface *with no boundary*, then

$$\iint_S \text{curl}(\mathbf{G}) \cdot \mathbf{n}\, dA = 0$$

Thus if we can find some boundaryless surface $S$ such that

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dA \neq 0 \tag{7}$$

then we will prove that $\mathbf{F}$ is not the curl of another vector field $\mathbf{G}$. A natural choice for a boundaryless surface is the unit-sphere $S^2$ centered at the origin. The outer normal vector to the surface of $S^2$ is just $\mathbf{r}$:

**Remark.** The (intuitively obvious) fact that $\mathbf{r}$ is the normal vector to $S^2$ can be proved in a few ways. First, we could consider the parametrization of the sphere

$$(\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$$

(see the end of this document if this confuses you). Then we could compute the cross product of the two tangent vectors to this surface, and show that it points in the same direction as $(\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$. I leave this method as an exercise.
The other way we could prove this fact is by noting that the unit sphere is defined by the equality $\langle \mathbf{r}, \mathbf{r} \rangle = 1$. In other words, $S^2$ is the level set of the function $f(\mathbf{r}) = \langle \mathbf{r}, \mathbf{r} \rangle$, and so it follows that $\mathbf{n}(\mathbf{r})$ is parallel to the gradient $\nabla f(\mathbf{r}) = 2\mathbf{r}$. Requiring that $\mathbf{n}(\mathbf{r})$ is normalized yields $\mathbf{n}(\mathbf{r}) = \mathbf{r}$.

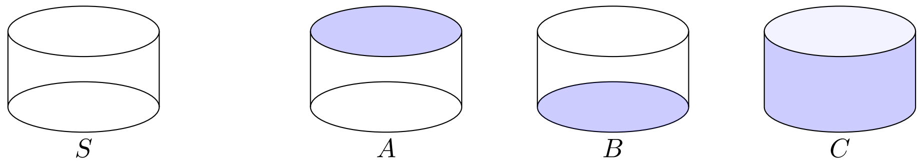
Since $\mathbf{F} = -\mathbf{r}/\|\mathbf{r}\|^3$, the surface integral in (7) becomes quite simple:

$$\iint_{S^2} \mathbf{F} \cdot \mathbf{n}\, dA = -\iint_{S^2} \frac{\mathbf{r} \cdot \mathbf{r}}{\|\mathbf{r}\|^3}\, dA = -\iint_{S^2} \frac{1}{\|\mathbf{r}\|}\, dA = -\iint_{S^2} dA = -4\pi$$

Where we have used the fact that $\|\mathbf{r}\| \equiv 1$ on $S^2$, and the fact that the area of $S^2$ is $4\pi$. Since $-4\pi \neq 0$, this proves that $\mathbf{F}$ cannot be the curl of a vector field $\mathbf{G}$. ⌟

**Problem 8.** Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$, where $\mathbf{F}(x, y, z) = \mathbf{i} + \mathbf{j} + z(x^2 + y^2)^2 \mathbf{k}$, and $S$ is the surface of the cylinder $x^2 + y^2 \leq 1$, $0 \leq z \leq 1$.

*Solution.* First lets look at a picture of $S$, which we write as a union of three pieces




In terms of the sets $A, B$ and $C$, we can write

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dA = \int_A \mathbf{F} \cdot \mathbf{n} \, dA + \int_B \mathbf{F} \cdot \mathbf{n} \, dA + \int_C \mathbf{F} \cdot \mathbf{n} \, dA$$

Now on $A$ and $B$, the outer normal vectors are $+\mathbf{k}$ and $-\mathbf{k}$, respectively. We conclude that

$$\int_A \mathbf{F} \cdot \mathbf{n} \, dA = \int_A z(x^2+y^2)^2 \, dA = \int_A (x^2+y^2)^2 \, dA$$

Where we have used the fact that $z \equiv 1$ on $A$. Parametrizing the disk with polar coordinates $(r, \theta) \mapsto (r\cos\theta, r\sin\theta, 1)$, we deduce that

$$\int_A (x^2+y^2)^2 \, dA = \int_0^1 \int_0^{2\pi} r^4 \, r \, dr \, d\theta = \frac{\pi}{3}$$

We do a similar computation for the integral over $B$.

$$\int_B \mathbf{F} \cdot \mathbf{n} \, dA = -\int_B z(x^2+y^2)^2 \, dA = 0$$

Where we have used the fact that $z \equiv 0$ on $B$.

Next, we turn to the integral over $C$. First we observe that the surface of the cylinder is defined by the equation $x^2+y^2 = 1$, which is a level set of the function $f(x, y, z) = x^2+y^2$. We know that the normal vector $\mathbf{n}(x, y, z)$ to a level set of a function $f$ is parallel to $\nabla f(x, y, z)$. It follows that the normal vector on $C$ is given by $\mathbf{n}(x, y, z) = x\mathbf{i} + y\mathbf{j}$. This vector is normalized since $x^2 + y^2 = 1$ on $C$. Now we plug this into the surface integral over $C$

$$\int_C \mathbf{F} \cdot \mathbf{n} \, dA = \int_C x + y \, dA$$

We parametrize $C$ by the following map $[0, 1] \times [0, 2\pi] \to \mathbb{R}^3$, $(z, \theta) \mapsto (\cos\theta, \sin\theta, z)$. The Jacobian of this linear transformation is 1, because the two tangent vectors are $(-\sin\theta, \cos\theta, 0)$ and $(0, 0, 1)$, and since they are orthogonal unit vectors their cross product is also a unit vector. We deduce

$$\int_C \mathbf{F} \cdot \mathbf{n} \, dA = \int_C x + y \, dA = \int_0^1 \int_0^{2\pi} \cos\theta + \sin\theta \, d\theta, \, dz = 0$$

Since $\int_0^{2\pi} \cos\theta \, d\theta = \int_0^{2\pi} \sin\theta \, d\theta = 0$. Combining all of our results, we conclude

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dA = \frac{\pi}{3}$$

There is another way to solve this problem. We invoke Gauss' Divergence theorem which states that

$$\int_S \mathbf{F} \cdot \mathbf{n} \, dA = \int_\Omega \operatorname{div}(\mathbf{F}) \, dV$$

where $\Omega$ is the solid cylinder. We parametrize $\Omega$ with cylindrical coordinates

$$(r, \theta, z) \mapsto (r\cos\theta, r\sin\theta, z) \qquad (r, \theta, z) \in [0,1] \times [0, 2\pi] \times [0,1]$$

A simple calculation shows that the Jacobian of this mapping is $r$. We calculate

$$\operatorname{div}(\mathbf{F}) = \frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(1) + \frac{\partial z(x^2+y^2)^2}{\partial z} = (x^2+y^2)^2$$

Then, in terms of our cylindrical coordinates, we have

$$\int_\Omega \operatorname{div}(\mathbf{F}) \, dV = \int_0^1 \int_0^{2\pi} \int_0^1 r^4 r \, dr \, d\theta \, dz = 2\pi \int_0^1 r^5 \, dr = \frac{\pi}{3}$$

So both ways to solve the problem give the same answer. ⌟

**Exercise.** This exercise explores a result we used in Problems 7 and 8. Prove the following:

**Theorem.** If $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function such that $f(\mathbf{a}) = c$ and $\nabla f(\mathbf{a}) \neq 0$, then prove that the normal vector $\mathbf{n}(\mathbf{a})$ to the $(n-1)$-dimensional surface defined by the equation $f(\mathbf{x}) = c$ is parallel to $\nabla f(\mathbf{a})$; that is, show that $n(\mathbf{a}) = \lambda \nabla f(\mathbf{a})$ for some non-zero $\lambda \in \mathbb{R}$. In the proof, assume the implicit function theorem.

**Hints.** Use the implicit function theorem to conclude a local parametrization of the surface $f(\mathbf{x}) = \mathbf{c}$. More precisely, use the theorem to show that there are two open sets $U \subset \mathbb{R}^{n-1}$ and $V \subset \mathbb{R}^n$ such that $0 \in U$ and $a \in V$, and a differentiable map $\Phi : U \to V$ such that

$$f(\Phi(x_1, \cdots, x_{n-1})) = \mathbf{c} \qquad \text{for all } (x_1, \cdots, x_{n-1}) \in U.$$

and $f(0) = \mathbf{a}$. Furthermore, show that we may assume that the rank of $D\Phi$ is $n-1$.

(a) Pick $\delta$ small enough that $(-\delta, \delta)^{n-1} \subset U$. Consider the $n-1$ curves $\gamma_i : (-\delta, \delta) \to V$ defined by $\gamma_i(t) = \Phi(0 + t\mathbf{e}_i)$.

(b) Since $\Phi$ has maximum rank, prove that the tangent vectors $\left\{\gamma_1'(0), \cdots, \gamma_{n-1}'(0)\right\}$ span an $(n-1)$-dimensional subspace (called the tangent space at $\mathbf{a}$ to the surface defined by $f(\mathbf{x}) = \mathbf{c}$).

(c) Prove that $\mathbf{n}(\mathbf{a}) \cdot \gamma_i'(0) = 0$ for each $i = 1 \cdots, n-1$. (Here use the fact that the curves $\gamma_i$ remain inside the surface). Deduce that $\left\{\gamma_1'(0), \cdots, \gamma_{n-1}'(0), \mathbf{n}(\mathbf{a})\right\}$ forms a basis for $\mathbb{R}^n$.

(d) Show that $\nabla f(\mathbf{a}) \cdot \gamma_i'(0) = 0$ for each $i$ by considering the rate of change of $t \mapsto f(\gamma_i(t))$. Deduce that we must have $\nabla f(\mathbf{a}) = \lambda \mathbf{n}(\mathbf{a})$ for some $\lambda$. Hint: if not then $\nabla f(\mathbf{a}) = \mathbf{n}(\mathbf{a}) + v$ for some $v$ which is orthogonal to every vector in $\left\{\gamma_1'(0), \cdots, \gamma_{n-1}'(0), \mathbf{n}(\mathbf{a})\right\}$.

**Problem 9.** Suppose that $\mathbf{F}$ is tangent to the closed surface $S = \partial W$ of a region $W$. Prove that

$$\iiint_W \operatorname{div}(\mathbf{F})\, dV = 0.$$

*Solution.* The thing we are trying to prove is certainly plausible: if we accept the interpretation of $\mathbf{F}$ as the vector field representing the velocities in a fluid, then $\iiint_W \operatorname{div}(\mathbf{F})\, dV$ represents the amount of fluid which diverges from the region $W$. If $\mathbf{F}$ is everywhere tangent to the surface $\partial W$, then it is intuitively true that no fluid can flow in or out, so the divergence should be zero.

Luckily, it is not hard for us to make the preceeding argument more precise. Gauss' Divergence theorem naturally applies to this problem, and we conclude that

$$\iiint_W \operatorname{div}(\mathbf{F})\, dV = \iint_{\partial W} \mathbf{F} \cdot \mathbf{n}\, dV$$

Since $\mathbf{F}$ is tangent to $\partial W$, it is orthogonal to $\mathbf{n}$, that is, $\mathbf{F} \cdot \mathbf{n} = 0$ everywhere on $\partial W$. Then it is immediate that

$$\iiint_W \operatorname{div}(\mathbf{F})\, dV = \iint_{\partial W} \mathbf{F} \cdot \mathbf{n}\, dV = 0$$

which is what we wanted to show. ⌟

**Problem 10.** Suppose that $f$ is a continuous function in $\mathbb{R}^3 - (0,0,0)$ and suppose that

$$\|\nabla f(X)\| \le \frac{1}{\|X\|}, \qquad f_{xx}(X) + f_{yy}(X) + f_{zz}(X) = \frac{1}{\|X\|^2}, \qquad \text{for all } X \in \mathbb{R}^3 - (0,0,0).$$

Denote $B_r = \left\{ \|X\|^2 \le r^2 \right\}$ for $r > 0$. Show that

$$\lim_{r \to 0} \int_{\partial B_r} \nabla f \cdot dS = 0.$$

Use this and the Divergence Theorem to evaluate

$$\int_{\partial B_1} \nabla f \cdot dS = 4\pi.$$

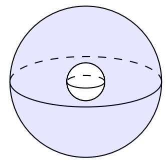
*Solution.* The rough outline for my solution is:

(i) Prove that $\lim_{r\to 0} \int_{\partial B_r} \nabla f \cdot dS = 0$.

(ii) Use the divergence theorem to conclude that

$$\int_{\partial B_1} \nabla f \cdot dS - \int_{\partial B_r} \nabla f \cdot dS = \iint_{B_1 - B_r} \operatorname{div}(\nabla f)\, dV$$

Since $\partial B_1 \cup \partial B_r$ is the boundary of $B_1 - B_r$ and the outer normal of $B_1 - B_r$ on $\partial B_r$ is actually the inner normal of $B_r$ on $\partial B_r$ (which explains the minus sign in the second term above). The figure below shows the region $B_1 - B_r$ in blue:



(iii) Use part (i) to take the limit $r \to 0$ in part (ii) to conclude that

$$\int_{\partial B_1} \nabla f \cdot dS = \int_{B_1} \Delta f \, dV \tag{8}$$

where $\Delta f = f_{xx} + f_{yy} + f_{zz}$.

(iv) Use the fact that $\Delta f(X) = 1/\|X\|^2$ to evaluate the integral (8).

The first part of our solution is showing that $\lim_{r\to 0} \int_{\partial B_r} \nabla f \cdot dS = 0$. We will use the fact that $\|\nabla f(X)\| \leq 1/\|X\|$.

**Intuition.** Why does $\lim_{r\to 0} \int_{\partial B_r} \nabla f \cdot dS = 0$ follow from $\|\nabla f(X)\| \leq 1/\|X\|$? The surface area of the ball of radius $r$ is $\sim r^2$. and the absolute value of the integrand is bounded above by $1/r$. Thus the integral is at least $\sim r$, and so when $r \to 0$ the integral vanishes. We prove this more rigourously below.

We first approximate

$$\left| \int_{\partial B_r} \nabla f(X) \cdot d\mathbf{S} \right| \leq \int_{\partial B_r} |\nabla f(X) \cdot n| \, dS \leq \int_{\partial B_r} \|\nabla f(X)\| \, \|n\| \, dS = \int_{\partial B_r} \|\nabla f(X)\| \, dS$$

Where we have used the fact that $\left|\int f \, dV\right| \leq \int |f| \, dV$ in the first inequality, and we have used the Cauchy-Schwarz inequality in the second inequality. In the final equality, we have used the fact that $\|n\| = 1$. Since

$$\|\nabla f(X)\| \leq \frac{1}{\|X\|}$$

and $\|X\| = r$ on $\partial B_r$, we deduce that for all $X \in \partial B_r$ we have $\|\nabla f(X)\| \leq 1/r$ and thus

$$\int_{\partial B_r} \|\nabla f(X)\| \, dS \leq \int_{\partial B_r} \frac{1}{r} \, dS = \frac{1}{r} \int_{\partial B_r} dS = \frac{1}{r} \left[4\pi r^2\right] = 4\pi r$$

Where we have used the fact that the surface area of a sphere is $4\pi r^2$. Combining all of these estimates, we deduce that

$$0 \leq \left| \int_{\partial B_r} \nabla f(X) \cdot d\mathbf{S} \right| \leq 4\pi r$$

and so when we take $r \to 0$, we must have $\lim_{r\to 0} \int_{\partial B_r} \nabla f(X) \cdot d\mathbf{S} = 0$ by the "squeeze theorem." This completes the first part of our solution.

For the second part, we refer to the discussion in (ii) at the beginning of the solution and to the divergence theorem to write

$$\iint_{B_1 - B_r} \operatorname{div}(\nabla f)\, dV = \int_{\partial B_1} \nabla f \cdot d\mathbf{S} - \int_{\partial B_r} \nabla f \cdot d\mathbf{S} \tag{9}$$

Using the fact that $\operatorname{div}(\nabla f(X)) = 1/\left\|X\right\|^2$ (since $\operatorname{div}(\nabla f(X)) = \Delta f(X)$), we can write the left hand side as

$$\iint_{B_1 - B_r} \frac{1}{\left\|X\right\|^2}\, dV$$

Now we use spherical coordinates to evaluate this integral, writing it as an iterated integral

$$\int_r^1 \int_0^\pi \int_0^{2\pi} \frac{1}{\rho^2} \left[\rho^2 \sin\phi\right]\, d\theta\, d\phi\, d\rho = \int_r^1 1\, d\rho \int_0^\pi \int_0^{2\pi} \sin\phi\, d\theta\, d\phi = 4\pi\,[1 - r]$$

I trust that the reader is familiar with spherical coordinates (if not, see the end of this document). Plugging this result back into (9) we conclude that

$$4\pi\,[1 - r] = \int_{\partial B_1} \nabla f \cdot d\mathbf{S} - \int_{\partial B_r} \nabla f \cdot d\mathbf{S}$$

Now when we take $r \to 0$ the second term on the right vanishes (we proved this in the first part of the solution!) so

$$4\pi = \int_{\partial B_1} \nabla f \cdot d\mathbf{S}$$

as we wanted to show. ⌟

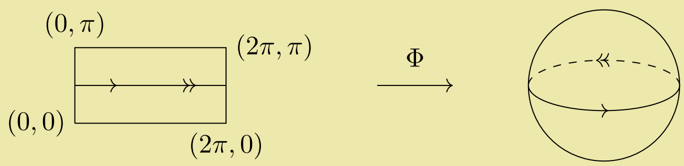
**Attention.** On the next page I include information about spherical coordinates, in case the reader wants to refresh his/her knowledge of them.

**Spherical coordinates.** I include this box since spherical coordinates are used multiple times in this assignment.

A particular parametrization which frequently occurs is the parametrization of the sphere and the ball known as *spherical coordinates*. For completeness, this parametrization of the sphere $S^2$ is a map $\Phi : [0, 2\pi] \times [0, \pi] \to S^2$ defined by

$$\Phi(\theta, \phi) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

The following figure might help visualize what this parametrization "looks like"




Using this parametrization, we can use the *change of variables formula* for parametrized surfaces to rewrite a surface integral $\int_{S^2} f\,dS$ as

$$\int_{S^2} f\,dS = \int_0^{\pi}\int_0^{2\pi} f(\Phi(\theta,\phi))\,|\Phi_\theta(\theta,\phi)\times\Phi_\phi(\theta,\phi)|\;d\theta\,d\phi$$

A simple calculation shows us that

$$|\Phi_\theta(\theta,\phi)\times\Phi_\phi(\theta,\phi)| = |\sin\phi| = \sin\phi,$$

where we are allowed to remove the absolute value signs since $\sin\phi \geq 0$ when $\phi \in [0, \pi]$.

A simple adjustment turns this parametrization into a parametrization $\Sigma : [0, 1] \times [0, 2\pi] \times [0, \pi] \to B_1$ of the unit ball. Namely,

$$\Sigma(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

Using this formula, we can use another change of variables theorem to rewrite the integral $\int_{B^1} f\,dV$ as

$$\int_{B^1} f\,dS = \int_0^1\int_0^{2\pi}\int_0^{\pi} f(\Sigma(\rho,\theta,\phi))\,|\det D\Sigma(\rho,\theta,\phi)|\;d\phi\,d\theta\,d\rho$$

A simple calculation shows that $|\det D\Sigma(\rho,\theta,\phi)| = \rho^2\sin\phi$.

We note one small subtlety. The change of variables formula for integrals is usually stated: Let $A$ and $B$ be bounded domains in $\mathbb{R}^n$, if $\Sigma : A \to B$ is a smooth *one-to-one* and *onto* function (that is it is a smooth *bijection*) and $f : B \to \mathbb{R}$ is an integrable function, then

$$\int_B f\,dV = \int_A f\circ\Sigma\;|\det D\Sigma|\;dV$$

The parametrization we use for the unit ball is *not* one-to-one (show this) and so, a priori, we cannot apply the change of variables theorem. The next exercise treats this subtlety.

**Exercise.** Let $\Sigma : [0,1] \times [0,2\pi] \times [0,\pi] \to B_1$ be the parametrization of the unit ball defined on the previous page.

(a) Prove that if we restrict $\Sigma$ to the domain $(0,1] \times (0,2\pi) \times (0,\pi)$, then it is a one-to-one function. Furthermore, prove that $\det D\Sigma$ is strictly positive on this set.

(b) Let $A = \Sigma((0,1] \times (0,2\pi) \times (0,\pi))$. Then $A \subset B_1$. Try to imagine what this subset looks like.

(c) Prove that if $f$ is any integrable function defined on the unit ball then

$$\int_A f\,dV = \int_{B_1} f\,dV$$

To do this, assume that $\int_{B_1} f\,dV = \int_{B_1 - A} f\,dV + \int_A f\,dV$, and show that $B_1 - A$ can be covered by open rectangles of arbitrarily small volume, so that $\int_{B_1 - A} f = 0$ for all integrable $f$.

(d) Apply the following change of variables theorem (and Fubini's theorem to get the iterated integral!) to deduce that

$$\int_{B_1} f\,dV = \int_0^1 \int_0^{2\pi} \int_0^{\pi} f(\Sigma(\rho,\theta,\phi))\rho^2 \sin\phi, d\phi\,d\theta\,d\rho$$

**Theorem.** Let $A \subset \mathbb{R}^n$ be an open set and $\Sigma : A \to B$ a continuously differentiable bijection such that $\det D\Sigma(x) \neq 0$ for all $x \in A$. If $f : B \to \mathbb{R}$ is integrable, then

$$\int_B f\,dV = \int_A (f \circ \Sigma)\,|\det \Sigma|\;dV$$

**Attention.** On the next page I include a somewhat challenging exercise for those students who are interested.

**Exercise.** The goal of this exercise is to strengthen the statement of Green's theorem. Prove the following:

**Theorem.** Let $\Omega \subset \mathbb{R}^2$ be a bounded region with oriented boundary $\partial\Omega$; suppose that Green's theorem applies to $\Omega$. Suppose there is a continuously differentiable bijection $\Phi : \Omega \to \Lambda$, with $\det D\Phi > 0$. Suppose further that $\Phi|_{\partial\Omega}$ maps $\partial\Omega$ onto $\partial\Lambda$. Prove that Green's theorem also applies to $\Lambda$.

*Proof.* In the proof, we label points in $\Omega$ by coordinates $(u, v)$, we label points in $\Lambda$ by coordinates $\Phi(u, v) = (x(u, v), y(u, v))$.

(a) Pick two $C^1$ functions $P : \Lambda \to \mathbb{R}$ and $Q : \Lambda \to \mathbb{R}$. Show that

$$\int_{\partial\Lambda} P(x,y)\,dx + Q(x,y)\,dy = \int_{\partial\Omega} [P(x,y)x_u + Q(x,y)y_u]\,du + [P(x,y)x_v + Q(x,y)y_v]\,dv$$

(b) Apply Green's theorem to the pair of functions $P(x,y)x_u+Q(x,y)y_u$ and $P(x,y)x_v+Q(x,y)y_v$ to conclude that the above integral is equal to

$$\int_\Omega \frac{\partial P(x,y)}{\partial u}x_v + \frac{\partial Q(x,y)}{\partial u}y_v - \frac{\partial P(x,y)}{\partial v}x_u - \frac{\partial Q(x,y)}{\partial v}y_u\,du\,dv$$

(c) Use the chain rule to conclude that the above integral is equal to

$$\int_\Omega \left[\frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y}\right](x_u y_v - x_v y_u)\,du\,dv$$

(d) Use the fact that $\det D\Phi > 0$ and the change of variables formula to conclude that the above integral is equal to $\int_\Lambda \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$, and thus

$$\int_{\partial\Lambda} P(x,y)\,dx + Q(x,y)\,dy = \int_\Lambda \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy$$

Which is what we want to show.

□

What we are showing here is that Green's theorem is a property of domains in $\mathbb{R}^2$ which is preserved under *diffeomorphisms* (a diffeomorphism is a continuously differentiable bijection with a differentiable inverse). This means that if we take any shape $\Omega$ and deform it in a smooth fashion, we do not change the conclusion of Green's theorem.