LIE GROUPS, ASSIGNMENT 1, DUE SEPTEMBER 21

In this assignment (and hopefully later in class), I will write $U(n) = \{A \in GL_n(\mathbb{C}) : AA^* = I\}$ and $O(n) = \{A \in GL_n(\mathbb{R}) : AA^{tr} = I\}$ (and with an S in front of them for determinant 1).

(1) Prove that $S^{2n-1} \cong SU(n)/SU(n-1)$. Conclude that S^3 has the structure of a Lie group.

(2) Recall that $GL_n(\mathbb{C})$ and $U(n)$ act transitively on $Fl(\mathbb{C}^n)$. Consider the standard flag

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \mathbb{C}^n$$

Find the stabilizers for the action of each of these two groups on the standard flag.

(3) Using (1) and $S^{n-1} \cong O(n)/O(n-1)$, find

$$\pi_0(O(n)),\ \pi_1(O(n)),\ \pi_0(SU(n)),\ \pi_1(SU(n))$$

(4) Let G be a connected Lie group. Let $H \subset G$ be a discrete, normal subgroup. Prove that H is central in G.

(5) Let G be a Lie group. Prove that $\pi_1(G)$ is abelian. (Hint: apply the previous exercise to $\tilde{G}$.)

(6) Consider the adjoint action of $SU(2)$ on its Lie algebra $\mathfrak{su}(2)$ (this action is given by $g \cdot X = gXg^{-1}$). Define a bilinear form on $\mathfrak{su}(2)$ by $(X, Y) = -tr(XY)$. Prove that this form is symmetric, positive definite, and invariant under $SU(2)$. Use this to prove that $SO(3) = SU(2)/\{\pm I\}$ and deduce that $SU(2)$ is the universal cover of $SO(3)$.

(7) Let G be a connected Lie group. Let $U \subset G$ be an open neighbourhood of $e \in G$. Prove that U generates G as a group (i.e. there is no proper subgroup of G containing U).

(8) Find the Lie algebra of the symplectic group $Sp_{2n}(\mathbb{R})$. Use this to find the dimension of $Sp_{2n}(\mathbb{R})$.